5.8.1. LIMITS AND CONTINUITY OF FUNCTIONS

Limit at a point

A function has a limit L in a point a if the value of the function approaches the value L when the input x approaches the value a.

Whenever you see *limit*, you can think of *approaching*.

The notation is as follows

$$\lim_{x \to a} f(x) = L$$

And mathematical definition is:

If for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, then the function $f(x)$ has a limit L in a point a and we can write $\lim_{x \to a} f(x) = L$.

Limit is mostly calculated in points where the function is not defined.

Here is an example. The function $f(x) = \frac{\ln(x+1)}{\ln(x)}$ is not defined for $x < 0$. Function is presented in Figure 5.86.

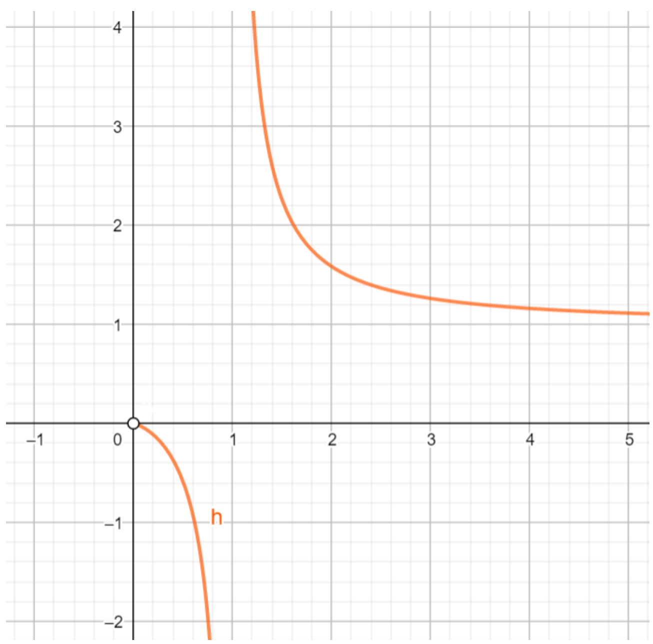

Figure 5.86

Hence, $a = 0$. If the input x approaches to zero, the value of the function approaches to $y = 0$ and we can write $\lim_{x \to 0} \frac{\ln(x+1)}{\ln(x)} = 0$

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Limit at infinity

If the value of a function approaches to the value L as x gets larger, then a function f has a limit L at infinity.

The notation is as follows

$$\lim_{x \to \infty} f(x) = L$$

Definition (limit):

If for every $\varepsilon > 0$, there exists $m > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x > m$, then the function $f(x)$ has a limit L in infinity and we can write $\lim_{x \to \infty} f(x) = L$.

Here is an example. The function $f(x) = \frac{1}{x}$.

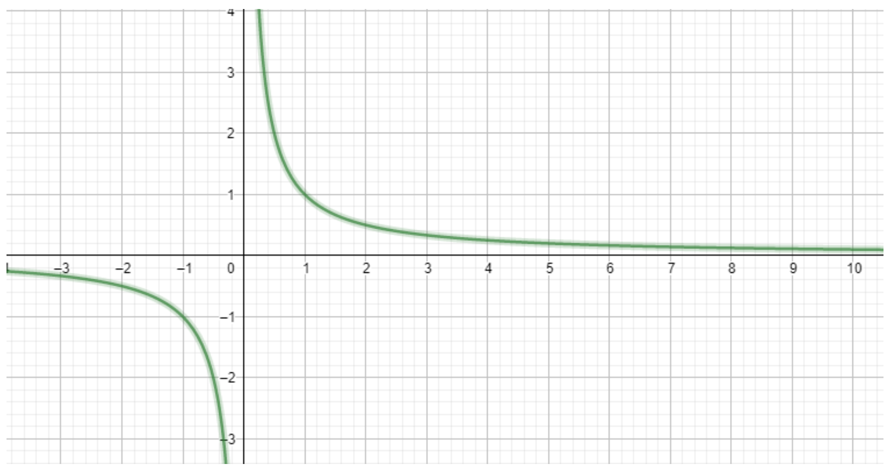

Figure 5.87

As the input value x gets larger, the value of a function approaches to zero.

x	$f(x)$
10	0,1
100	0,01
1000	0,001

Also, we can see that as the value of x gets smaller, the value of a function also approaches to zero.

x	$f(x)$
–10	–0,1
–100	–0,01
–1000	–0,001

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And we can conclude that $\lim_{x\to\pm\infty} f(x) = 0$.

Infinite limit

If the value of a function gets larger as x approaches to value a, then a function f has a limit infinity in a point a.

The notation is as follows

$$\lim_{x\to a} f(x) = \infty$$

Definition:

If for every $M > 0$, there exists $\delta > 0$, such that $f(x) > M$ whenever $0 < |x - a| < \delta$, then the function $f(x)$ has a limit infinity in a point a and we can write $\lim_{x\to a} f(x) = +\infty$.

If for every $M < 0$, there exists $\delta > 0$, such that $f(x) < M$ whenever $0 < |x - a| < \delta$, then the function $f(x)$ has a limit infinity in a point a and we can write $\lim_{x\to a} f(x) = -\infty$.

Sometimes if we are not concerned whether $f(x)$ approaches to $+\infty$ or $-\infty$ we use this notation $\lim_{x\to a} f(x) = \pm\infty$ or just $\lim_{x\to a} f(x) = \infty$.

If we look at the function $f(x) = \frac{1}{x}$ in Figure 3, we can notice that as input value x approaches to zero the value of function approaches to $+\infty$ or $-\infty$. We can conclude that $\lim_{x\to 0} f(x) = \infty$.

Properties of limits

Suppose that $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = N$.

Then

P1 $\lim_{x\to a}\big(f(x) \pm g(x)\big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = M \pm N$

Addition property - the limit of a sum/difference is equal to the sum/difference of limits

P2 $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = M \cdot N$

Multiplication property - the limit of a product is equal to the product of limits

Note that this implies the following:

P2* $\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)\, , c \in R$I.e.

$\lim_{x\to 0} 2x = 2 \lim_{x\to 0} x$

P3 $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{M}{N}, N \neq 0$

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Division property - the limit of a quotient is equal to the quotient of limits

P4 $\lim_{x \to a} f(x)^k = M^k, M > 0$

Important limits are also called **common limits** and are usually listed in tables and/or known by heart.

The first common limit to remember is:

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

The second, more general limit is even more important:

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0, \qquad \text{where } n \in N$$

Also, as a side note, the limit of a constant is always equal to that constant:

$$\lim_{x \to \pm\infty} c = c$$

Let us now show the calculation of a more complex limit:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x}$$

Note that the technique described below is suitable only when calculating the limit at infinity.

The most common technique when dealing with limits is to divide the numerator and denominator of the fraction by the highest order polynomial in the denominator.

The expression can be written as:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x} : \frac{x^3}{x^3}$$

Divide the numerator and denominator separately:

$$\lim_{x \to \pm\infty} \frac{(x^3 + 3x^2 + 3) : x^3}{(x^3 - 7x) : x^3}$$

Since the parenthesis is divided by x^3, we can divide each term in the parenthesis separately:

$$\lim_{x \to \pm\infty} \frac{x^3 : x^3 + 3x^2 : x^3 + 3 : x^3}{x^3 : x^3 - 7x : x^3}$$

Remember the division property of powers? $x^m : x^n = x^{m-n}$
Now let us apply that property to every term in the numerator and denominator:

$$\lim_{x \to \pm\infty} \frac{x^{3-3} + 3x^{2-3} + 3x^{-3}}{x^{3-3} - 7x^{1-3}}$$

Calculating the differences, we obtain:

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$$\lim_{x\to\pm\infty} \frac{x^0 + 3x^{-1} + 3x^{-3}}{x^0 - 7x^{-2}}$$

Finally, let us calculate each term, and rewrite using the following power property: $ax^{-n} = \frac{a}{x^n}$.

Also, remember that $x^0 = 1$ if $x \neq 0$.

$$\lim_{x\to\pm\infty} \frac{1 + \frac{3}{x} + \frac{3}{x^3}}{1 - \frac{7}{x^2}}$$

Only now are we able to use the property P.3. (the limit of a quotient is the quotient of limits)

$$\frac{\lim\limits_{x\to\pm\infty}\left(1 + \frac{3}{x} + \frac{3}{x^3}\right)}{\lim\limits_{x\to\pm\infty}\left(1 - \frac{7}{x^2}\right)}$$

After using the property P.1. (the limit of a sum is the sum of limits) the expression simplifies further:

$$\frac{\lim\limits_{x\to\pm\infty} 1 + \lim\limits_{x\to\pm\infty}\frac{3}{x} + \lim\limits_{x\to\pm\infty}\frac{3}{x^3}}{\lim\limits_{x\to\pm\infty} 1 - \lim\limits_{x\to\pm\infty}\frac{7}{x^2}}$$

We now use P.2*. (the constant can be written before the limit):

$$\frac{\lim\limits_{x\to\pm\infty} 1 + 3\lim\limits_{x\to\pm\infty}\frac{1}{x} + 3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty} 1 - 7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}}$$

Finally, we can use the common limits written above:

$$\frac{\lim\limits_{x\to\pm\infty} 1 + 3\lim\limits_{x\to\pm\infty}\frac{1}{x} + 3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty} 1 - 7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}} = \frac{1 + 3\cdot 0 + 3\cdot 0}{1 - 7\cdot 0} = \frac{1+0+0}{1-0} = \frac{1}{1} = 1$$

Therefore, we conclude that:

$$\lim_{x\to\pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x} = 1$$

The procedure for calculating limits may seem complex right now, but we assure the reader that the procedure is quite straightforward:

1) Divide by the greatest order polynomial of the denominator
2) Simplify the expression in the numerator and denominator.

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This part was most of the work, but the work done has nothing to do with limits, just carefully dividing polynomials and using power properties. Here steps may be omitted if the student is confident with basic polynomial algebra.

3) Using the properties of limits, reduce the limit to a series of common limits, and calculate separately each common limit.

Let us once again stress that the following procedure is only useful for calculating limits when x approaches infinity.

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