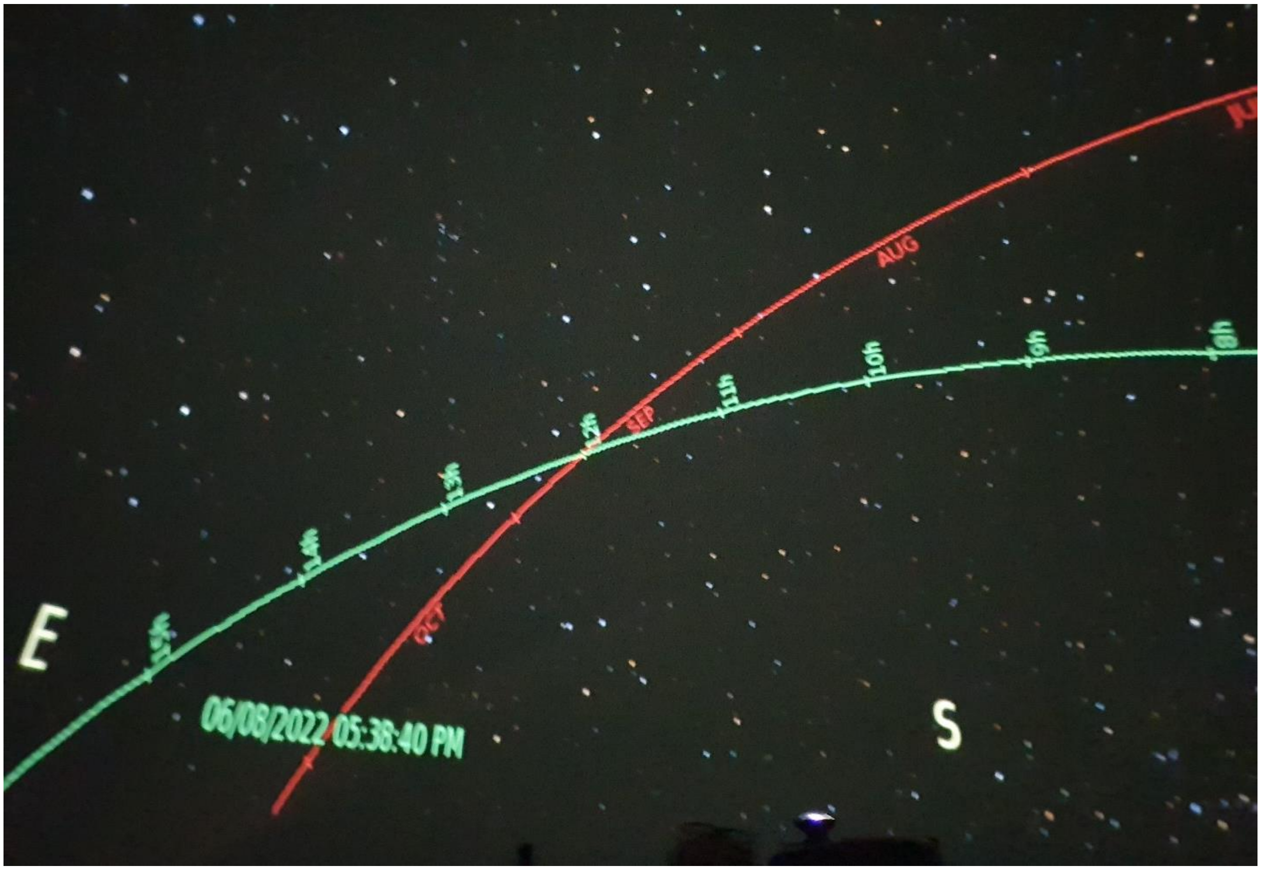
5. FUNCTIONS



The Ecliptic: The orbit of the Earth around the Sun looks like a circuit. Positions of bodies on the celestial sphere with respect to the ecliptic is a function of celestial longitude and latitude

ABSTRACT:

The function is one of the basic concepts in mathematics. The description starts with the definition of a function and introductory properties of functions: a sign of a function, intervals of monotonicity, odd and even functions, and limits of a function. Elementary functions, like linear, quadratic, exponential, logarithmic and trigonometric functions will be introduced.

This chapter provides examples of equations with terms involving these functions, and illustrates the algebraic techniques necessary to solve them and the importance of maritime issues. In short, it is essential to be familiar and comfortable with the fundaments of functions before proceeding to the formal introduction of calculus in the next chapter.

AIM: To learn basic concepts in functions, like a sign of a function, increasing or decreasing function and limit of a function and to solve maritime problems using functions.

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Learning Outcomes:

1. Understanding the definition of function and recognize whether the mapping is a function
2. Determine whether the function is injection, surjection and bijection
3. Find the intervals of monotonicity of a function
4. Solve the limits of a function
5. Use the appropriate function in solving maritime problems.

Prior Knowledge: algebraic expressions, algebraic identities, linear equations and inequalities.

Relationship to real maritime problems:

Functions are widely used in solving many engineering problems. Practical application of functions is in navigation theory, calculating the set and drift, the velocity of a vehicle, the travelling time, and distance.

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Content

5.1. FUNCTION. BASICS. GRAPH. FUNCTION TABLE

The function is one of the basic concepts in mathematics. In this section, a formal definition of a function is provided. Functions can be represented through tables, formulas, and graphs The formal notation and terms related to functions are provided.

The notion of notation of functions becomes clear with the help of examples. The section contains several maritime examples of using functions, using GeoGebra and MS Excel.

Definition (function):

Function f is a relation in which each object from the set of inputs X is associated to exactly one object from the set of outputs Y. Each function must have three elements defined:

1. Domain X – a set of inputs, i.e. a set of all arguments of the function

2. Mapping rule f – the way this data is transformed - functional equation

3. Codomain Y – a set of possible outputs

Element x is an ***argument*** (input) and element y is an ***image*** of x (the value of function or output).

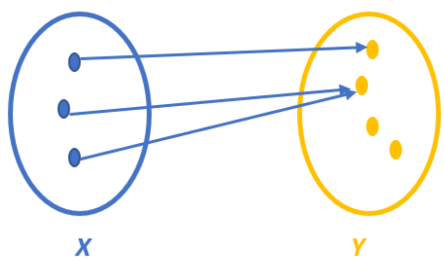

Figure 5.1

The statement can be written

$$f: X \to Y$$

Set X is called the domain, set Y is called the codomain.

Functions where domain and codomain are subsets of real number sets are commonly used in real life problems.

Functions $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$ and $g: \langle 0, \infty\rangle \to \langle -1, \infty\rangle$ defined by $g(x) = x^2 - 1$ are not the same functions, as we can see on Figure 5.2

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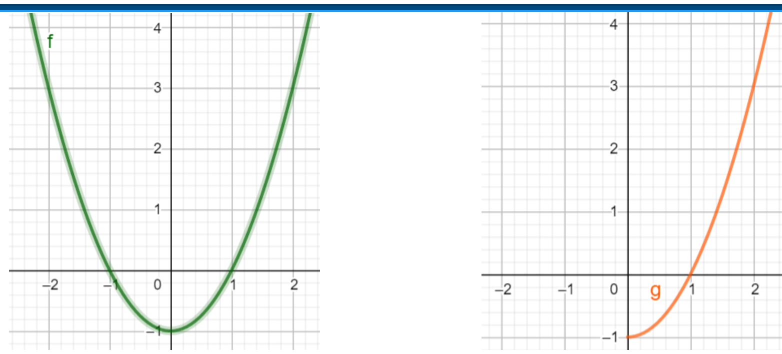
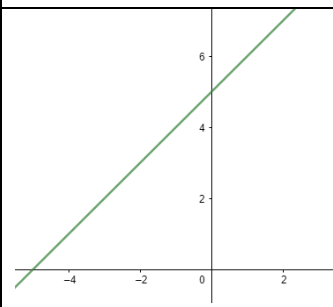


a) b)

Figure 5.2. Graph of functions: a) $f(x) = x^2 - 1$; *b)* $g(x) = x^2 - 1$

There are various ways to assign functions. The table below shows three different ways to set the function $f(x) = x + 5$.

<table>
<tr><th>Descriptive</th><th>Table</th><th>Graph</th></tr>
<tr><td>Add the number 5 to each real number.</td><td><table><tr><th>X</th><th>y</th></tr><tr><td>1</td><td>6</td></tr><tr><td>2</td><td>7</td></tr><tr><td>3</td><td>8</td></tr><tr><td>4</td><td>9</td></tr></table></td><td></td></tr>
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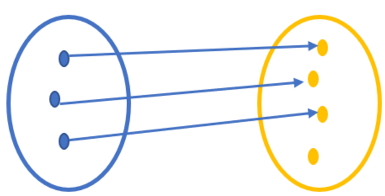
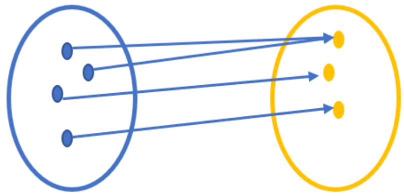
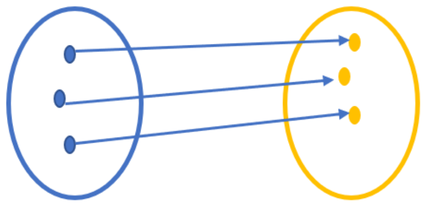
Injection is a one-to-one function. It is a function that maps distinct elements of the domain to distinct elements of the codomain.



A function is called ***surjection*** if every element of the codomain is mapped by at least one element of the domain.



A function is called ***bijection*** if a function is both injection and surjection.



Example 5.1

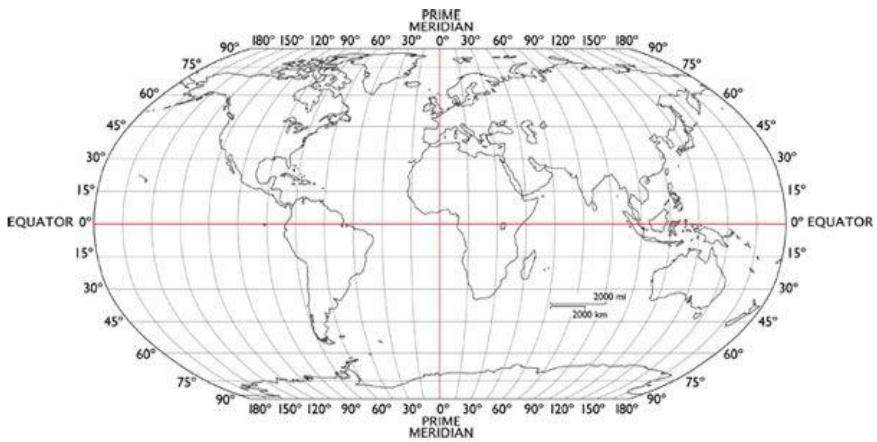


Figure 5.3

Global positioning using longitude and latitude is an example of a function. Each point on the Earth's surface is associated with a unique combination of two coordinates.

For example, the city of Split is associated with coordinates (43.5081°N, 16.4402°E) (43° 30’ 29’’ N and 16° 26’ 25’’ E).

On Google maps https://www.google.com/maps choose the place you want to travel to next summer and read its longitude and latitude. Convert the values to degrees, minutes and seconds.

Is the mapping that accompanies each point on the Earth's surface an ordered pair of number coordinates:

a) function

b) injection

c) surjection

d) bijection.

Solution:

Computer instructions: right click on a place on the map, select *What's here*? Latitude and longitude in degrees will be displayed.

Mobile phone instructions: Press and hold the selected place on the screen. The latitude and longitude in degrees will be displayed in the data entry bar.

You can check the calculation on your computer by clicking on the amount of latitude or longitude. Latitude and longitude in degrees, minutes, and seconds will be displayed in the left corner of the data entry bar.

The mapping that accompanies each point on the Earth's surface an ordered pair of number coordinates is a function, injection, surjection and bijection.

a) The mapping is a function, because each point on the surface has corresponding coordinates.

b) The function is an injection because two different places on the Earth's surface will surely have different coordinates. There cannot be two different cities with the same latitude and longitude.

c) The function is a surjection because all possible values of latitude and longitude are hit.

If we would somehow add another meridian and have 181° of longitude, that would be a problem. Then no place on Earth would be mapped to that 181° longitude, and the function would no longer be a surjection.

d) Since the function is an injection and surjection, it is a bijection.

This is important because it means we can go "backwards". Knowing the coordinates of one place, for instance (43.5081°N, 16.4402°E), we can distinctively say which point on the surface it is (Split).

Example 5.2

In a port, 300 passengers can board one of the 3 ships (A, B or C) that depart at the same time. Each ship has at least 2 passengers.

Is the mapping in which each passenger is associated to a ship:

a) function

b) injection

c) surjection

d) bijection

e) Is the reverse mapping assigned to each ship a list of passengers on it a function?

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Solution:

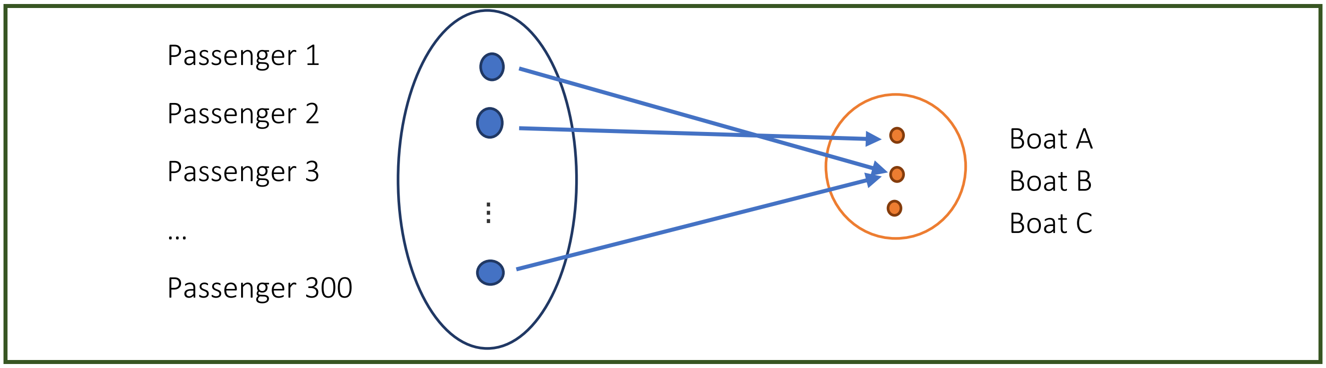

Figure 5.4

a) The mapping is a function because every passenger will board on exactly one ship.

b) The mapping is not an injection, because it is not one-to-one mapping. There are 300 hundred passengers and only 3 ships, so several passengers' board on the same ship.

c) The mapping is a surjection, because every element of the codomain (every ship) has at least one associated element of the domain (passenger).

d) The mapping is not a bijection, because it is not an injection.

e) Reverse mapping is not a function because every ship is not associated to exactly one passenger, but to many passengers.

Example 5.3

The GPS system receives messages about the coordinates of the Nautilus ship every hour during the four-hour voyage.

The last voyage of the ship is given by the table

Time	X coordinate (Northern latitude)	Y coordinate (Eastern longitude)
12:00	44.52	14.51
13:00	44.52	14.62
14:00	44.52	14.69
15:00	44.52	14.81
16:00	44.52	14.89

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The ship's voyage was sketched in Geogebra. https://www.geogebra.org/m/shtqu5kq In which direction did the ship move? What was its course?

If we know that the total length of the voyage was 24 km, what was the average speed of the ship?
Did the ship have a steady speed during the voyage?
Then did the ship go the fastest and when the slowest?
Determine the speed of the ship in each of the intervals.
Try to determine the Y coordinate of the ship in each time interval when the ship would be moving at a constant velocity of 6 km/h.
g) If 1 knot = 1.85 km/h, what was the average velocity of the ship in knots

Solution:

a) from Rovenska Nova to Novalja

b) 6 km/h

c) No

d) Ship had the fastest interval between 14:00 and 15:00, and slowest between 13:00 and 14:00.

Time	ΔY coordinate (longitude)	ΔY coordinate (km)	v (km/h)
12:00 – 13:00	0.11	6.95	6.95
13:00 – 14:00	0.07	4.42	4.42
14:00 – 15:00	0.12	7.58	7.58
15:00 – 16:00	0.08	5.05	5.05

f)

Time	Y coordinate (Eastern longitude) (real values)	Y coordinate (Eastern longitude) (with constant velocity 6/km/h)
12:00	14.51	14.51
13:00	14.62	14.605
14:00	14.69	14.70
15:00	14.81	14.795
16:00	14.89	14.89

g) $v = \frac{6}{1.85} = 3.24\ knots$

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5.2. ANALYSING THE GRAPH OF A FUNCTION

For a given function $f(x)$, set of all pairs $\big(x, f(x)\big)$ is called the graph of the function.

5.2.1. Sign of a function value

The positive regions of a function are intervals where the function is above the x-axis. Mathematically speaking, function is positive on interval $\langle a, b\rangle$ if $f(x) > 0$ for every $x \in \langle a, b\rangle$.

The negative regions of a function are intervals where the function is below the x-axis. Function is negative on interval $\langle a, b\rangle$ if $f(x) < 0$ for every $x \in \langle a, b\rangle$.

All points for which $f(x) = 0$ are called zeros.

Example 5.4 https://www.geogebra.org/calculator/rvnautep

Find the positive and negative regions of a function.

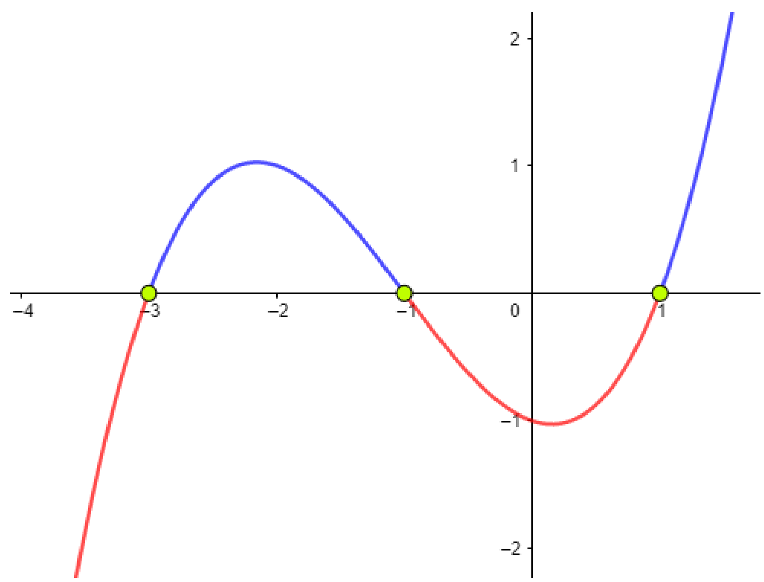

Figure 5.5

Solution:

Recall that we read function values on the y axis, so for a positive sign we are interested in x values where the y coordinate of that point is greater than 0.

All points colored blue have a positive functional value, so we say that the function is positive at these intervals. In this task these are intervals $< -3, -1 > \cup < 1, +\infty >$.

All points colored red have a negative functional value, so we say that the function is negative at these intervals. In this task these are the intervals $< -\infty, -3 > \cup < -1, 1 >$.

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5.2.2. Increasing and decreasing functions

Function is increasing if when x increases, then y also increases. When $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ we say that function is increasing.

Function is decreasing if when x increases, then y decreases. When $x_1 < x_2$ then $f(x_1) \geq f(x_2)$ we say that function is decreasing.

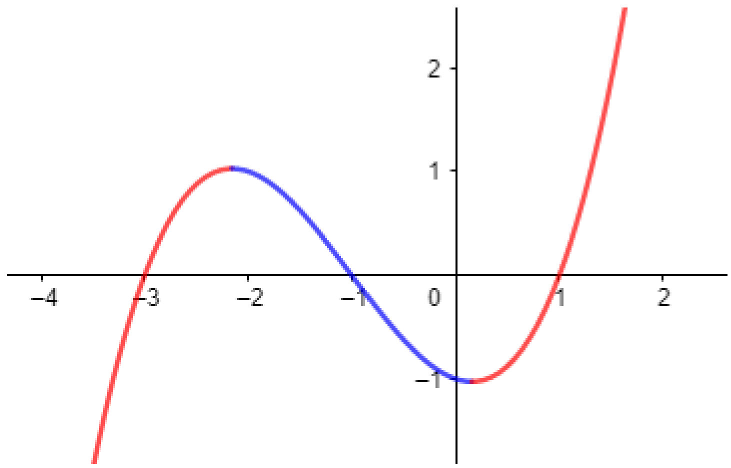

Figure 5.6 https://www.geogebra.org/calculator/mufgrbvs

For all values colored red the function is increasing.

For all values colored blue the function is decreasing.

Example 5.5

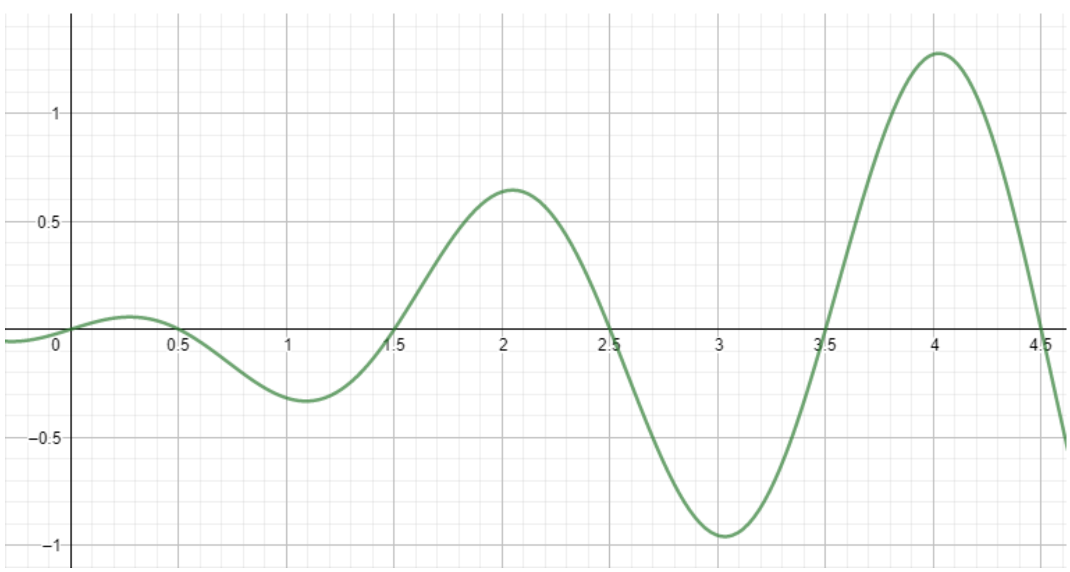

Figure 5.7

Find in interval $[0,4.5]$

All intervals where the function is positive (all x values for which the function value is positive)
All zero points
All intervals where the function is decreasing.

Solutions:

a) $x \in \langle 0,0.5\rangle \cup \langle 1.5,2.5\rangle \cup \langle 3.5,4.5\rangle$
b) Zero points are $(0,0), (0.5,0), (1.5,0), (2.5,0), (3.5,0), (4.5,0)$
c) $x \in \langle 0.5,1.5\rangle \cup \langle 2.5,3.5\rangle$

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5.3. ODD AND EVEN FUNCTIONS

Aims

1) Introducing concept of odd/even function
2) Students learn how to differentiate odd/ even function analytically and graphically.
3) Understand the concept of drawing function graphics.
4) Introducing algebraic and polar forms

5.3.1. Even function

Definition:

A function is "**even**" when $f(-x) = f(x)$ for all x. When you plug in $-x$, you get back the same function with which you started.

Example 5.6

Are given function $f(x)= 3x^2 – 5x^4$ odd or even?

$f(- x) = 3(-x)^2 – 5(-x)^4 = 3x^2 – 5x^4$

$f(-x) = f(x)$ given function is even.

Even functions are symmetric about the y-axis. Graphics remain unchanged when reflected across the *y*-axis.

Graphic 1: Even function $y= x^4 + x^2$

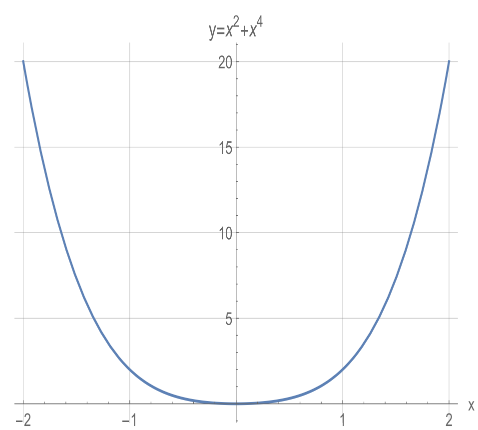

Figure 5.8 Even function $y= x^4 + x^2$

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5.3.2. Odd function

Definition:

A function is "odd" when f (- x) = - f (x) for all x. When you plug in -x, you get back the negation of the function with which you started.

Example 5.7

Are given function $f(x)= 4x^3 + 7x$ odd or even?

$f(-x) = 4(-x)^3 + 7(- x) = -4x^3 – 7x = - (4x^3 + 7x)$

Not even   Odd

Odd functions are symmetric about the origin. The graph remains unchanged after a rotation of 180 degrees about the origin.

Graphic 2: Odd function $y= x^3 - x$

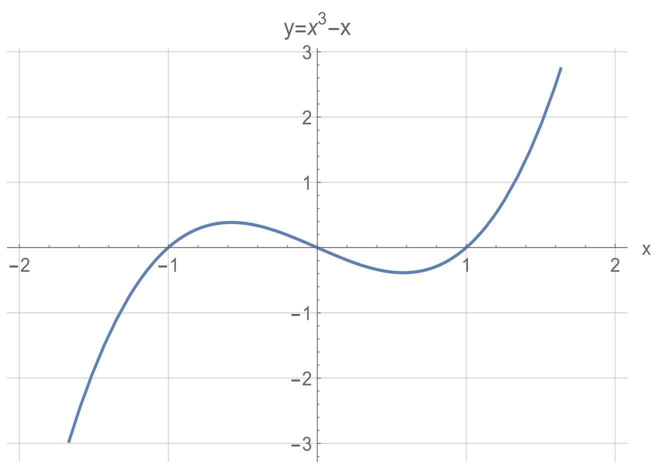

Figure 5.9 Odd function $y= x^3 - x$

1) *Neither odd nor even functions*

Majority of functions are neither odd nor even functions. All of these "other" functions are referred to as "neither", when being compared to the odd and even function definitions.

Example 3: Are given function $f(x) = 5x^2 + 3x + 5$ odd or even?

$f(-x) = 5(-x)^2 + 3 (- x) + 5 = 5x^2 - 3x + 5 = - (5x^2 + 3x – 5)$

$f(-x) \neq f(x) \Longrightarrow 5x^2 + 3x + 5 \neq 5x^2 - 3x + 5 \Longrightarrow$ Not even

$f(-x)\neq -f(x) \Longrightarrow (5x^2 + 3x – 5) \neq - (5x^2 + 3x + 5) \Longrightarrow$ Not odd

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This function is neither.

5.3.3. Exercises

Task 5.1 Is this function odd, even or neither

1) $y=x^2+5x$; 2) $y=-x^2+2$; 3) $y=x^2+2x-3$;
4) $y=5x-2$; 5) $y=x^4+2x^2$; 6) $y=x^3-5x+3$;
7) $y=x+\frac{1}{x}$; 8) $y=\frac{2x^5}{3x^3+x}$; 9) $y=\frac{3}{x^2-5}$;
10) $y=x^4+x^2-3x$; 11) $y=x^3+x^{-1}$; 12) $y=3x^4+x^2+4$.

Task 5.2 Is this function odd, even or neither

a) $y=5x$;
b) $y=5x^2+1$;
c) $y=2x^3$;
d) $y=7x^2+x$;
e) $y=5x^2$;
f) $y=6x$;
g) $y=5x^2+1$;
h) $y=3x^4+x^2+4$;
i) $y=5x-1$;
j) $y=6x^2+4$;
k) $y=x^2-2x+3$;
l) $y=x^3-5x-3$;
m) $y=\frac{3}{x}$;
n) $y=6x+1$;
o) $y=3x^4$;
p) $y=x+\frac{1}{x}$.

5.3.4. Homework

1. $y = 2x^2 - 3x + 6$
2. $y = x^2 + 3x + 1$
3. $y = 3x^2 + 5x - 4$
4. $y = x^2 + x + 2$
5. $y = 3x^2 - 7x - 6$

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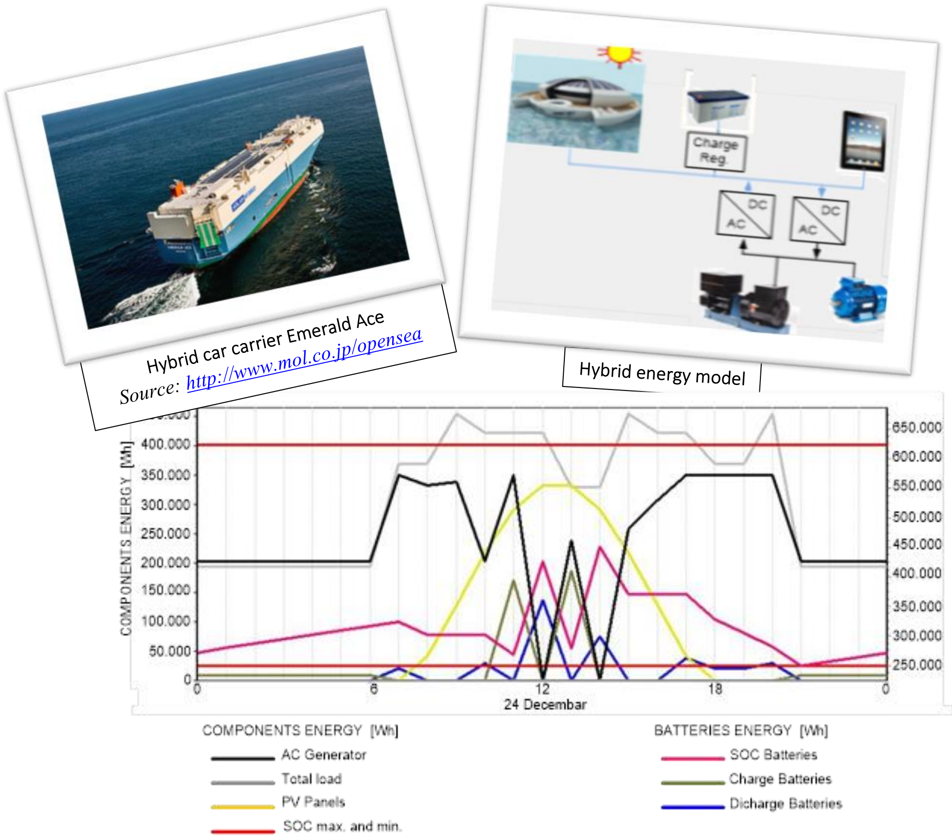
5.4. ELEMENTARY FUNCTIONS



Elementary functions have implementations in solving different real-world problems as calculating energy load, consumption and distribution in the ship power plant with renewable resources

ABSTRACT:

This chapter describes elementary functions, their properties and applications. Elementary functions are the functions which we can meet in most of the calculus and basic math applications. They include polynomial functions - linear, quadratic, etc., rational functions, trigonometric functions, exponential functions and their inverse functions, including the inverse trigonometry functions and logarithms, and, also functions defined as a sum, product and/or composition of finitely many elementary functions. Lots of functions which appear in technical or economic applications are elementary functions.

In this chapter, we will focus on understanding these functions and transforming those using basic algebraic operations. Examples (and diagrams/graphs) are given throughout the text to provide guidance on how to approach and solve various problems. Exercises are also included with solutions at the end of each task. Activities for individual learning and assessing your knowledge are available.

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AIM: To acquire skills in plotting graphs of functions, solving equations, systems of equations, to understand the concepts behind making computations. It turns out that elementary functions are important not only for solving mathematical problems but also in many real applications.

Learning Outcomes:

1. Solving linear, quadratic, exponential, logarithmic equations and inequalities.
2. Plotting graphs of functions.
3. Be familiar with the properties of those functions.
4. Learn applications of those functions in real life and technical problems

Prior Knowledge: basic knowledge from primary and high school. Acquaintance of basic rules and formulae.

Relationship to real life problems: We apply linear functions to solve problems about rates of change, curve fitting, linear regression formula. There are many real-world situations that deal with quadratics and parabolas. Throwing a ball, shooting a cannon, diving from a platform and hitting i.e., a golf ball are examples of situations that can be modeled by quadratic functions. In many of these situations you will want to know the highest or lowest point of the parabola. Applications of exponential decay can be found with the following examples: medications/caffeine leaving the body, radioactive decay, half-life, carbon dating, depreciation of material objects, etc. Applications of exponential growth can be found with the following examples: Population growth, bacteria growth, appreciation of material objects, compound interest. Scientists agree that the greenhouse effect is approximately logarithmic. Square roots are used in many jobs. They are used by engineers, carpenters, architects, designers, construction managers, technicians, statisticians, drafters, economists, lawyers, social scientists, timekeepers, agricultural workers, floor installers, medical assistants and many others jobs.

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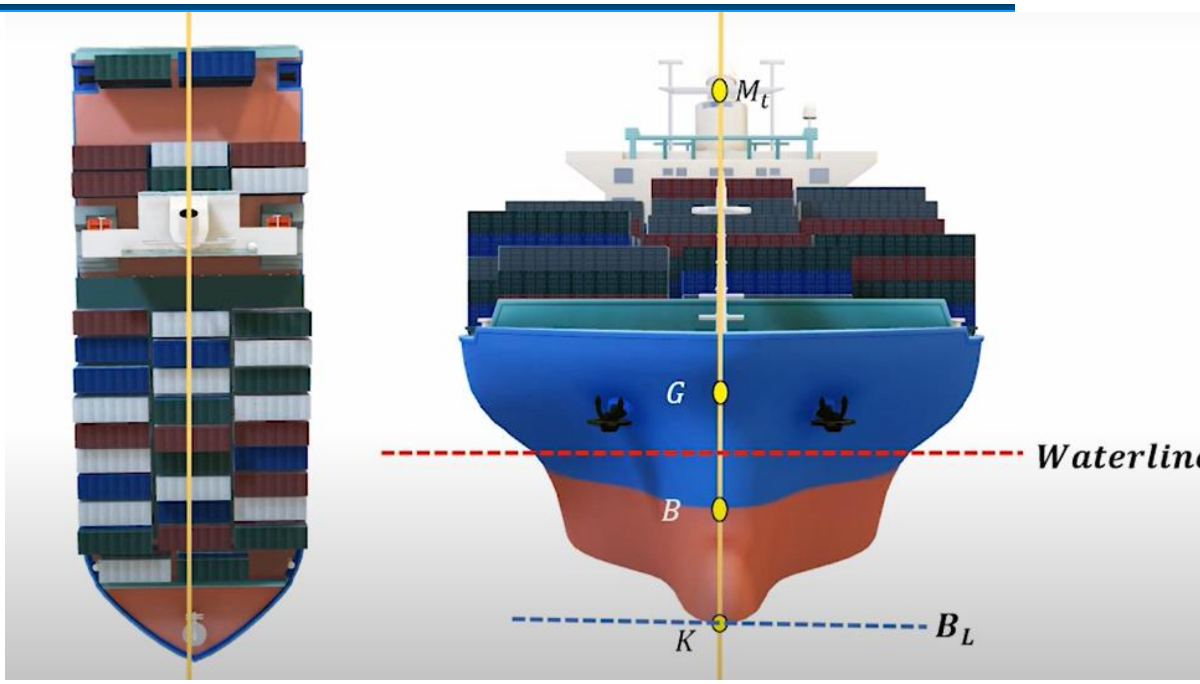

Ship Stability | LIST Angle Calculation for Ship is one of many different problems where a linear function is used

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- Worked examples and exercises
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    - Graph and general form of a quadratic function
    - Domain and range of a quadratic function
    - Finding the $\boldsymbol{x}$– and $\boldsymbol{y}$-intercepts of a quadratic function
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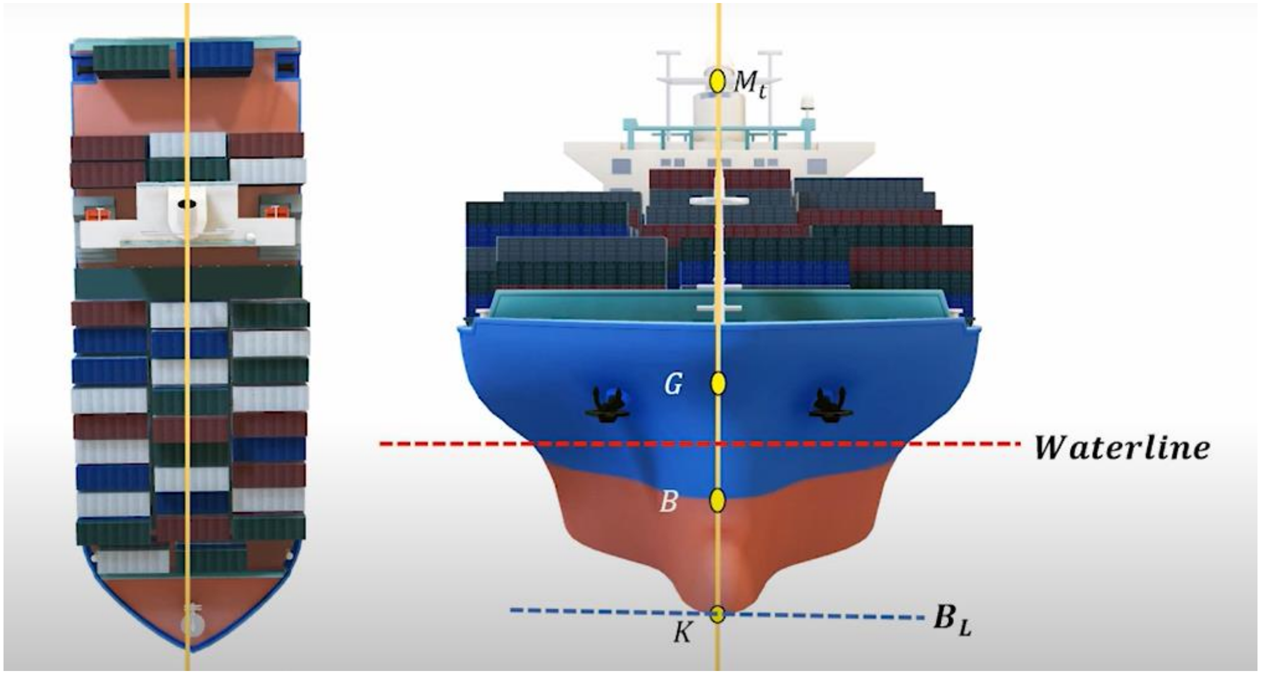
5.4.1. Linear function and straight line



Ship Stability | LIST Angle Calculation for Ship is one of many different problems where a linear function is used

Introduction

The linear function is popular in various branches of science (i.e., economics). It is attractive because it is simple and easy to handle mathematically. It has many important applications.

Linear functions are those whose graph is a straight line. The origin of the name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane.

A linear function has the following form

$$\boldsymbol{y} = \boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}, \text{ (slope } \boldsymbol{a}, \ \ \boldsymbol{y} -\text{intercept } \boldsymbol{b}).$$

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y.
b is the *constant* term or the y ***−intercept***. It is the value of the dependent variable when $x = 0$.

a is the *coefficient* of the independent variable. It is also known as the ***slope*** and gives the *rate of change* of the dependent variable.
The slope of a line is a number that describes both the *direction* and the *steepness* of the line. Slope is often denoted by the letter a . Recall the slop-intercept form of a line, $y = ax + b$. Putting the equation of a line into this form gives you the slope (a) of a line, and its y-intercept (b).

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The steepness, or incline, of a line is measured by the absolute value of the slope. A slope with a greater absolute value indicates a steeper line. In other words, a line with a slope of -9 is steeper than a line with a slope of 7.

Slope is calculated by finding the ratio of the "vertical change" to the "horizontal change" between any two distinct points on a line. This ratio is represented by a quotient ("rise over run"), and gives the same number for any two distinct points on the same line. It is represented by $a = \frac{rise}{run}$.

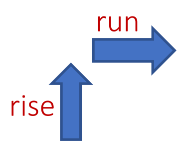

Figure 5.10 Visualization of a slope. The slope of a line is calculated as "rise over run."

Slope describes the direction and steepness of a line, and can be calculated given two points on the line:

$$\boldsymbol{a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}}$$

When $a > 0$ then the linear function is increasing (y-value increases as the x-value increases - it is easy to see that $y = f(x)$ tends to go up as it goes along - the blue line on Figure 5.10 has a positive slope of $\frac{1}{2}$ and a y-intercept of -3); when a < 0 then the linear function is decreasing (y-value decreases as the x-value increases- it is easy to see that $y = f(x)$ tends to go down as it goes along – the red line on Figure 5.11 has a negative slope of -1 and a y-intercept of 5.

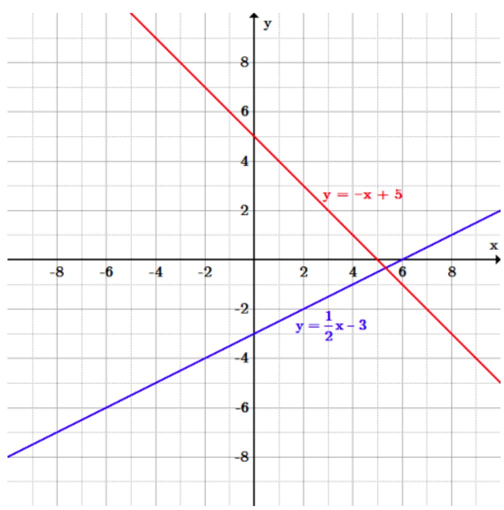

Figure 5.11 Graph of linear functions when $\boldsymbol{a > 0}$ *and* $\boldsymbol{a < 0}$.

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Vertical and Horizontal Lines

Vertical lines have an undefined slope, and cannot be represented in the form $y = ax + b$, but instead as an equation of the form $x = c$ for a constant c, because the vertical line intersects a value on the x-axis, c. For example, the graph of the equation $x = 3$ includes the same input value of 4 for all points on the line, but would have different output values, such as $(3, -3), (3, 0), (3, 1), (3, 6), (3, 2), (3, -1)$, etc. Vertical lines are **NOT functions**, however, since each input is related to more than one output. If a line is vertical the slope is undefined.

Horizontal lines have a slope of zero and they are represented by the form, $y = b$, where b is the y-intercept. A graph of the equation $y = 3$ includes the same output value of 3 for all input values on the line, such as (−1, 3), (0, 3), (2, 3), (3, 3), (6,3) etc. Horizontal lines **ARE functions** because the relation (set of points) has the characteristic that each input is related to exactly one output.

Graphing a linear function

To graph a linear function, we have to:

1. Find 2 points which satisfy the equation;
2. Plot them;
3. Connect the points with a straight line.

Example 5.8

$y = 5x + 25$:

Let $x = 1$ then $y = 5\,(1) + 25 = 30$, let $x = 3$ then $y = 5(3) + 25 = 40$

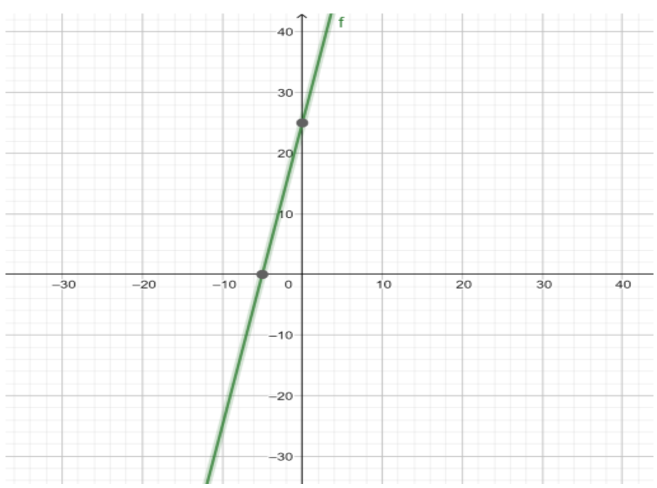

Figure 5.12 Graph of the linear function

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In the linear function graphs above (*Figure 5.12*) the constant, $a = 5$, determines the slope or gradient of that line, and the constant term, $b = 25$, determines the point at which the line crosses the y-axis, otherwise known as the y-intercept.

Point-slope form of the equation of a line

The equation of the line through $(x_0,\ y_0)$ with slope a is

$$\boldsymbol{y - y_0 = a(x - x_0)}.$$

The equation of the line through two points: $(x_1,\ y_1)$, $(x_2,\ y_2)$ is

$$\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}.$$

Worked examples and exercises

Example 5.9

The total cost in euros of manufacturing x units of a certain commodity is

$f(x) = 30x + 1000$.

a) Compute the cost of manufacturing 10 units.
b) Compute the cost of manufacturing 10th unit.

Solution

a) Substitute $x = 10$ into formula $f(x) = 30x + 1000$: $f(10) = 30\,(10) + 1000 = 1300$ [€].
b) Subtract the cost of manufacturing 9 units from the cost of manufacturing 10 units:

$$f(10) - f(9) = 1300 - 1270 = 30\ [€].$$

Example 5.10

Find the slope and y –intercept of the line $3x + 2y = 6$ and draw the graph.

Solution

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First solve for y: $y = -\frac{3}{2}x + 3$.

Compare this for the slope-intercept form $\boldsymbol{y = ax + b}$ and conclude that $\boldsymbol{a} = -\frac{3}{2}$ and $b = 3$. To draw the graph, plot the y –intercept $(0, 3)$ and any other convenient point that satisfies the equation, say $(2, 0)$, and draw the line through them.

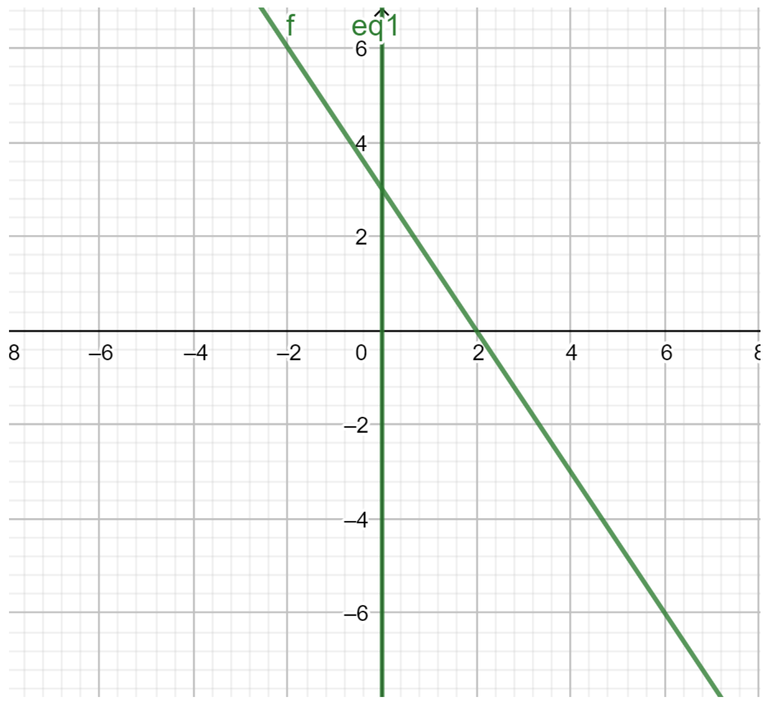

Figure 5.13 ***Graph of the function*** $\boldsymbol{3x + 2y = 6}$

Example 5.11

Find the equation of the line through $(2, 5)$ and $(1, -2)$.

Solution

First compute the slope: $a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-5}{1-2} = 7$.

Then use one of the given points, say (2, 5) as $(x_0,\ y_0)$ in the point-slope formula

$$y - y_0 = a(x - x_0):$$

$$y - 5 = 7(x - 2) \quad => \quad y = 7x - 9.$$

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Example 5.12

Since the beginning of the month, a local reservoir has been losing water at constant rate. On the 12th of the month, the reservoir held 400 million liters of water and on the 22th, it held 250 million liters.

a) Express the amount of water in the reservoir as a function of time.
b) How much water was in the reservoir on the eighth of the month?

Solution

a) Let

$x =$ number of days since the first of the month;

$y =$ amount of water in the reservoir (in million litres units).

Since the rate of change of the water is constant, y is a linear function of x. To find this function, use the point-slope form to get the equation of the line through the points $(12, 400)$ and $(22, 250)$:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{250 - 400}{22 - 12} = \frac{-150}{10} = -15.$$

$$y - 400 = -15(x - 12) \quad => \quad y = -15x + 580.$$

b) To calculate the amount of water in the reservoir on the 8th of the month, substitute $x = 8$ into the formula of y : $y = -15(8) + 580 = 460$ [million liters].

Example 5.13

Find the point of intersection of the lines: $y = -x + 5$ and $y = 4x + 3$.

Solution

Set two expressions for y equal to each other and solve for x:

$$-x + 5 = 4x + 3$$

$$-5x = -2 \quad => \quad x = 0.4.$$

To find y, substitute $x = 0.4$ into one of the original equations, say $y = -x + 5$:

$y = -0.4 + 5 = 4.6.$ Thus, the point of intersection is $(0.4;\ 4.6)$.

Example 5.14

Membership in a private fitness club costs $800€$ per year and entitles the member to use the courts for a fee of 3€ per hour. At a competing club, membership costs 650€ per year and the charge for the use of the gym is 5€ per hour. If only financial considerations are to be taken into account, how should a client choose which club to join?

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Solution

Let

$x =$ number of hours of using the gym during a year;

$f_1(x) =$ total cost at the first club

$f_2(x) =$ total cost at the second club.

Then,

$$f_1(x) = 800 + 3x$$

while

$$f_2(x) = 650 + 5x.$$

To find the point of intersection, set $f_1(x) = f_2(x)$ and solve

$$800 + 3x = 650 + 5x$$

$$150 = 2x \Rightarrow \quad x = 75.$$

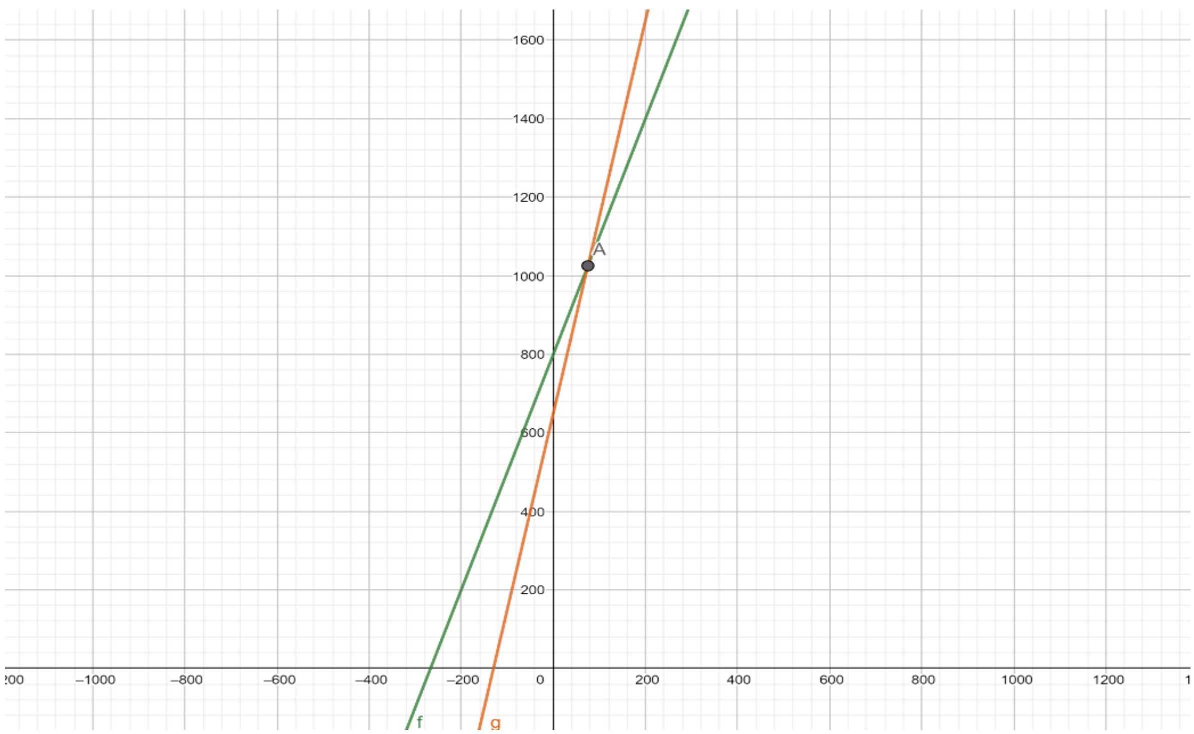

Figure 5.14

Conclude that if $x < 75$, the client should join the second club and if $x > 75$ the client should join the first club.

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Example 5.15

A river ship with a drive does not stop along its way along the Vistula River from the city of Włocławek in Poland to the city of Gdańsk also in Poland for three days, while from Gdańsk to Włocławek for four days. How many days will the raft sail (without a drive) from Włocławek to Gdańsk?

Solution

Let v_1 be a speed of the ship, v_2 be a speed of the raft. The ship while travels downstream has the speed

$$v_1 + v_2$$

and while travels upstream has the speed:

$$v_1 - v_2.$$

When we compare the distance s, defined by

$$s = v \cdot t$$

from Włocławek to Gdańsk and from Gdańsk to Włocławek we get the equation:

$$3(v_1 + v_2) = 4(v_1 - v_2).$$

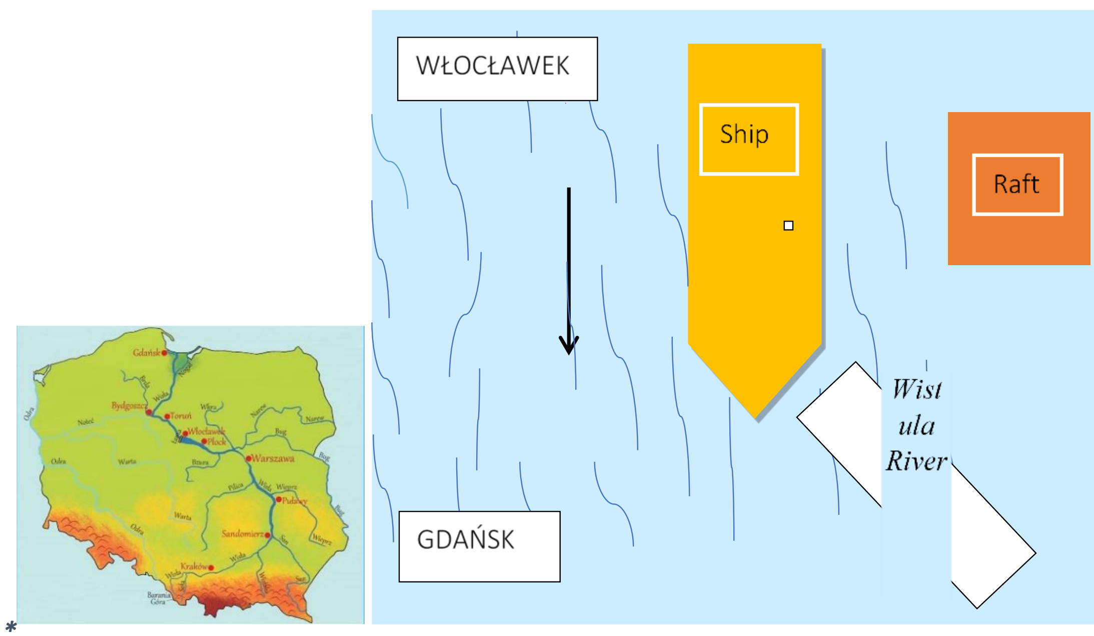

Figure 5.15

Hence

$$v_1 = 7v_2.$$

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The speed of the raft is the same as the river current i.e., v_2.

Therefore, the number of days the raft will pass the distance

$$s = 3(v_1 + v_2) = 3(7v_2 + v_2) = 24v_2$$

will be

$t = \frac{24v_2}{v_2} = 24.$

The raft will sail from Włocławek to Gdańsk 24 days.

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Exercises

Task 5.3 If $f(x) = 3x - 2$, compute $f(2), f(0), f(-1)$.

Task 5.4 If $f(x) = 2x + 6$, find the value of x for which $a)\ f(x) = 6, \quad b)\ f(x) = 4$.

Task 5.5 The cost of renting a car is $f(x) = 14 + 0.12x$ €, where x is a number of kilometres driven.

a) What is the cost of renting a car for 60 km trip?
b) How much is charged for each additional kilometer?
c) If the total rental cost was $23{,}60$ €, how far was the car driven?

Task 5.6 Find the slope and y intercept of each of the following lines:

a) $2y = 6x + 4,$ b) $5x - 4y = 20,$ c) $2y + 3x = 0.$

Task 5.7 Find the equation of the line which passes through the origin $(0, 0)$ and has slope -2.

Task 5.8 Find the equation of the line through the points $(2, 4)$ and $(1, -3)$.

Task 5.9 A doctor owns 1,500 € worth of medical books which for tax purposes, are assumed to depreciate at a constant rate over the 10-year period so that at the end of the 10-year period, their value will have been reduced to zero.

a) Express the value of the books as a function of time.
b) By how much does the value decrease each year?

Task 5.10 Find the point of intersection (if any) of the lines:

a) $y = x + 4$ and $y = -2x + 1,$
b) $y = 4x + 9$ and $y = 4x - 6.$

Task 5.11 First plumber charges 16 € plus 6 € per half hour. A second charges 21 € plus 4 € per half hour. If only financial considerations are to be taken into account, how should you decide which plumber to call?

Task 5.12 A ship sails from Włocławek to Gdańsk two days, while from Gdańsk to Włocławek it sails six days. How many days does the river flow from Włocławek to Gdańsk?

Task 5.13

a) As a dry air moves upward, it expands and cools. If the ground temperature is $20°C$ and the temperature at the height of 1 km is $10°C$, express the temperature T (in $°C$) as a function of the height h (in km) assuming the function is linear.

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b) Draw the graph of the function in part a). What does the slope represent?

c) What is the temperature at a height of 2.5 km?

Answers

5.3. $(2) = 4, f(0) = -2,\ f(-1) = -5$

5.4. $x = 0,\ x = -1.$

5.5. a) 21.20 €, b) 12 cents, c) 80 kilometers.

5.6. a) $a = 3,\ b = 2$ b) $a = 1.25,\ b = -5$ c) $a = -\frac{3}{2},\ \ b = 0$.

5.7. $y = -2x.$

5.8. $y = 7x - 10.$

5.9. 7. a) $y = -150x + 1500$ b) 150 €

5.10. a) $(-1, 3)$ b) none.

5.11. Call the first plumber if work will take less than 75 minutes and call the second if work will take more than 75 minutes.

5.12. 10. 6 days.

5.13. a) $T(h) = -10h + 20$

b) The rate of change of temperature with respect to height.

c) -5℃.

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Sample chapter exam

1. It is estimated that t years from now, the population of a certain community will be

$$f(t) = 600t + 12000.$$

a) What will the population be 8 years from now?

b) What is the current population?

c) By how much does the population increase each year?

d) When will the population be 15000?

2. Find the slope and y intercept of the given lines: $5x - 4y = 20$, $\frac{x}{3} + \frac{y}{4} = 4$.

3. Write an equation for the line with a given properties:

a) through $(1, 4)$ and $(5, 0)$;

b) through $(-1, 3)$ with slope -5.

4. Under the provisions of a proposed property tax bill a homeowner will pay 100 € plus 8% of the assessed value of the house. Under the provisions of a competing bill, the homeowner will pay 1900 € plus 2% of the assessed value of the house. If only financial considerations are taken into account, how should a homeowner decide which bill to support?

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5.4.2. Quadratic functions

In this section, we will investigate quadratic functions. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Graph and general form of a quadratic function

The graph of a quadratic function is a U-shaped curve called a **parabola**. Important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens upward, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 5.16.

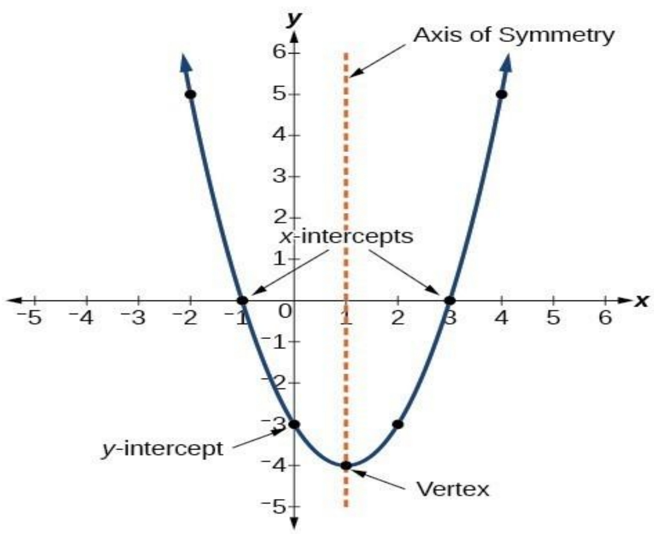

Figure 5.16 The graph of a quadratic function

The $\mathbf{y}$-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the ***zeros***, or ***roots*** of the quadratic function i.e., the values of x at which $y = \mathbf{0}$.

The general form of a quadratic function is

$$\boldsymbol{f(x) = ax^2 + bx + c}$$

Where a, b, and $\mathbf{c}$ are real numbers and $a \neq 0$.

If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. We can use the general form of a parabola to find the equation for the axis of symmetry.

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The axis of symmetry is defined by $x = -\frac{b}{2a}$.

If we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of xhalfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 5.17 shows the graph of the quadratic function written in general form as

$y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, $c = 3$. As $a > 0$, the parabola opens upwards. The axis of symmetry is $x = -\frac{4}{2} = -2$. We can see from the graph that the vertical line

$x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upwards, the vertex occurs at the lowest point on the graph, in this example: $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x –axis, occur at $(-3,0)$ and $(-1,0)$.

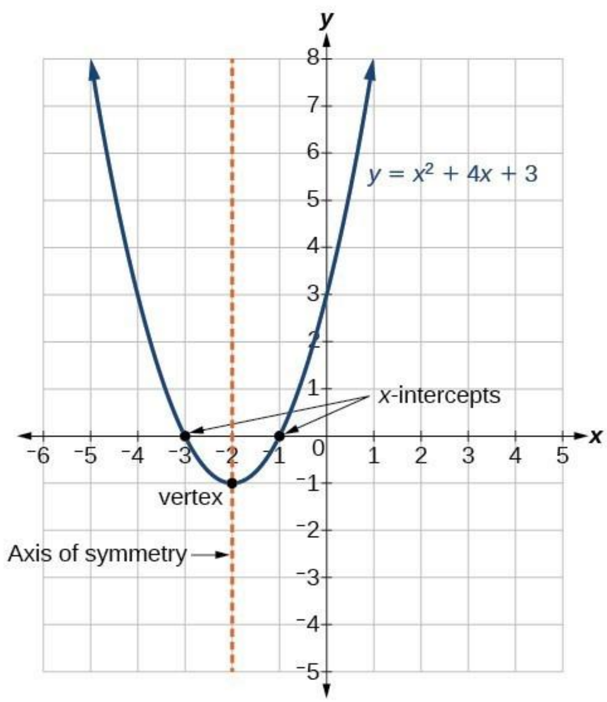

Figure 5.17 The graph of $y = x^2 + 4x + 3$

The **standard form** of a quadratic function is represented by

$$f(x) = a(x - p)^2 + q$$

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where $(p,\ q)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**. The function presented in *Figure 5.17* has the standard form: $y = (x+2)^2 - 1$.

As with the general form, if $a > 0$, the parabola opens upwards and the vertex is a minimum. If $a < 0$, the parabola opens downwards, and the vertex is a maximum.

Figure 5.18 shows the graph of the quadratic function written in standard form as

$$y = -3(x+2)^2 + 4\ .$$

Since $x - p = x + 2$ in this example, $p = -2$. In this form, $a = -3$, $p = -2$ and $q = 4$. Because $a = -3 < 0$, the parabola opens downward. The vertex is at $(-2,\ 4)$.

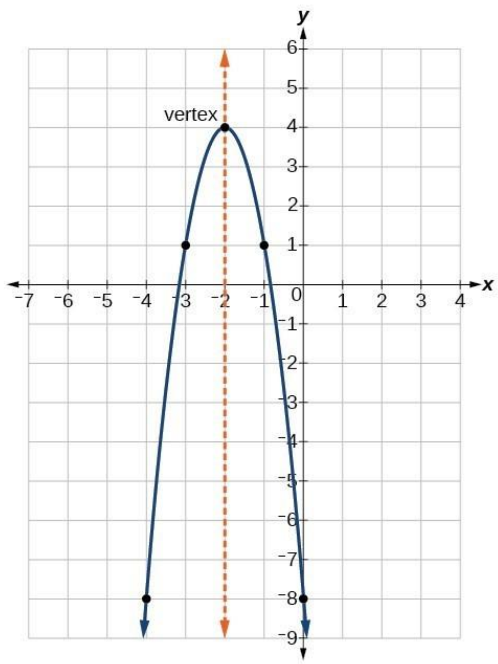

Figure 5.18 The graph of $\boldsymbol{y = -3(x+2)^2 + 4}$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. *Figure 5.19* is the graph of this basic function.

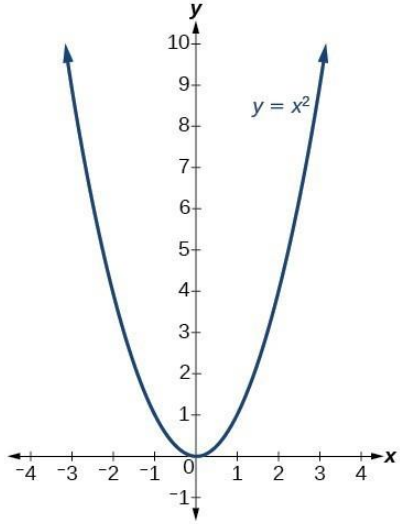

Figure 5.19 Graph of $\boldsymbol{y = x^2}$.

If $q > 0$, the graph shifts upward, whereas if $q < 0$, the graph shifts downward. In Figure 5.18 $q > 0$, so, the graph is shifted 4 units upward. If $p > 0$, the graph shifts toward the right and if $p < 0$, the graph shifts to the left. In Figure 5.18 $p < 0$, so the graph is shifted 2 units to the left.

The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch.

If $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In *Figure 5.18* $|a| > 1$, so, the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x-p)^2 + q = a(x^2 - 2xp + p^2) + q = ax^2 - 2axp + ap^2 + q = ax^2 + bx + c$$

For the quadratic expressions to be equal, the corresponding coefficients must be equal.

$$-2ap = b, \text{ so } \quad \boldsymbol{p = \frac{-b}{2a}}.$$

This gives us the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$ap^2 + q = c \;\; q = c - ap^2 = c - a\left(\frac{-b}{2a}\right)^2 = c - \frac{b^2}{4a} = -\frac{b^2 - 4ac}{4a}$$

In practice, though, it is usually easier to remember that $\boldsymbol{q}$ is the output value of the function when the input is $\boldsymbol{p}$, so $f(p) = f\left(\frac{-b}{2a}\right) = q$.

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Note

The expression $b^2 - 4ac$ usually denoted as the upper-case Greek letter, Δ , is defined as

$$\Delta = b^2 - 4ac$$

and it is called **discriminant** of a square trinomial $ax^2 + bx + c$.
Quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is a formula for solving quadratic equations in terms of the coefficients.

The number of $\mathrm{x-}$intercepts of a quadratic function depends on the sign od the discriminant
$\boldsymbol{\Delta = b^2 - 4ac}$:

- If $\Delta < 0$ then there is no $\mathrm{x-}$intercepts of a quadratic function;
- If $\Delta = 0$ then there is one $\mathrm{x-}$intercept of a quadratic function and $x_0 = \frac{-b}{2a}$;
- If $\Delta > 0$ then there are two $\mathrm{x-}$intercepts a quadratic function and

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Example 5.16

Find the vertex of a quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$p = \frac{-b}{2a} = \frac{-(-6)}{2 \cdot 2} = \frac{6}{4} = \frac{3}{2}$$

The vertical coordinate of the vertex will be at

$$q = f(p) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}.$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts, $f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$

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Domain and range of a quadratic function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is the set of all real numbers. As parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $\left[f\left(-\frac{b}{2a}\right), \infty\right)$; the range of a quadratic function written in general form with a negative a value is $\left(-\infty, f\left(\frac{-b}{2a}\right)\right]$.

The range of a ***quadratic function*** written in standard form

$$f(x) = a(x-p)^2 + q$$

with a positive a value is $[q, \infty)$; the range of a quadratic function written in standard form with a negative a value is $(-\infty, q]$.

How to find the domain and range of a given quadratic function:

The domain of any quadratic function is the set of all real numbers.

Determine whether a is positive or negative. If a is positive, the quadratic function has a minimum. If a is negative, the quadratic function has a maximum.

Determine the maximum or minimum value of the quadratic function, q.

If the parabola has a minimum, the range is $[q, \infty)$.

If the parabola has a maximum, the range is $(-\infty, q]$.

Example 5.17

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is a set of all real numbers. Because a is negative, the parabola opens downwards and has a maximum value. We want to determine the maximum value. We start by finding the x −value of the vertex:

$$p = -\frac{b}{2a} = -\frac{9}{-10}.$$

The maximum value q is given by $q = f(p)$:

$$q = f\left(\frac{9}{10}\right) = -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 = \frac{61}{20}.$$

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The range is $\left(-\infty,\ \frac{61}{20}\right]$.

Figure 5.20 presents two parabolas with their extreme values: minimum and maximum.

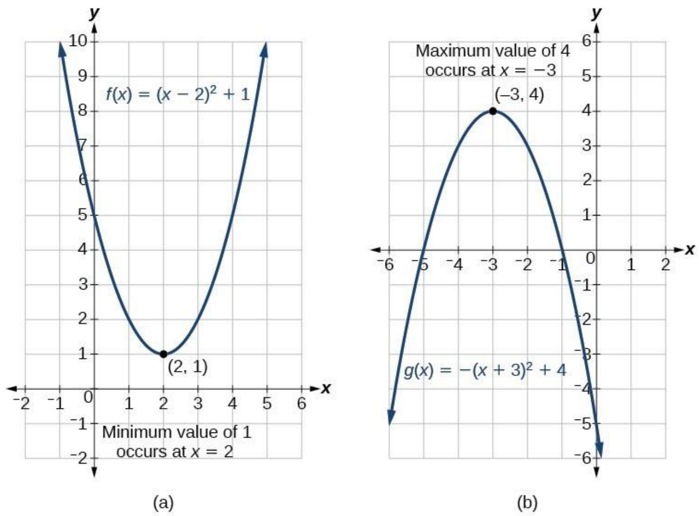

Figure 5.20 Examples of minimum and maximum values of the quadratic functions

Finding the x– and y-intercepts of a quadratic function

We need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x –intercepts at locations where the output is zero. Notice that the number of

x-intercepts can vary depending upon the location of the graph- see *Figure 5.21*.

How to find the $y-$ intercept and $x-$ intercepts of a given a quadratic function $f(x)$.

Evaluate $f(0)$ to find the $y-$ intercept.

a. Solve the quadratic equation $f(x) = 0$ to find the $x-$ intercepts.

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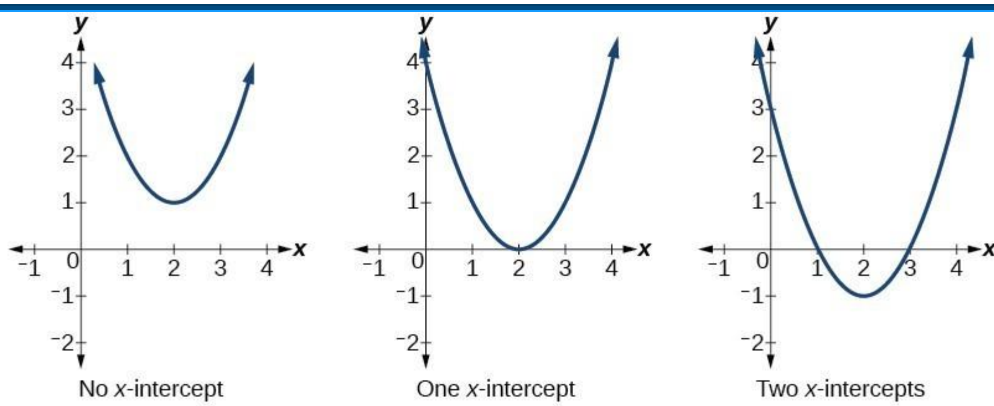

Figure 5.21 Number of x −intercepts of a parabola.

Example 5.18

Find the $y-$ and x −intercepts of a parabola $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y −intercept by evaluating $f(0)$:

$$f(0) = 3 \cdot 0^2 + 5 \cdot 0 - 2 = -2$$

So, the y −intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$3x^2 + 5x - 2 = 0.$$

In this case, using the quadratic formula , $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{\Delta}}{2a}$, as

$\Delta = b^2 - 4ac = \quad 5^2 - 4 \cdot 3 \cdot (-2) = 49,\ \sqrt{\Delta} = \sqrt{49} = 7 \quad \text{thus}$

$x_1 = \frac{-5-7}{6} = -2 \ \text{ and } \ x_2 = \frac{-5+7}{6} = \frac{1}{3}.$

So, the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2,0)$.

By graphing the function, we can confirm that the graph crosses the y −axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at at $\left(\frac{1}{3}, 0\right)$ and $(-2,0)$.

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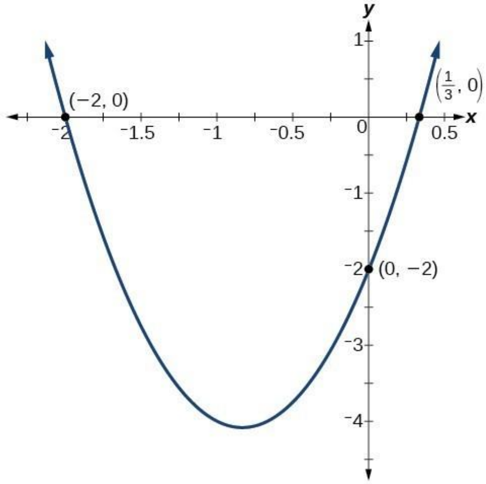

Figure 5.22 The $\boldsymbol{y}-$ *and* $\boldsymbol{x}-$*intercepts of a parabola* $\boldsymbol{f(x) = 3x^2 + 5x - 2}$

Vieta's formulas

Vieta's formulas give a simple relation between the roots of a polynomial and its coefficients. In the case of the quadratic equation, they take the following form:

$$\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$$

$$\boldsymbol{x_1 \cdot x_2 = \frac{c}{a}}.$$

These results follow immediately from the relation:

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0,$$

which can be compared term by term with

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0.$$

The first formula above yields a convenient expression when graphing a quadratic function. Since the graph is symmetric with respect to a vertical line through the vertex, when there are two real roots the vertex's x-coordinate is located at the average of the roots (or intercepts). Thus the x-coordinate of the vertex is given by the expression

$$p = \frac{x_1 + x_2}{2} = -\frac{b}{2a}.$$

The y-coordinate can be obtained by substituting the above result into the given quadratic equation, giving

$$q = -\frac{b^2}{4a} + c = -\frac{b^2 - 4ac}{4a}.$$

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Solving quadratic equations

The Quadratic Formula

As for $ax^2 + bx + c = 0$, the values of x which are the solutions of the equation are given by:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

or

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b-\sqrt{\Delta}}{2a},$$

we can also solve a quadratic equation using factorization:

$$ax^2 + bx + c = a(x-x_1)(x-x_2),$$

then $ax^2 + bx + c = a(x-x_1)(x-x_2) = 0,\ (x-x_1) = 0$ and $(x-x_2) = 0$,

so $x = x_1,\ x = x_2$.

Example 5.19

Solve the equation: $x^2 - 16 = 0$.

Solution

Factorize $x^2 - 16 = 0,\ (x-4)(x+4) = 0$.

We have two solutions $(x-4) = 0$ and $(x+4) = 0$, so $x = 4$, $x = -4$ respectively.

Example 5.20

Solve the equation: $x^2 + 6x = 0$.

Solution

Factorize $x^2 + 6x = 0,\ x(x+6) = 0$.

We have two solutions $x = 0$ and $x + 6 = 0$, so $x = 0$, $x = -6$ respectively.

Example 5.21

Solve the equation $2x^2 - 8x + 6 = 0$.

Solution

First, we divide the entire equation by 2 as a common factor of the coefficients, so

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$2x^2 - 8x + 6 = 0$ can be written as $x^2 - 4x + 3 = 0$. Now we compute the discriminant

$$\Delta = (-4)^2 - 4 \cdot 1 \cdot 3 = 4 > 0, \quad \sqrt{\Delta} = 2,$$

then

$$x_1 = \frac{4+2}{2} = 3\,, \qquad x_2 = \frac{4-2}{2} = 1.$$

Example 5.22

Solve the equation $x^2 + 6x + 9 = 0$.

Solution

After factorizing $x^2 + 6x + 9 = 0$ we have $(x+3)^2 = 0$, so we have double root of the equation $x_1 = x_2 = x_0 = -3$.

Example 5.23

Give the example of a quadratic equation with integer coefficients which has two roots:

$(5 - 2\sqrt{3})^{-1}$ and $(5 + 2\sqrt{3})^{-1}$.

Solution

There are given the roots $x_1 = \frac{1}{5-2\sqrt{3}}$ and $x_2 = \frac{1}{5+2\sqrt{3}}$.

As

$$(x - x_1)(x - x_2) = 0 \quad \Leftrightarrow \quad x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0,$$

so, we compute

$$x_1 + x_2 = \frac{1}{5 - 2\sqrt{3}} + \frac{1}{5 + 2\sqrt{3}} = \frac{5 + 2\sqrt{3} + 5 - 2\sqrt{3}}{(5 + 2\sqrt{3})(5 - 2\sqrt{3})} = \frac{10}{25 - 12} = \frac{10}{13},$$

$$x_1 \cdot x_2 = \frac{1}{5 - 2\sqrt{3}} \cdot \frac{1}{5 + 2\sqrt{3}} = \frac{1}{13}.$$

Hence, we obtain:

$$x^2 - \frac{10}{13}x + \frac{1}{13} = 0 \text{ and } 13x^2 - 10x + 1 = 0\,.$$

Example 5.24

Solve the equation $4x^2 + 2x + 1 = 0$.

As $\Delta = (2)^2 - 4 \cdot 4 \cdot 1 = -12$, so this equation has no real roots.

But we can find two complex roots knowing that $\sqrt{-1} = i$. Then $\sqrt{\Delta} = i\sqrt{12} = 2i\sqrt{3}$ and

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$$x_1 = \frac{-2-2i\sqrt{3}}{8} = \frac{-1-i\sqrt{3}}{4}, \quad x_2 = \frac{-1+i\sqrt{3}}{4}.$$

Solving quadratic inequalities

Quadratic inequalities can be of the following forms:

$$ax^2 + bx + c > 0,$$

$$ax^2 + bx + c \geq 0,$$

$$ax^2 + bx + c < 0,$$

$$ax^2 + bx + c \leq 0.$$

To solve a quadratic inequality, we must determine which part of the graph of a quadratic function lies above or below the x-axis. An inequality can therefore be solved graphically using a graph or algebraically using a table of signs to determine where the function is positive and negative.

Example 5.25

Solve for x: $x^2 - 5x + 6 \geq 0$.

Solution

Let us factorize the quadratic: $(x-3)(x-2) \geq 0$,

then we determine the critical values of x.

From the factorized quadratic we see that the values for which the inequality is equal to zero are $x = 3$ and $x = 2$. These are called the critical values of the inequality and they are used to complete a table of signs. To do we must determine where each factor of the inequality is positive and negative on the number line:

to the left (in the negative direction) of the critical value

equal to the critical value

to the right (in the positive direction) of the critical value

In the final row of the table, we determine where the inequality is positive and negative by finding the product of the factors and their respective signs.

Critical values		$x = 2$		$x = 3$	
$x-3$	$-$	$-$	$-$	0	$+$
$x-2$	$-$	0	$+$	$+$	$+$
$f(x) = (x-3)(x-2)$	$+$	0	$-$	0	$+$

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From the table we see that $f(x)$ is greater than or equal to zero for $x \leq 2$ or $x \geq 3$.

The graph in *Figure 5.23* does not form part of the answer and is included for illustration purposes. A graph of the quadratic helps us determine the answer to the inequality. We can find the answer graphically by seeing where the graph lies above or below the x-axis.

From the standard form, $x^2 - 5x + 6$, $a > 0$ and therefore the parabola opens up and has a minimum turning point.

From the factorized form, $(x-3)(x-2)$, we know the x-intercepts are $(2;0)$ and $(3;0)$.

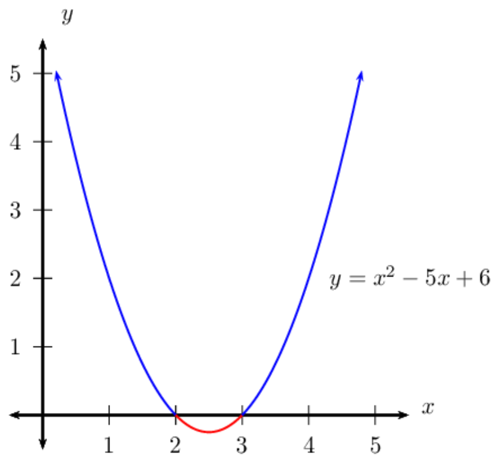
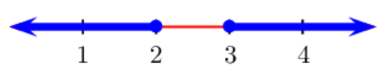


Figure 5.23 Graph of $f(x) = x^2 - 5x + 6$.

The parabola is above or on the x −axis for $x \leq 2$ or $x \geq 3$.

Final answer and presentation on a number line is:

$$x^2 - 5x + 6 \geq 0 \text{ for } x \in (-\infty, 2] \cup [3, \infty)$$



Example 5.26

Suppose that you head out on a river boat cruise that takes 4 hours to go 20 km upstream and then turn around and go 20 km back downstream. When you get back, you notice that the speedometer of the boat wasn't working during the cruise, so you want to calculate the boat's speed. The river has a current of 3 kilometers per hour.

Solution

Assume that v_b-speed of the boat. As $s = v \cdot t$, $t = \frac{s}{v}$ we can create an equation:

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$$\frac{20}{v_b-3}+\frac{20}{v_b+3}=4.$$

Multiply both sides by $(v_b-3)(v_b+3)$ and get

$$20(v_b-3)+20(v_b+3)=4(v_b-3)(v_b+3)$$

then

$$40v_b=4{v_b}^2-36\ \ ,\ \ 4{v_b}^2-40v_b-36=0$$

$$\Delta=(-40)^2-4(4)(-36)=2176$$

$$\sqrt{\Delta}=\sqrt{2176}$$

$$x{v_b}_1=\frac{40-\sqrt{2176}}{8}\approx-0.83\quad {v_b}_2=\frac{40+\sqrt{2176}}{8}\approx 10.83$$

We see that $v_b\ =\ -0.83$ or $v_b\ =\ 10.83$. Since we are talking about a speed, the negative answer makes no sense, so the answer is $v_b\ =\ 10.83$. In other words, the boat was traveling at a speed of 10.83 km/h.

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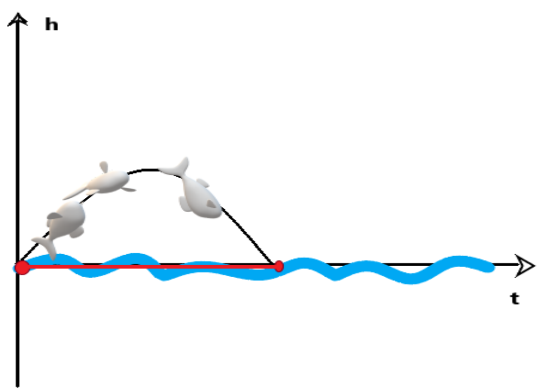
Exercises

Task 5.14 Solve the quadratic equations and inequalities:

a) $9x^2 + 42x + 49 = 0.$
b) $12x^2 - 25x + 12 = 0.$
c) $x^2 + 4x + 1 = 0.$
d) $x^4 - 3x^2 + 2 = 0$ Hint: Substitute $x^2 = t,\ \ t > 0.$
e) $9 - x^2 \leq 0.$
f) $x^2 - 8x > 0.$
g) $-x^2 + 4x - 3 < 0.$
h) $-x^2 + x - \sqrt{2007} < 0.$
i) $x^2 + \sqrt{1999}x + 500 < 0.$
j) $x(x+1) \leq 0.$

Task 5.15 A dolphin jumps out of the sea with an initial velocity of 20 feet per second (assume its starting heigh is 0 feet). Use the vertical motion model, $h = -16t^2 + vt + s$, where v is initial velocity in feet/second and s is the dolphin starting height in feet, to calculate the amount of time the dolphin is in the air before it hits the water again (see the Figure below). Round you answer to the nearest tenth if necessary.



Task 5.16 An osprey, a fish-eating bird of prey, dives toward the water to a salmon.

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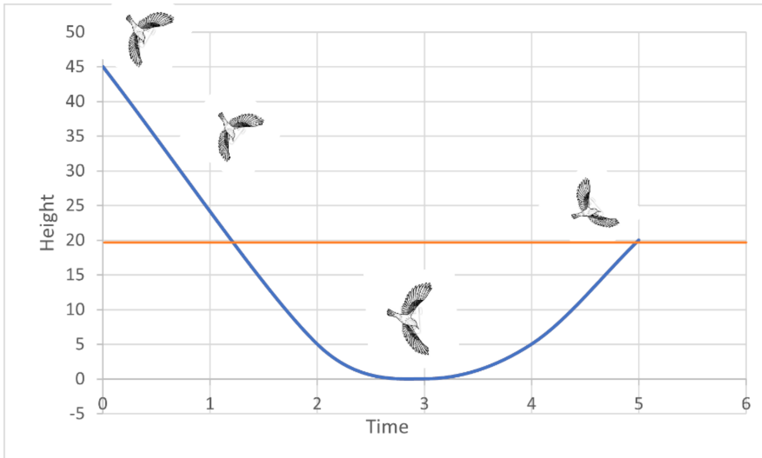
The height h, in meters of the osprey above the water t seconds after it begins its dive can be approximated by the function $h(t) = 5t^2 - 30t + 45$. Determine the time it takes for the osprey to reach a return height of 20 m. See figure below.



Task 5.17 A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, is given the equation $h = -4.9t^2 + 52t + 376$. At what time does the rocket hit the sea? (Round answer to 2 decimal places).

Answers

5.14.

a. $x = -\frac{7}{3}$
b. $x_1 = \frac{3}{4}, \quad x_2 = \frac{4}{3}$
c. $x_1 = -2 - \sqrt{3}, \quad x_2 = -2 + \sqrt{3}$
d. $x_1 = 1, \ x_2 = -1, \ x_3 = \sqrt{2}, \ x_4 = -\sqrt{2}$
e. $x \in (-\infty, -3] \cup [3, \infty)$
f. $x \in (-\infty, 0) \cup (8, \infty)$
g. $x \in (-\infty, 1) \cup (3, \infty)$
h. $x \in \mathbb{R}$
i. No solutions
j. $x \in [-1, 0]$.

5.15. The dolphin is in the air for $\frac{5}{4} = 1.25 \approx 1.3$ seconds.
5.16. It takes 5 seconds for the osprey to reach a return height of 20 m.

5.17. $t = 15{,}55$ seconds.

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Sample chapter exam

1. Sketch the following polynomials:

   a) $f(x) = (x-3)(x+6)$
   b) $f(x) = x^2 + 4x + 1$.

2. Find the $y-$ and $x-$intercepts of a parabola $f(x) = 2x^2 - 3x - 2$.

3. Find the domain and range of $f(x) = 3x^2 + 9x - 1$.

4. Find the vertex of a quadratic function $f(x) = x^2 - x - 2$. Rewrite the quadratic in standard form (vertex form).

5. Solve the equations:

   a) $x^2 - 2x - 15 = 0$.
   b) $x^2 + 4x + 15 = 0$.

6. Solve the inequalities:

   a) $x^2 - x < 2$
   b) $x^2 + 1 \geq 2x^2 - x$.

7. The seaman launch the flare from a crow's nest of a height of 5 meters. The height (h, in meters) of the flare t seconds after taking off is given by the formula:

$$h = -3t^2 + 14t + 5.$$

   a) How long will it take for the flare to hit the sea?
   b) Find the time when the flare is 5 meters form hitting the sea.

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5.4.3. Exponential functions

The exponential functions are mathematical functions used in many real-world situations. They are mainly used to find the exponential decay or exponential growth or to compute investments, model populations, radioactive decay and so on. In this section, you will learn about exponential function formulas, rules, properties, graphs, derivatives, exponential series and examples.

Definition and properties

Let a be given real number auch that, $a > 0, \ \ a \neq 1.$

A function of the form $\boldsymbol{y = a^x}$ is called exponential, x is any number.

a is called the base, while x is called the *exponent*.

The domain of the exponential function is $\mathbb{R}$, while the range is $(0, \infty)$.

The exponential function we define in four stages:

a) If $x = n,$ a positive integer, then

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}};$$

b) If $x = 0,$ then $a^0 = 1$;

c) If $x = -n,$ where n is a positive integer, then $a^{-n} = \frac{1}{a^n}$;

d) If x is a rational number, $x = \frac{p}{q},$ where p, q are inegers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}\,.$$

Figure 5.34 presents the graphs of the exponential functions $y = 2^x$, $y = \left(\frac{1}{2}\right)^x$, $y = 4^x$, $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{3}{2}\right)^x$, $y = \left(\frac{2}{3}\right)^x$. Notice that both graphs pass through the same point $(0, 1)$, because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential functions grows more rapidly (for $x > 0$). When the base $a > 1$ $\left(a = 1,\ a = 4,\ a = \frac{3}{2}\right)$:

- the exponential function is increasing;
- the graph is asymptotic to the x-axis as x approaches negative infinity;
- the function increases without bound as x approaches positive infinity;
- the graph is continuous;
- the graph is smooth.

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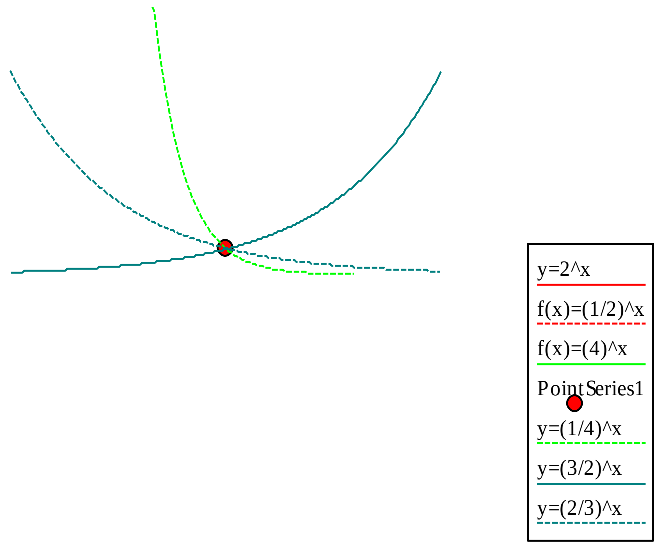

Figure 5.24 $y = 2^x$, $y = \left(\frac{1}{2}\right)^x$, $y = 4^x$, $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{3}{2}\right)^x$, $y = \left(\frac{2}{3}\right)^x$

The properties of the exponential functions and their graphs when the base $0 < a < 1$ are given.

- they form decreasing graphs
- the lines in the graph above are asymptotic to the x −axis as x approaches positive infinity
- the lines increase without bound as x approaches negative infinity
- they are continuous graphs
- they form smooth graphs.

Among the infinity number of exponential functions, two of them play the most important role: $y = 10^x$, $y = e^x$, where the number

$$e = 2{,}71828182845904523536028747135266249775724 \ldots$$

is irrational (i.e. it cannot be represented as ratio of integers) and transcendental (i.e. it is not a root of any non-zero polynomial with rational coefficients). It is enough to admit that

$$e \approx 2{,}72.$$

The number e is sometimes called Euler's number, after the Swiss mathematician Leonhard Euler or as Napier's constant. However, Euler's choice of the symbol e is said to have been retained in his

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honor. The constant was discovered by the Swiss mathematician Jacob Bernoulli while studying compound interest.

Figure 5.25 shows the graph of $y = 10^x$, while in *Figure 5.26* you can see the graph of $y = e^x$.

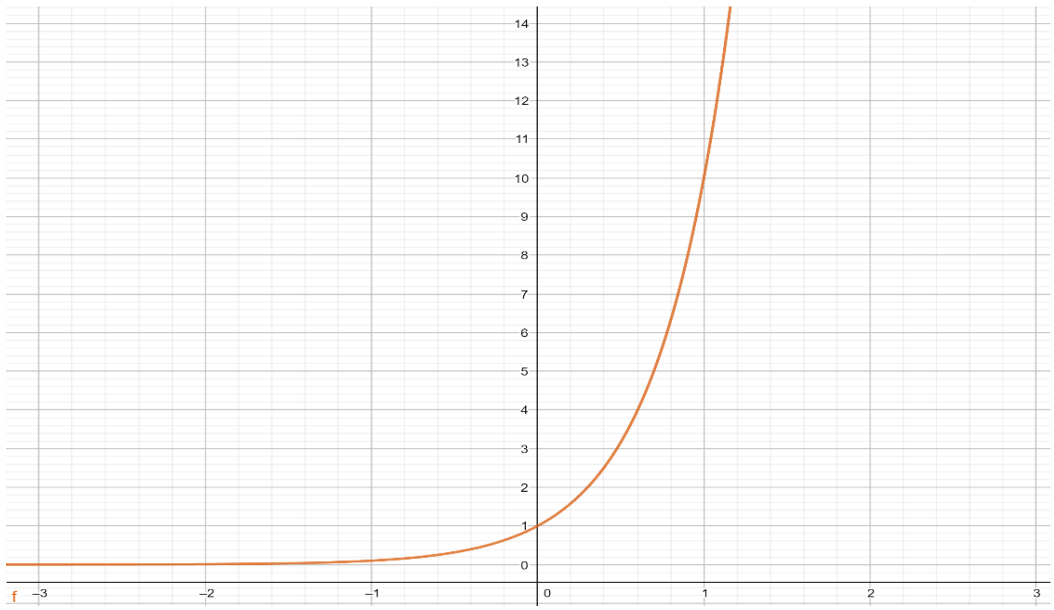

Figure 5.25 The graph of $\boldsymbol{y = 10^x}$

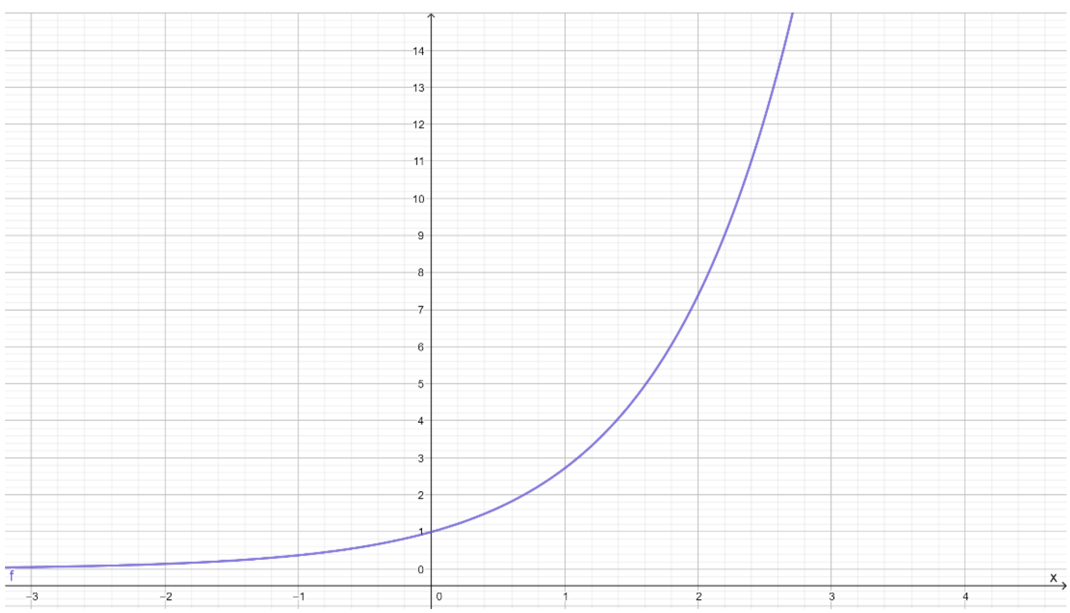

Figure 5.26 The graph of $\boldsymbol{y = e^x}$.

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Basic rules for exponentials

Rule or special case	Formula $a > 0, \quad x \in \mathbb{R}, y \in \mathbb{R}$	Example
Product	$a^x a^y = a^{x+y}$	$2^3 2^4 = 2^7$
Quotient	$\frac{a^x}{a^y} = a^{x-y}$	$\frac{2^4}{2^3} = 2^{4-3} = 2$
Power of power	$(a^x)^y = a^{x \cdot y}$	$(2^3)^2 = 2^{2 \cdot 3} = 2^6 = 64$
Power of a product	$(a \cdot b)^x = a^x b^x$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144$
Power of one	$a^1 = a$	$2^1 = 2$
Power of zero	$a^0 = 1$	$2^0 = 1$
Power of negative one	$a^{-1} = \frac{1}{a}$	$2^{-1} = \frac{1}{2}$
Change sign of exponents	$a^{-x} = \frac{1}{a^x}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
Fractional exponents	$a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$	$4^{\frac{3}{2}} = \sqrt{4^3} = (\sqrt{4})^3 = 2^3 = 8$

Solving exponential equations and inequalities

Example 5.27

Solve the following equations and inequalities.

a) $\mathbf{2^x + 2^{x+1} + 2^{x+2} = 6^x + 6^{x+1}}$

Solution

Assumption: $x \in \mathbb{R}$. Using the rules of exponentials we have:

$$2^x + 2 \cdot 2^x + 2^2 \cdot 2^x = 6^x + 6 \cdot 6^x \iff 2^x(1+2+4) = 6^x(1+6) \iff$$
$$2^x = 6^x \iff 1 = \frac{6^x}{2^x} \iff 1 = \left(\frac{6}{2}\right)^x \iff 1 = 3^x \iff x = 0.$$

b) $\left(\frac{7}{11}\right)^{7x-11} \geq \left(\frac{11}{7}\right)^{11x-7}$

Solution

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Assumption: $x \in \mathbb{R}$. Using the rules of exponentials we have:

$$\left[\left(\frac{11}{7}\right)^{-1}\right]^{7x-11} \geq \left(\frac{11}{7}\right)^{11x-7} \quad \Leftrightarrow \quad \left(\frac{11}{7}\right)^{11-7x} \geq \left(\frac{11}{7}\right)^{11x-7}$$

As the base $a = \frac{11}{7} > 1$, so the last inequality is equivalent to the following inequality:

$$11 - 7x \geq 11x - 7 \quad \Leftrightarrow \quad -18x \geq -18 \quad \Leftrightarrow \quad x \leq 1.$$

c) $\mathbf{6 \cdot 9^x + 5 \cdot 6^x - 6 \cdot 4^x \leq 0}$.

Solution

Assumption: $x \in \mathbb{R}$. Dividing the inequality by positive 4^x we obtain

$$\mathbf{6 \cdot \frac{9^x}{4^x} + 5 \cdot \frac{6^x}{4^x} - 6 \leq 0} \quad \Leftrightarrow \quad 6 \cdot \left[\left(\frac{3}{2}\right)^2\right]^x + 5 \cdot \left(\frac{3}{2}\right)^x - 6 \leq 0 \quad \Leftrightarrow$$

$$6 \cdot \left[\left(\frac{3}{2}\right)^x\right]^2 + 5 \cdot \left(\frac{3}{2}\right)^x - 6 \leq 0\,.$$

Now we substitute $\left(\frac{3}{2}\right)^x = t, \quad t > 0$. Then the last inequality takes the form:

$$6t^2 + 5t - 6 \leq 0\,.$$

As $\Delta = 25 + 144 = 169, \ \sqrt{\Delta} = 13\,,\ t_1 = \frac{-5-13}{12} = -\frac{3}{2},\ t_2 = \frac{-5+13}{12} = \frac{2}{3}$.

Therefore $6t^2 + 5t - 6 \leq 0 \ \Leftrightarrow \ -\frac{3}{2} \leq t \leq \frac{2}{3} \ \Leftrightarrow \ -\frac{3}{2} \leq \left(\frac{3}{2}\right)^x \leq \frac{2}{3}$.

Hence $\left(\frac{3}{2}\right)^x \leq \frac{2}{3} \ \Leftrightarrow \ \left(\frac{3}{2}\right)^x \leq \left(\frac{3}{2}\right)^{-1} \ \Leftrightarrow \ x \leq -1.$

d) $\mathbf{2^{2x} \leq 3 \cdot 2^{x+\sqrt{x}} + 4 \cdot 2^{2\sqrt{x}}}$.

Solution

Assumption: $x \geq 0$. Let us divide both sides of the inequality by positive $2^{2\sqrt{x}}$ and obtain

$$2^{2x-2\sqrt{x}} \leq 3 \cdot 2^{x+\sqrt{x}-2\sqrt{x}} + 4 \quad \Leftrightarrow \quad 2^{2(x-\sqrt{x})} \leq 3 \cdot 2^{x-\sqrt{x}} + 4\,.$$

Now we substitute $2^{x-\sqrt{x}} = t,\ t > 0$. Then we get

$$t^2 \leq 3t + 4 \quad \Leftrightarrow \quad t^2 - 3t - 4 \leq 0 \quad \Leftrightarrow \quad (t-4)(t+1) \leq 0\,.$$

As we see in *Figure 5.27* $-1 \leq t \leq 4 \ \Leftrightarrow \ -1 \leq 2^{x-\sqrt{x}} \leq 4$.

Hence $2^{x-\sqrt{x}} \leq 4 \ \Leftrightarrow \ 2^{x-\sqrt{x}} \leq 2^2 \ \Leftrightarrow \ x - \sqrt{x} \leq 2.$

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Now we use substitution $\sqrt{x} = u,\ u \geq 0$ and obtain

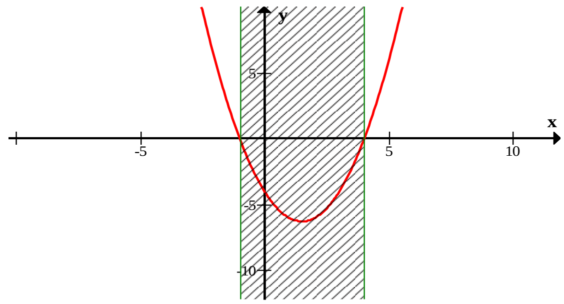

Figure 5.27 The illustration of $\boldsymbol{x^2 - 3x - 4 \leq 0}$.

$$u^2 - u \leq 2 \quad \Leftrightarrow \quad u^2 - u - 2 \leq 0 \quad \Leftrightarrow \quad (u+1)(u-2) \leq 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -1 \leq u \leq 2 \quad \text{(see } \textit{Figure 5.28}\text{)}$$

$$\Leftrightarrow \quad -1 \leq \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \begin{cases} x \geq 0 \\ x \leq 4 \end{cases} \quad \Leftrightarrow \quad x \in [0, 4].$$

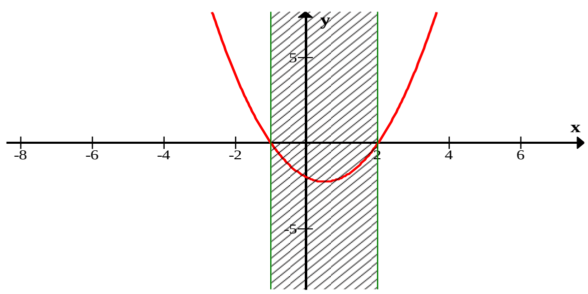

Figure 5.28 The illustration of $\boldsymbol{x^2 - x - 2 \leq 0}$.

Example 5.28 Compound interest

Most people who have a savings account with a bank or other financial institution leave their deposits for a period of time expecting to accrue money as time passes. If the deposits are made in an account carrying **simple interest** (flat rate of interest) the interest received is calculated on the original deposit for the duration of the account.

This would mean that if one invested $1\ 000$ € at a **flat interest rate** of 3.5% then in the first year he or she would have accrued:

$$\text{total earned = principal + } \mathbf{3.5\%} \text{ of principal over } \mathbf{1} \text{ year}$$
$$= 1000 + \frac{3{,}5}{100} \cdot 1000 \cdot 1 = 1000(1 + 0.035) = 1035$$

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Or $1\,035$ €.

We could perform the same calculations over five years, shown in the table below.

YEAR	TOTAL IN ACCOUNTS €	
Year 0	1000	1000
Year 1	$(1+0.035)$ of amount earned in year 0	$1000(1+0.035) = 1035$
Year 2	$(1+0.035)$ of amount earned in year 1	$(1+0.035)1000(1+0.035) = 1071{,}225$
Year 3	$(1+0.035)$ of amount earned in year 2	$= 1000\cdot(1+0.035)^3 = 1108{,}71788$
Year 4	$(1+0.035)$ of amount earned in year 3	$1000\cdot(1+0.035)^4 = 1147{,}523$
Year 5	$(1+0.035)$ of amount earned in year 4	$1000\cdot(1+0.035)^5 = 1187{,}686$

The compound interest formula is as follows

$$A = P\left(1+\frac{r}{100}\right)^n,$$

where

- A is the total amount returned, P is the principal (initial amount)
- r is the rate as a percentage returned in each investment period and n is the number of investment periods.

Example 5.29 Compound interest

Suppose in 2010 a man purchased a yacht Delfia 47S/Y 4 Breeze valued at $\$275\,000$. We know that yacht depreciate at 11.2% each year. What would the value of the yacht be after a period of time? Examine the table below for calculations for 5 years.

Solution

It is a special case of the depreciation formula:

$$D = P\left(1-\frac{r}{100}\right)^n,$$

where D is final value of the asset, P is the initial value of the asset, r is the rate of depreciation per period and n is the number of depreciation periods.

YEAR		
Year 0	275000	275000
Year 1	$(1-0.112) of\ year\ 0$	$(1-0.112)\cdot 275000 == 244200$

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Year 2	$(1-0.112) of\ year\ 1$	$275000\cdot(1-0.112)^2 == 216849.6$
Year 3	$(1-0.112) of\ year\ 2$	$275000\cdot(1-0.112)^3 == 192\,562{,}4$
Year 4	$(1-0.112) of\ year\ 3$	$275000\cdot(1-0.112)^4 == 170995{,}5$
Year 5	$(1-0.112) of\ year\ 4$	$275000\cdot(1-0.112)^5 == 151844$

Example 5.30

Exponential decrease can be modeled as:

$$N(t) = N_0 e^{-\lambda t}$$

where N is the quantity, N_0 is the initial quantity, λ is the decay constant (specific for each element), and t is time.
Oftentimes, half-life is used to describe the amount of time required for half of a sample to decay. It can be defined mathematically as:

$$t_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$$

where $t_{\frac{1}{2}}$ is half-life.
Half-life can be inserted into the exponential decay model as such:

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}.$$

Find how much carbon and iodine are present after a set period of time (t) given the information provided in the following table.

Element	$\boldsymbol{\lambda}$	$\boldsymbol{N_0}$ (in grams)	$\boldsymbol{t}$
Carbon	$1{,}203\cdot 10^{-4}\ years^{-1}$	3	$5760\ years$
Iodine	$0{,}08666\ days^{-1}$	5	$8\ days$

Solution

Using the decay function

$$N(t) = N_0 e^{-\lambda t}$$

for Carbon, the amount left after $t = 5\,760$ years is

$$N(t) = 3e^{-1{,}203\cdot 10^{-4}\cdot 5760} = 3e^{-6929} \approx 1{,}5\ g.$$

For Iodine, the amount left after $t = 8$ days is

$$N(t) = 5e^{-0{,}08666\cdot 8} = 5e^{-0{,}6933} \approx 2{,}5\ g.$$

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Notice that in each of the cases above the resultant mass is half of the initial mass of each element. This is an important notion in nuclear research. The time taken for a quantity of a specific element to be reduced to one half of its original mass is known as the **half-life** of the element. The half-life of carbon is $5\ 760$ years and the half-life of iodine is 8 days.

Remark

- **half-life**: The time it takes for a substance (drug, radioactive nuclide, or other) to lose half of its pharmacological, physiological, biological, or radiological activity.

  Imagine we have 100 kg of a substance with a half-life of 5 years. Then in 5 years half the amount (50 kg) remains. In another 5 years there will be 25 kg remaining. In another 5 years, or 15 years from the beginning, there will be 12.5. The amount by which the substance decreases, is itself slowly decreasing.
- **isotope**: Any of two or more forms of an element where the atoms have the same number of protons, but a different number of neutrons. As a consequence, atoms for the same isotope will have the same atomic number but a different mass number (atomic weight).

Example 5.31

A certain substance decays exponentially over time and is modelled by the function

$$N(t) = 4e^{-\lambda t},$$

where $\lambda = \frac{1}{5771}\ years^{-1}$ and $N(t)$ is measured in grams. Find how much of the substance is present initially and how much is present $4\ 000$ years later. Use the findings to comment on the half-life of this particular substance.

Solution

Given the function: $N(t) = 4e^{\left(-\frac{t}{5771}\right)}$,
to find the mass present initially put $t = 0$.

$$N(0) = 4e^{\left(-\frac{0}{5771}\right)} = 4 \cdot e^0 = 4\ .$$

Initial mass is 4 grams.

When $t = 4000$

$$N(4000) = 4e^{\left(-\frac{4000}{5771}\right)} = 4e^{(-0{,}69312)} = 2{,}000052 \approx 2$$

After 4000 years the mass is 2 grams.
The mass has halved after $4\ 000$ years so the half-life of the substance must be $4\ 000$ years.

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Exercises

Task 5.18 Solve the equations:

a) $3^{x+2} - 3^{x-1} = \frac{26}{9}$.
b) $3 \cdot 5^x - 2 \cdot 5^{x-1} = 5^{x+1} - \frac{12}{5}$.
c) $\frac{3}{10} \cdot \left(\frac{3}{2}\right)^{x-2} = \frac{6}{5}\left(\frac{3}{2}\right)^{x-3} - \frac{1}{2}$.
d) $4^{\frac{1}{2}x-1} = 2^{3(x+1)}$.
e) $2^{x^2-6x-\frac{5}{2}} = 16\sqrt{2}$.
f) $2^{2x} + 2^x = 20$.
g) $3^{2x} - 4 \cdot 3^x + 3 = 0$.
h) $\sqrt{2^x} \cdot \sqrt{3^x} = 6^x - 30$
i) $2^x + 3^x = 3^{x+1} - 2^{x+1}$.

Task 5.19 Solve inequalities:

a) $\left(\frac{8}{9}\right)^{8x^2-9} \geq \left(\frac{9}{8}\right)^{9x^2-8}$.
b) $\left(\frac{1}{2}\right)^{2x^2+x-1} > \left(\frac{1}{4}\right)^{\frac{1}{2}x^2+x-\frac{1}{8}}$.
c) $\left(\frac{1}{3}\right)^{|x-3|} \leq \frac{1}{9}$.
d) $5 \cdot 4^x + 2 \cdot 25^x \leq 7 \cdot 10^x$.
e) $2^{x+3} - 5^x < 7 \cdot 2^{x-2} - 3 \cdot 5^{x-1}$.
f) $7^{-x} - 3 \cdot 7^{x+1} > 4$.

Task 5.20 Suppose in 2020 a man purchased a motor yacht Chris Craft Launch 25GT valued at $\$234\ 750$. We know that yacht depreciate at 11.2% each year. What would the value of the yacht be in 2026?

Task 5.21 The taxation department allows depreciation of 25% pa on the diminishing value of some kind of computer devices installed on yachts produced by a boatyard. If a boatyard installs computers valued at $\$120\ 000$, construct a depreciation schedule for the next five years presenting the information in the table.

Task 5.22 The voltage (V measured in volts) across a capacitor is modelled by the equation $V = 10e^{\frac{-t}{3}}$, where t is measured in seconds. Find V when $t = 5$.

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Task 5.23 The decay of radium is modelled by the function $R = R_0 e^{-0,077t}$, where R is the amount remaining (g), t is time (weeks) and R_0 is the original amount.
Generate a table of values to find the half-life of 10 g of radium.(Remember that half-life means time to reach half of the original amount).

Task 5.24 Carbon dating involves the measurement of concentration of carbon remaining in an object. The decay function $C = 100 \cdot 2^{-0,1786t}$ is used to determine the age of a bone taken from an archaeological dig, where C is the concentration remaining and t is time in thousands of years. It is found that 60% of the original carbon remains in the samples.
Estimate the age of the bone. (Hint: Develop a table of values for the inverse function and find when $C = 60$).

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Answers

5.18.

a) $x = -1$
b) $x = 0$
c) $x = 2$
d) $x = -\frac{5}{2}$
e) $x = -1$ or $x = 7$
f) $x = 2$
g) $x = 0$ or $x = 1$
h) $x = 2$
i) $x = 1.$

5.19.

a) $x \in [-1,\ 1]$
b) $x \in \left(-\frac{1}{2},\ \frac{3}{2}\right)$
c) $x \in (-\infty, 1] \cup [5, \infty)$
d) $x \in [0,\ 1]$
e) $x \in (3, \infty)$
f) $x \in (-\infty, -1]$

5.20. $\$\ 115102.14$

5.21.

Year 1	$\$90000$
Year 2	$\$67500$
Year 3	$\$50625$
Year 4	$\$37968.75$
Year 5	$\$28476.56$

5.22. $V = 1.89.$

5.23.

Original function	
Weeks t	Radium $f(t)$ (g)
0	10,00
1	9,26
2	8,57
3	7,94
4	7,35
5	6,80
6	6,30
7	5,83
8	5,40
9	**5,00**

From the table we see that 10 grams of radium is reduced to 5 grams in 9 weeks.

5.24.

Original function		Inverse function	
Thousands of years t	Carbon	Carbon	Thousands of years t
0	100	100	0
1	88	88	1
2	78	78	2
3	69	69	3
4	61	61	4
5	54	54	5

If we start with $100\ g$ then 60% will occur when we have $60\ g$. From either table we can see that we have $60\ g$ when the bone is $4\ 000$ years old.

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Sample chapter exam

1. Solve the equation: $2 \cdot 4^{\sqrt{x}} = \sqrt[4]{2} \cdot 8^{x-1}$.

2. Solve the inequality: $5^x - 20 > 5^{3-x}$.

3. Find all the values of x for which $f(x) > 0$, if $f(x) = \left(\frac{3}{5}\right)^{x^2-x-6} - 1$.

4. There are given functions: $f(x) = 4^{x+1} - 7 \cdot 3^x$ and $g(x) = 3^{x+2} - 5 \cdot 4^x$. Solve the inequality $f(x) \leq g(x)$.

5. Find the domain and a range of a function $f(x) = e - e^x$.

6. The equation $P = 20 \cdot 10^{0,1n}$ can be used to convert any number of decibels (n) to the corresponding number of micropascals (P) used to measure loudness. Show that a 60 decibel sound is 10 times as loud as a 50 decibel sound, and 100 times as loud as a 40 decibel sound.

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5.4.4. Logarithmic functions

Logarithmic functions are the inverses of exponential functions, and any exponential function can be expressed in logarithmic form. Similarly, all logarithmic functions can be rewritten in exponential form. Logarithms are useful in permitting us to work with very large numbers while manipulating numbers of a much more manageable size.

Definition:

A logarithmic function is a function of the form

$$\boldsymbol{y = \log_a x, \quad x > 0, \quad a > 0, \quad a \neq 1,}$$

which is read " y equals the log of x, base a".

$$\boldsymbol{y = log_a\, x \quad \text{is equivalent to } x = a^y.}$$

There are no restrictions on y.

Example 5.32

Evaluate: a. $\log_3 81$, b. $\log_{25} 5$, c. $\log_{10} 0.001$.

Solution

a) $\log_3 81 = 4$ because $3^4 = 81$;

b) $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$;

c) $\log_{10} 0.001 = -3$ because $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$.

1. $\log_a(a^x) = x$ for every $x \in \mathbb{R}$
2. $a^{\log_a x} = x$ for every $x > 0$.

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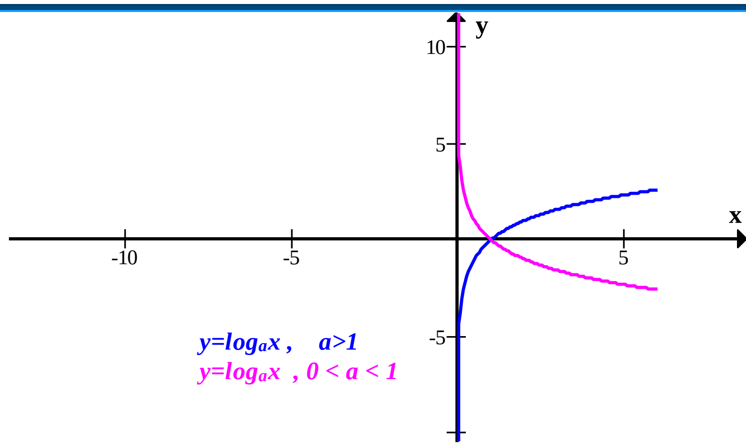

Figure 5.29 Graphs of logarithmic function when the base $\boldsymbol{a > 1}$ and $\boldsymbol{0 < a < 1}$.

The logarithmic function $y = \log_a x$ has domain $(\mathbf{0}, \infty)$ and range $\mathbb{R}$ and it is continuous since it is the inverse of a continuos function, namely, the exponential function.

As we see in *Figure 5.29* when the base $a > 1$ the logarithmic function is increasing while for $0 < a < 1$ the function is decreasing. *Figure 5.30* shows the graphs of $y = \log_a x$ with various values of the base a. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

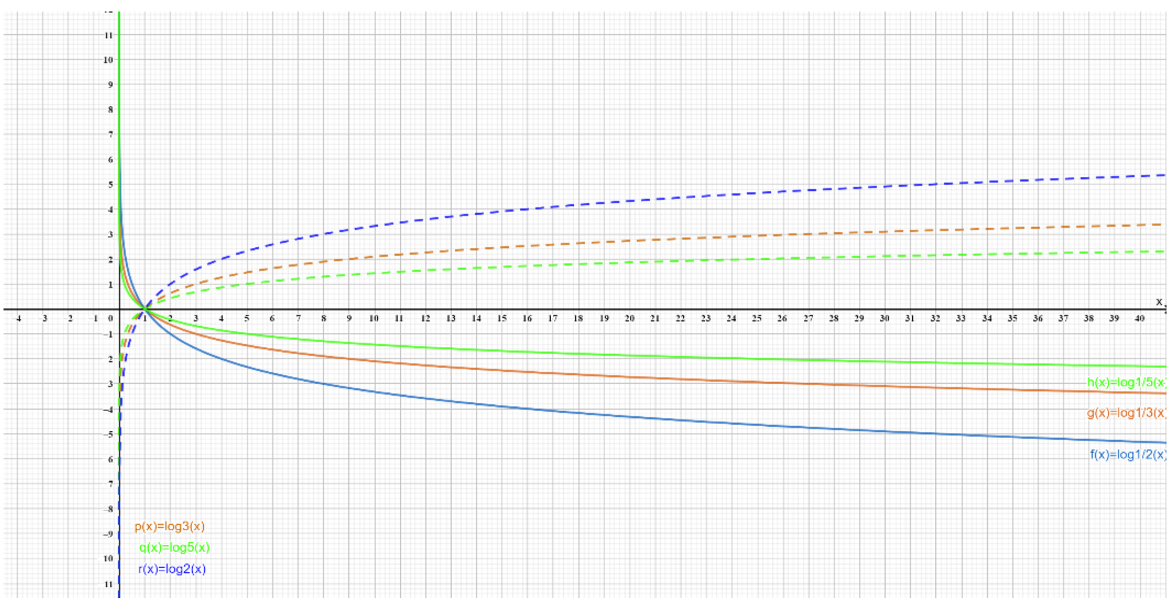

Figure 5.30 The graphs of $\boldsymbol{y = log_a x}$ with various values of the base $\boldsymbol{a}$.

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We need to introduce some special logarithms that occur on a very regular basis. They are the **common logarithm** and the **natural logarithm**. Here are the definitions and notations that we will be using for these two logarithms:

- common logarithm: $\mathbf{log_{10}\,x = log\,x}$, the base $a = 10$;
- natural logarithm: $\mathbf{log_e\,x = ln\,x}$, where e is Euler's number.

So, as we see the common logarithm is simply the log base 10, except we drop the “base 10” part of the notation. Similarly, the natural logarithm is simply the log base e with a different notation and where e is the same number that we saw in the previous section and is defined to be $e = 2.718281828\ldots$.*Figure 5.31* presents the graphs of the logarithms.

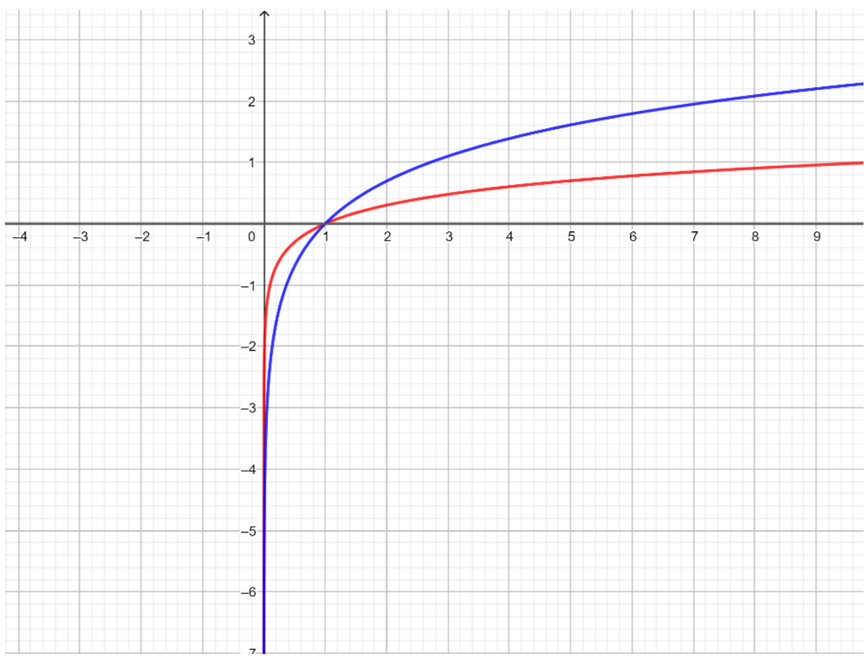

Figure 5.31 Graphs of $y = log\,x$, $y = ln\,x$

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Remember

If we put: $a = 10$ and $\log_{10} x = \log x$, $a = e$, $\log_e x = \ln x$, then the **defined properties** of the common logarithm and the natural logarithm functions become

$$y = \log x \ \Leftrightarrow \ 10^y = x \ ,$$

$$y = \ln x \ \Leftrightarrow \ e^y = x \ ,$$

and also

$$\log(10^x) = x \,, \quad x \in \mathbb{R} \ ,$$

$$10^{\log x} = x, \qquad x > 0,$$

$$\ln(e^x) = x \,, \quad x \in \mathbb{R} \ ,$$

$$e^{\ln x} = x, \qquad x > 0.$$

In particular, if we set $x = 10$, $x = e$, we get

$$\log 10 = 1, \ \ln e = 1.$$

Example 5.33

a) Find x if $\ln x = 5$.

Solution

$\ln x = 5$ means $e^5 = x$.
Therefore $x = e^5$.

b) Solve the equation $e^{5-3x} = 10$.

Solution :

We take natural logarithm of both sides of the equation:

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \frac{1}{3}(5 - \ln 10).$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution to four decimal places: $x \approx 0.8991$.

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The following theorem summarizes the properties of logarithmic functions.

Theorem 4.1.

If $a > 1$, the function $f(x) = \log_a x$ is one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$, then

1) $\mathbf{\log_a (xy) = \log_a x + \log_a y}$
2) $\mathbf{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y}$
3) $\mathbf{\log_a (x^y) = y\log_a x}$

Example 5.34

a) Evaluate $\log_4 2 + \log_4 32$.
b) Evaluate $\log_2 80 - \log_2 5$.
c) Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Solution:

a) Using Property 1 in Theorem 4.1., we have

$$\log_4 2 + \log_4 32 = \log_4 (2 \cdot 32) = \log_4 64 = 3,$$

since $4^3 = 64$.

b) Using Property 2 we have

$$\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = 4,$$

since $2^4 = 16$.

c) Using Properties 3 and 1 of logarithms, we have

$$\ln a + \frac{1}{2}\ln b = \ln a + \ln b^{\frac{1}{2}} = \ln a + \ln \sqrt{b} = \ln\left(a\sqrt{b}\right).$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

For any positive number a, $a \neq 1$, we have

4) $\mathbf{\log_a x = \frac{\ln x}{\ln a}}$.

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Generally, if we need to change the base a for another, let say b , we can do it as follows

$$5)\ \boldsymbol{\log_a x = \frac{\log_b x}{\log_b a}}.$$

Example 5.35 *Evaluate* $log_8 5 = \frac{ln\,5}{ln\,8}$.

Solution

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976.$$

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 4.4.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is a continuous, increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote.

Example 5.36

Determine which of the numbers are greater: $\log_3 222$ or $\log_2 33$.

Solution:

As the function $f(x) = \log_a x$, $a > 1$ is increasing, we have

$$\log_3 222 < \log_3 243 = \log_3 3^5 = 5 = \log_2 2^5 = \log_2 32 < \log_2 33.$$

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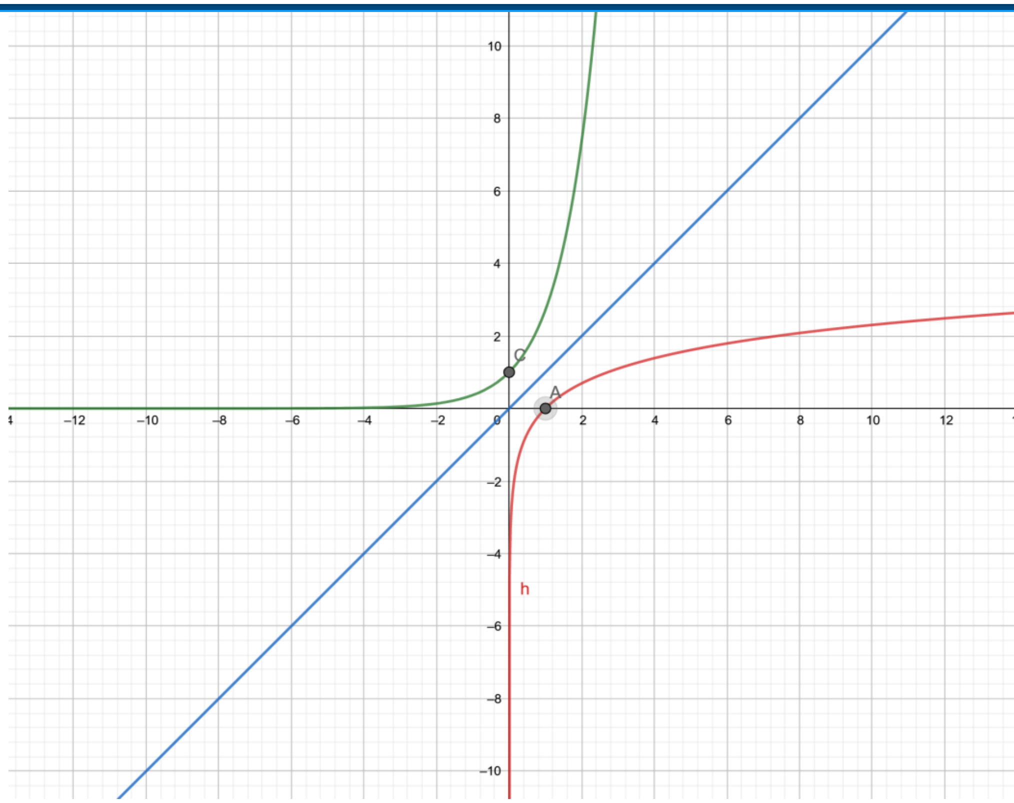

Figure 5.32 Graphs of $\boldsymbol{y = \ln x}$, $\boldsymbol{y = e^x}$ are symmetric with regard to $\boldsymbol{y = x}$.

Example 5.37 Solve the equations

a. $\log(3-x)(x-5) = \log(x-3) + \log(5-x)$.
b. $\log_3(4 \cdot 3^{x-1} - 1) = 2x - 1$.
c. $\log(5x^2 + 2x - 1) - \log(x+2) = 1$.

Solutions

a. As $(3-x)(x-5) = (x-3)(5-x)$, so we can write

$$\log(x-3)(5-x) = \log(x-3) + \log(5-x).$$

So the equation is satisfied if and only if

$$(x-3>0 \;\text{ and }\; 5-x>0) \Leftrightarrow (x>3 \;\text{ and }\; x<5) \Leftrightarrow 3<x<5.$$

b. Assumption: $4 \cdot 3^{x-1} - 1 > 0$, $3^{x-1} > \frac{1}{4}$ and we take $\log$ base 3 of both sides of the inequality and we get

$$x - 1 > \log_3 \frac{1}{4},$$
$$x - 1 > -\log_3 4,$$

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$$x > 1 - \log_3 4,$$
$$x > \log_3 3 - \log_3 4,$$
$$x > \log_3 \tfrac{3}{4}.$$

Now:

$$\mathrm{Log}_3(4 \cdot 3^{x-1} - 1) = \log_3(3^{2x-1})$$

$$4 \cdot 3^{x-1} - 1 = 3^{2x-1} \quad |\cdot 3$$

$$4 \cdot 3^x - 3 = 3^{2x} \;\Leftrightarrow\; (3^x)^2 - 4 \cdot 3^x + 3 = 0.$$

Let $3^x = t,\ t > 0$, then

$$t^2 - 4t + 3 = 0 \;\Leftrightarrow\; (t-1)(t-3) = 0 \;\Leftrightarrow\; t = 1 \text{ or } t = 3.$$

Hence $3^x = 1$, $3^x = 3 \;\Leftrightarrow\; x = 0,\ x = 1$.

Both $x = 0,\ x = 1$ satisfy the condition $x > \log_3 \frac{3}{4}$.

c. Assumption 1: $5x^2 + 2x - 1 > 0$.
Assumption 2: $x + 2 > 0$.

The equation can be written as follows:

$$\log(5x^2 + 2x - 1) = \log(x+2) + 1 \;\Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log(x+2) + \log(10) \;\Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log[10(x+2)] \;\Leftrightarrow$$

$5x^2 + 2x - 1 = 10(x+2)$ for all x satisfying Assumptions: 1 and 2.

Notice that if x_0 satisfies the equation and the Assumption 2 then x_0 satisfies also Assumption 1. Therefore, after solving the equation, it is enough to verify if its solutions satisfy Assumption 2 – it is much easier.

Now we solve the quadratic equation:

$$5x^2 - 8x - 21 = 0:$$
$$\Delta = 64 + 420 = 484,\ \sqrt{\Delta} = 22,\ x_1 = \frac{8-22}{10} = -\frac{7}{5},\ x_2 = \frac{8+22}{10} = 3.$$

It is easy to check that both solutions satisfy Assumption 2, thereby Assumption 1 which means that

$$x_1 = -\frac{7}{5},\ x_2 = 3$$

are solutions of the equation

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$$\log(5x^2 + 2x - 1) - \log(x + 2) = 1.$$

If we need to solve the logarithmic inequalities we will use the following facts:

- If $a > 1$, $g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \Leftrightarrow f(x) \geq g(x)$.
- If $0 < a < 1$, $f(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \Leftrightarrow f(x) \leq g(x)$.

Example 5.38 Solve the inequalities

a. $\log(x-4) + \log x \leq \log 21$.
b. $\log(2^x + x - 13) > x - x\log 5$.
c. $3^{(\log_3 x)^2} + x^{\log_3 x} \leq 162$.
d. $\log_{(x-2)} \frac{x-1}{x-3} \geq 1$.

Solutions

a. Assumption: $(x - 4 > 0\,,\ \ x > 0) \Leftrightarrow\ x > 4$.
For $x > 4$ we have

$$\log(x-4) + \log x \leq \log 21 \ \Leftrightarrow\ \log x(x-4) \leq \log 21 \ \Leftrightarrow\ x(x-4) \leq 21$$

$$\Leftrightarrow x^2 - 4x - 21 \leq 0.$$

Compute $\Delta = 16 + 84 = 100$, $\sqrt{\Delta} = 10$, $x_1 = -3$, $x_2 = 7$.

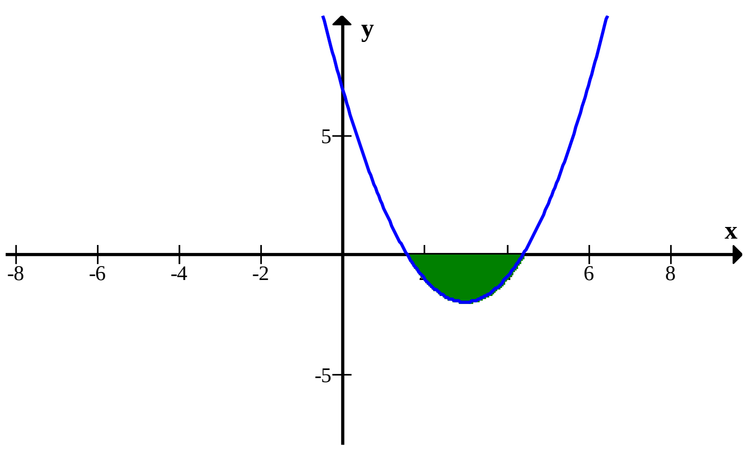

Figure 5.33

Thus $x \in [-3, 7]$ and taking included the assumption $x > 4$, finally we obtain

$$x \in (4,\ 7].$$

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b. We write the right side of the inequality as common logarithm, namely

$$x - x\log 5 = x(1 - \log 5) = x(\log 10 - \log 5) = x\log 2 = \log 2^x.$$

Now the inequality b. can be presented as

$$\log(2^x + x - 13) > \log 2^x.$$

We know that

if $a > 1$, $g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \;\Leftrightarrow\; f(x) \geq g(x)$.

We have $a = 10$, $g(x) = 2^x > 0$ for $x \in \mathbb{R}$, so

$$\log(2^x + x - 13) > \log 2^x \;\Leftrightarrow\; 2^x + x - 13 > 2^x \;\Leftrightarrow\; x > 13.$$

Then $x \in (13, \infty)$.

c. Assumption: $x > 0$.

As $3^{(\log_3 x)^2} = \left(3^{\log_3 x}\right)^{\log_3 x} = x^{\log_3 x}$, so the inequality takes the form of

$$2 \cdot x^{\log_3 x} \leq 162 \;\Leftrightarrow\; x^{\log_3 x} \leq 81.$$

Both sides of the inequality are positive, so take both sides log base 3 we get

$$\log_3 x^{\log_3 x} \leq \log_3 81 \;\Leftrightarrow\; (\log_3 x)^2 \leq 4 \;\Leftrightarrow\; |\log_3 x| \leq 2 \;\Leftrightarrow$$

$$\Leftrightarrow -2 \leq \log_3 x \leq 2 \;\Leftrightarrow\; \log_3 3^{-2} \leq \log_3 x \leq \log_3 3^2 \;\Leftrightarrow$$

$$\Leftrightarrow 3^{-2} \leq x \leq 3^2 \;\Leftrightarrow\; \frac{1}{9} \leq x \leq 9.$$

These numbers satisfy the assumption $x > 0$, so $x \in \left[\frac{1}{9},\ 9\right]$.

d. Assumption: $\begin{cases} x - 2 > 0 \\ x - 2 \neq 1 \\ \frac{x-1}{x-3} > 0 \end{cases} \;\Leftrightarrow\; \begin{cases} x > 2 \\ x \neq 3 \\ x > 3 \end{cases} \;\Leftrightarrow\; x > 3\,.$

Now we have

$$\log_{(x-2)} \frac{x-1}{x-3} \geq \log_{(x-2)}(x-2).$$

As for $x > 3$, $a = x - 2 > 1$, $g(x) = x - 2 > 0$, then on the basis of the fact:
“If $a > 1$, $g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \;\Leftrightarrow\; f(x) \geq g(x)$”, we obtain

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$$\left(\frac{x-1}{x-3} \geq x-2 \ \ , \ \ x>3\right) \Leftrightarrow \begin{cases} x>3 \\ (x-2)(x-3) \ \leq x-1 \end{cases} \Leftrightarrow \begin{cases} x>3 \\ x^2-6x+7 \leq 0 \end{cases}$$

Now we compute $\Delta = 36-28=8$, $\sqrt{\Delta} = 2\sqrt{2}$, $x_1 = 3-\sqrt{2}$, $x_2 = 3+\sqrt{2}$. From Fig. 4.6. it follows that $x \in \left[3-\sqrt{2}\,,\ 3+\sqrt{2}\right]$ and $x>3$. Finally

$$x \in (3\,,\ 3+\sqrt{2}].$$

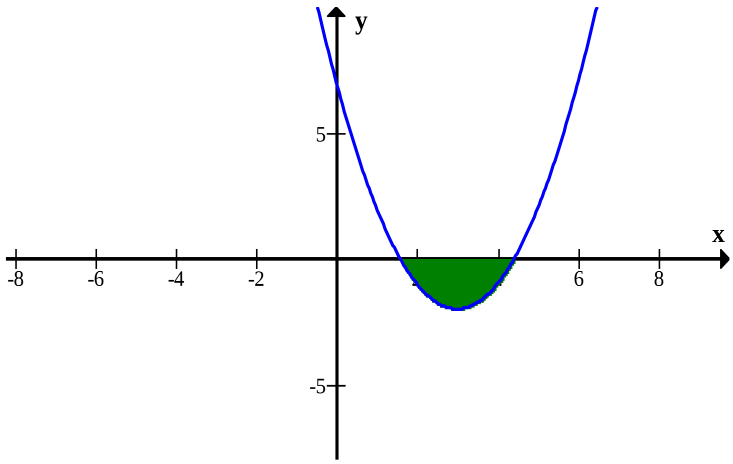

Figure 5.34

Example 5.39 ***Measuring loudness***

Sounds can vary in intensity from the lowest level of hearing (a ticking watch 7 meters away) to the pain threshold (the roar of a jumbo jet). Sound is detected by the ear as changes in air pressure measured in micropascals (μP). The ticking watch is about $20\ (\mu P)$, conversational speech about $20\,000\ (\mu P)$, a jet engine close up about $200\,000\,000\ (\mu P)$, an enormous range of values. A scale was required to compress the range of 20 to $200\,000\,000$ into a more manageable and useful form from 0 to 140. The decibel scale was invented for this purpose. If P is the level of sound intensity to be measured and P_0 is a reference level, then

$$n = 20\log\left(\frac{P}{P_0}\right),$$

where n is the decibel scale level.
If we assume $20\ (\mu P)$ to be the threshold level, then the equation would be:

$$n = 20\log\left(\frac{P}{20}\right)$$

and the graph of the relationship would resemble the one below (Figure 5.35).

Decibel scale for loudness of sound

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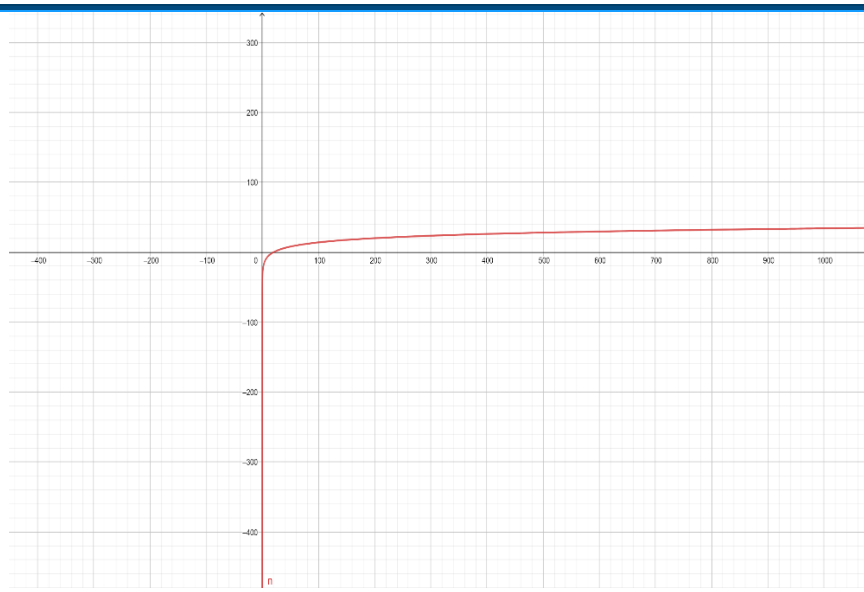

Figure 5.35 Sound level (μP)

As the sound is measured in a logarithmic scale using a unit called a decibel then we can use the following formula:

$$d = 10\log\left(\frac{P}{P_0}\right)$$

where P is the power or intensity of the sound and P_0 is the weakest sound that the human ear can hear.

One hot water pump has a noise rating of 50 decibels. One device in engine room, however, has a noise rating of 62 decibels. How many times is the noise of the device in engine room more intense than the noise of the hot water pump?

Solution

We can't easily compare the two noises using the formula, but we can compare them to P_0. Start by finding the intensity of noise for the hot water pump. Use h for the intensity of the hot water pump's noise:

$$50 = 10\log\left(\frac{h}{P_0}\right),$$

$$5 = \log\left(\frac{h}{P_0}\right),$$

$$10^5 = \frac{h}{P_0}$$

$$h = 10^5 P_0,$$

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then repeat the same process to find the intensity of the noise for the device in engine room

$$62 = 10\log\left(\frac{d}{P_0}\right),$$
$$6{,}2 = \log\left(\frac{d}{P_0}\right),$$
$$10^{6{,}2} = \frac{d}{P_0},$$

$$d = 10^{6{,}2}P_0.$$

To compare d to h, we divide $\frac{d}{h} = \frac{10^{6.2}P_0}{10^5 P_0} = 10^{1.2}$

Answer: The noise of the device in engine room is $10^{1.2}$ (or about 15.85) times more intense of the noise of the hot water pump.

Remark

The conditions of people on board the ships are particularly difficult, because even after working in crew cabins are at risk of being in high-level areas vibration and noise. During the cruise, the crew cannot escape to the forest or park in areas of peace and quiet. The crew (especially the mechanics) is exposed to danger related to hearing loss. In addition, excessive levels of noise and vibration cause others ailments such as cardiovascular and nervous system diseases. Ailments syndrome health related to noise, infrasonic noise and vibrations is called vibroacoustic disease. For example, at room of marine power plant a permissible noise level is 90 decibels ($90\ dB$), at control room, navigation cabin – $65dB$.

Example 5.40

The stellar magnitude of a star is negative logarithmic scale, and the quantity measured is the brightness of the star. If $SM = -\log B$, where SM is stellar magnitude and B is brightness, answer the following questions:

a) What is the stellar magnitude of star A which has a brightness of 0.7943?
b) Star B has a magnitude of 2.1, what is its brightness?
c) Compare the brightness of the two stars.

Solution

a) If the star has a brightness of 0.7943 from the formula $SM = -\log B$ it will have a stellar magnitude of $-\log(0.7943)$ or 0.1. This can be confirmed from the graph.
b) If the magnitude is 2.1 then

$$2.1 = -\log B$$
$$-2.1 = \log B$$

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$B = 10^{-2.1}$
$B \approx 0.007943.$

The brightness is 0.007943.

c) Comparing the star A with the star B : A is about 100 times brighter then B.

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Exercises

Task 5.25 Evaluate:

a. $\log_{\sqrt{2}} 16$
b. $\log_2 \frac{1}{8}$
c. $\log_4 0.5$
d. $\log_{\sqrt{2}} 0.25$
e. $\log_{\frac{2}{3}} 2.25$
f. $\log_{\frac{1}{9}} 3\sqrt[3]{3}$
g. $16^{\log_2 3}$.

Task 5.26 Evaluate $\log_{35} 28$ if we know that $\log_{14} 2 = a, \quad \log_{14} 5 = b.$

Task 5.27 Solve the equations:

a. $\ln(5x - e) = 1$
b. $\log_{1.5}\left(2x - \sqrt[3]{1.5}\right) = \frac{1}{3}$
c. $\log_x 3\sqrt{3} = \frac{1}{2}$
d. $\log_{\frac{3}{4}}\left(1 - \frac{x-2}{2x-5}\right) = -1$
e. $\ln(\log_2 x) = 0.$

Task 5.28 Solve the equations:

a. $\log_3\left(x + \sqrt{3}\right) = -\log_3\left(x - \sqrt{3}\right)$
b. $\log_3(5x + 1) - \log_3(x - 1) = 2$
c. $\log_4 x + \log_4(12 - 2x) = 2$
d. $\log(5 - x) + 2\log\sqrt{x - 3} = 0$
e. $\frac{1}{2}\log(2x + 7) + \log\sqrt{7x + 5} = 1 + \log\frac{9}{2}$
f. $\log_3 x + \log_5 x = \frac{\log 15}{\log 3}$
g. $(\log_3 x)^2 = \frac{1}{2}\log_3 x.$

Task 5.29 Solve inequalities

a. $\log(x - 3) - \log(27 - x) \leq -\log 5 - 1$
b. $\log_{\frac{1}{2}}(\log_5 x) \geq 0$
c. $\log_{\frac{1}{3}}(|x| - 1) > -2$
d. $3^{\log_{\frac{1}{5}}(x^2-4x-4)} < 1$
e. $\log_{x^2}(x + 6) \geq 1$

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f. $\log_{\frac{1}{2}} \frac{2x+1}{3x+2} > 3$

Task 5.30 A particular dangerous bacteria culture, that threatens the marine fauna of the Maldives, doubles every 20 minutes and follows the exponential function $N(t) = 200 \cdot 2^{3t}$, where $N(t)$ is the number of bacteria in the culture after t hours. After how many hours will be $1\,000\,000$ bacteria in the culture?

Task 5.31 Rearrange the following formula to make x the subject: $y = 1.4e^{-0.6x} - 3.$

Answers

4.24

a. 8 c. $-\frac{1}{2}$ d. -4 f. $-\frac{2}{3}$ g. 81
b. -3 e. -2

4.25. $\frac{a+1}{b-a+1}$

4.26.

a. $x = \frac{2}{5}e$ b. $x = \sqrt[3]{1.5}$ c. $x = 27$ d. $x = \frac{11}{5}$ e. $x = 2$

4.27.

a. $x = 2$ c. $x_1 = 2,\ x_2 = 4$ f. $x = 5$ h.
b. $x = \frac{5}{2}$ d. $x = 4$ g. $x_1 = 1,\ x_2 = \sqrt{3}$
e. $x = 10$

4.28.

a. $x \in \ (3, \frac{59}{17}]$ c. $x \in (-10, -1)\ \cup (1, 10)$ f. $x \in \left(-\frac{1}{2}, -\frac{6}{13}\right).$
b. $x \in (1, 5]$ d. $x \in (-\infty, -1)\ \cup (5, \infty)$
e. $x \in [-2,\ \ -1) \cup (1,\ \ 3]$

4.29. After approximately 4.096 hours there will be $1\,000\,000$ bacteria in the culture.

4.30. $x = -\frac{5}{3}\ln\left(\frac{y+3}{1.4}\right).$

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Sample chapter exam

1. Prove the following statements:

   a. $\log_{\sqrt{a}} x = 2\log_a x$,

   a. $\log_{\frac{1}{\sqrt{a}}} \sqrt{x} = -\log_a x$,

   c. $\log_{a^4} x^2 = \log_a \sqrt{x}$.

2. Solve the equation: $\log_{\frac{1}{2}}[\log_2(\log_4 x)] = -1.$

3. Solve the inequality: $\log_{\frac{1}{\sqrt{5}}}(6^{x+1} - 36^x) \geq -2.$

4. Find the domain of the function: $f(x) = \log_{x^2-3}(x^2 + 2x - 3).$

5. Draw the graph of each of the following logarithmic functions and analyze each of them completely (i.e., domain, range, zeros, y −intercept, sign, maximal intervals of monotonicity):

   a. $f(x) = \log(-x)$,

   b. $f(x) = -\log(x - 3)$.

6. If $y = 3(\mu e)^k$ show that $k = \frac{\ln y - \ln 3}{\ln \mu + 1}$.

7. If $A = P(1 + i)^n$, find n in terms of A, P and i.

8.* Solve the inequality without using a calculator: $\log_{2008}(x^2 - 2007x) \leq 1.$

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5.4.5. Square root function

Square roots are often found in math and science problems. Students can easily understand the rules of square roots and answer any questions involving them, whether they require direct calculation or just simplification.

A square root asks you which number, when multiplied by itself, gives the result after the " $\sqrt{\ }$ " symbol. So $\sqrt{4} = 2$ and $\sqrt{25} = 5$.

The " $\sqrt{\ }$ " symbol tells you to take the square root of a number and you can find this on most calculators. The symbol " $\sqrt{\ }$ " is called the radical and x is called the radicand.

We can factor square roots just like ordinary numbers, so $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ or

$\sqrt{6} = \sqrt{2}\,\sqrt{3}$.

Square root of a negative number

The definition of a square root means that negative numbers should not have a square root (because any number multiplied by itself gives a positive number as a result). The imaginary number i is used to mean the square root of -1,

$$\sqrt{-1} = i, \quad i^2 = -1.$$

Any other negative roots are expressed as multiples of i, $\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \mp 2i$.

Square root function

The square root function is a type of power function $f(x) = x^{\alpha}$, with fractional power as it can be written

$$\boldsymbol{f(x) = x^{\frac{1}{2}},\ f(x) = \sqrt{x}}.$$

Its domain D_f is the set of non-negative real numbers:

$$D_f = [0, \infty) = \mathbb{R}_+ \cup \{0\}.$$

Its range is also the set of non-negative real numbers: $[0, \infty)$.

The graph of the square root function is shown in *Figure 5.36.* with some points.

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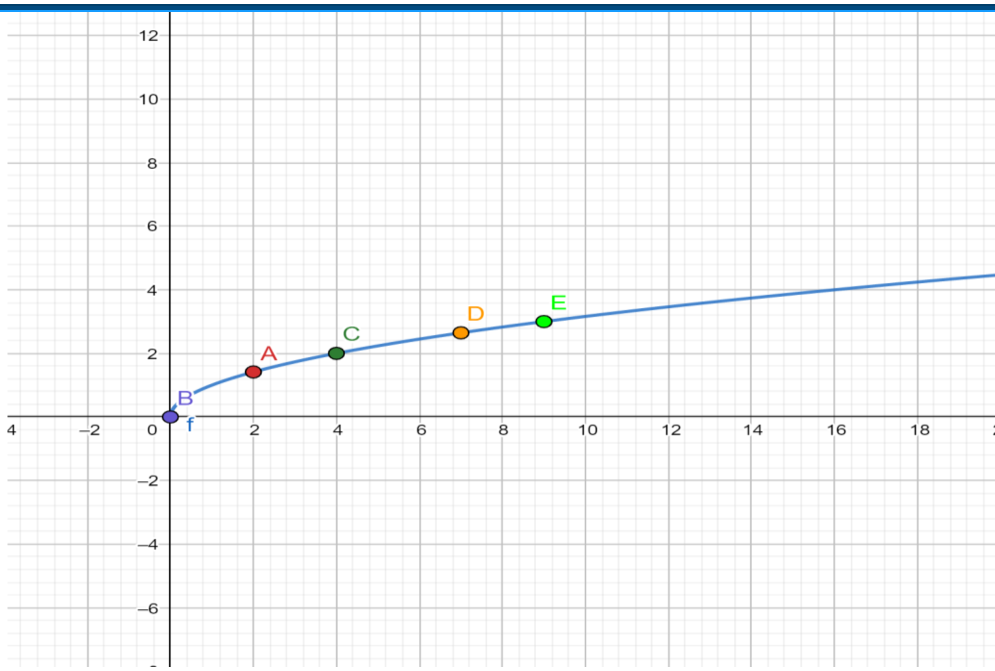

Figure 5.36 The graph of $f(x) = \sqrt{x}$

- **It is wrong** to write $\sqrt{25} = \mp 5$. The radicand is the symbol of the square root function and a function has only one output which as defined above is equal to the positive root.

Correct answer: $\sqrt{\mathbf{25}} = \mathbf{5}$

- **It is wrong** to write $\sqrt{\boldsymbol{x}^{\mathbf{2}}} = \boldsymbol{x}$. The output of the square root is non-negative and x in the given expression may be negative, zero or positive.

Correct answer: $\sqrt{\boldsymbol{x}^{\mathbf{2}}} = |\boldsymbol{x}|$

Properties of square root function

Some of the properties of the square root function may be deduced from Fig.5.1.

1. x and y intercepts are both at $(0,0)$.
2. the square root function is an increasing function
3. the square root function is a one-to-one function and has an inverse.

Square root functions of the general form:

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$$\boldsymbol{f(x) = a\sqrt{x-c} + d.}$$

Figure 5.37 presents how the graph of $f(x) = a\sqrt{x-c} + d$ looks like when the parameters a, c, d change.

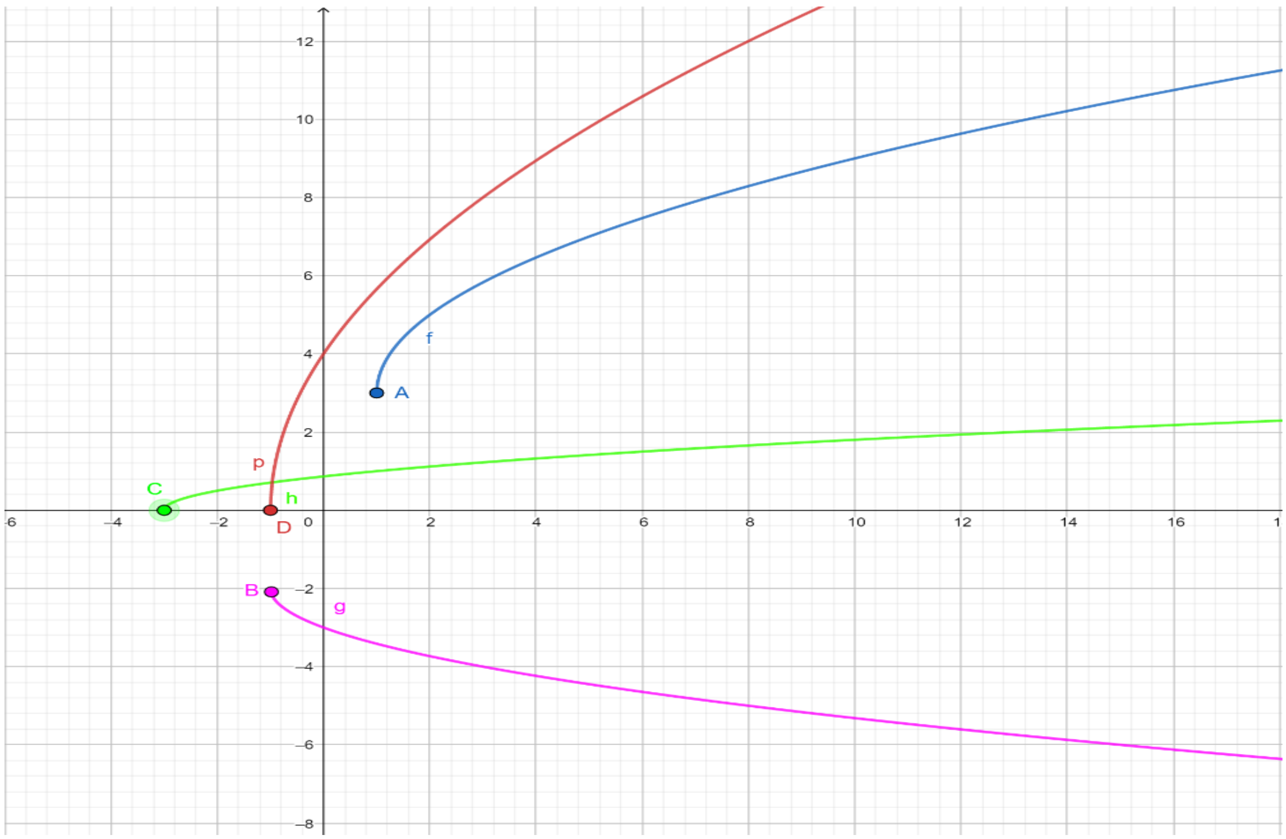

Figure 5.37 Graphs of $\boldsymbol{f(x) = 2\sqrt{x-1} + 3}$, $\boldsymbol{g(x) = -\sqrt{x+1} - 2}$, $\boldsymbol{h(x) = \frac{1}{2}\sqrt{x+3}}$,

$p(x) = 4\sqrt{x+1}$.

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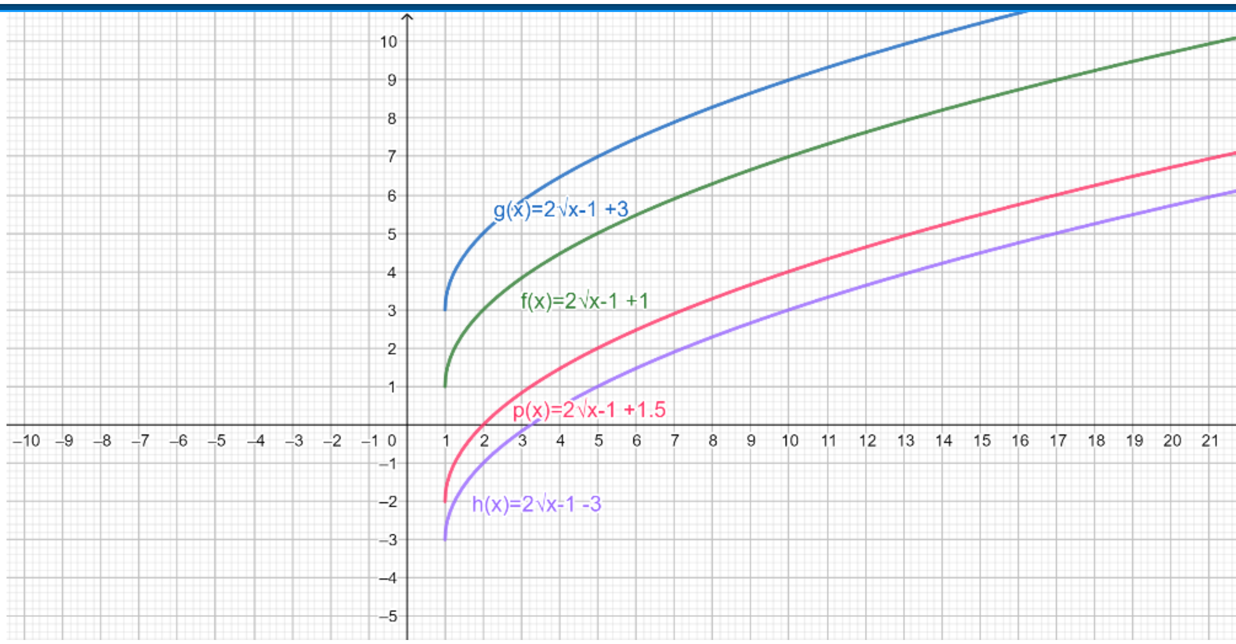

Figure 5.38 What happens to the graph when the value of parameter $\boldsymbol{d}$ *changes?*

From **Figure 5.37** to **Figure 5.39**.-5.4. we can conclude that

- changes in the parameter d affect the y coordinates of all points on the graph hence the vertical translation or shifting. When d increases, the graph is translated upward and when d decreases the graph is translated downward.

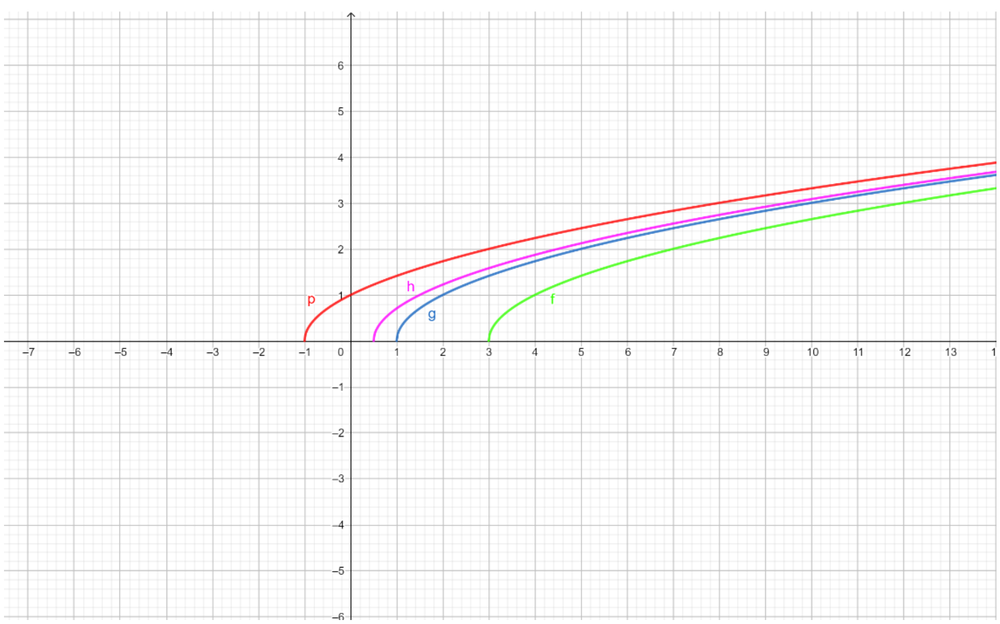

Figure 5.39 What happens to the graph when the value of parameter $\boldsymbol{c}$ *changes?*

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$p(x) = \sqrt{x+1}$, $h(x) = \sqrt{x-0.5}$, $g(x) = \sqrt{x-1}$, $f(x) = \sqrt{x-3}$.

- When c increases, the graph is translated to the right and when c decreases, the graph is translated to the left. This is also called horizontal shifting.
- Parameter a is a multiplicative factor for the y coordinates of all points on the graph of function $f(x)$.

  If a be greater than zero and larger than 1, the graph stretches (or expands) vertically. If a gets smaller than 1, the graph shrinks vertically. If a changes sign, a reflection of the graph on the x axis occurs.

- Only parameter c affects the domain. The domain of $f(x) = a\sqrt{x-c} + d$ may be found by solving the inequality $x - c \geq 0$. Hence, the domain is the interval $[c, \infty)$.
- Only parameters a and d affect the range. The range of function $f(x)$ may be found as follows: with x in the domain defined by interval $[c, \infty)$, the $\sqrt{x-c}$ is always positive or equal to zero hence $\sqrt{x-c} \geq 0$ and if parameter a is positive then $a\sqrt{x-c} \geq 0$.

  If we add d to both sides, we obtain $a\sqrt{x-c} + d \geq d$. Hence the range of the square root function defined above is the set of all values in the interval $[d, \infty)$.

  If parameter a is negative then $a\sqrt{x-c} \leq 0$. If we add d to both sides we obtain

$$a\sqrt{x-c} + d \leq d$$

  Hence the range of the square root function defined above is the set of all values in the interval $(-\infty, d]$.

Solving square roots equations and inequalities

We will use the following theorem:

Theorem 5.1.

If $a \geq 0,\ b \geq 0, n \in N$, then

- $a = b \ \Leftrightarrow\ a^n = b^n$
- $a < b \ \Leftrightarrow\ a^n < b^n$
- $a \leq b \ \Leftrightarrow\ a^n \leq b^n$.

Example 5.41

1. Solve the equation $\sqrt{10x+6} = 9 - x$.

Solution

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Assume that $10x + 6 \geq 0,\ x \geq -\frac{3}{5}$.

The left side of the equation is non-negative, so to be sure that the equation is not contradictory we have to assume additionally that $9 - x \geq 0$.

Finally, we get the assumption: $x \in \left[-\frac{3}{5}, 9\right]$.

Then using Theorem 5.1 we have

$$\sqrt{10x+6} = 9 - x \Leftrightarrow\ 10x + 6 = (9-x)^2\ \Leftrightarrow\ x^2 - 28x + 75 = 0.$$

$$\Delta = 28^2 - 4 \cdot 75 = 484,\quad \sqrt{\Delta} = 22,\quad x_1 = \frac{28-22}{2} = 3,\quad x_2 = \frac{28+22}{2} = 25.$$

Due to the assumption $x \in \left[-\frac{3}{5}, 9\right]$, $x_1 = 3$ is the solution of the equation.

Example 5.42 Solve the inequalities:

a. $\sqrt{x+3} < -2$
b. $\sqrt{2-x} > -5$
c. $\sqrt{5-x} < 3$
d. $\sqrt{11-x} > x - 9$
e. $\sqrt{3-2x-x^2} < 2x^2 + 4x - 3$

Solution

a. Assume that $x + 3 \geq 0,\ i.e.\ x \geq -3$.

Left side of $\sqrt{x+3} < -2$ is non-negative while the right one is negative, so we conclude that the inequality is a contradiction. It means that the inequality has no solutions.

b. Assume that $2 - x \geq 0\ i.e.\ x \leq 2$.

Left side of $\sqrt{2-x} > -5$ is non-negative while the right one is negative, so we conclude that the inequality is satisfied for all $x \leq 2$. Hence $x \in (-\infty, 2]$.

c. Assume that $5 - x \geq 0,\ i.e.\ x \leq 5$.

Both sides of $\sqrt{5-x} < 3$ are non-negative, so according to Theorem 5.1., for $x \leq 5$ we have

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$$\sqrt{5-x} < 3 \;\Leftrightarrow\; \left(\sqrt{5-x}\right)^2 < 3^2 \;\Leftrightarrow\; 5-\mathrm{x} < 9 \;\Leftrightarrow\; x > -4.$$

Having regard to the assumption $x \le 5$ we get $x \in (-4,\ 5]$.

d. Assume that $11 - x \ge 0,\ i.e.\ x \le 11.$

As both sides of $\sqrt{11-x} > x - 9$ do not have the permanent sign at $(-\infty,\ 11]$ we have to consider two cases:

1. $x < 9,$ then the left side of the inequality is non-negative while the right one is negative, so the inequality holds for all $x < 9$;

2. $9 \le x \le 11,$ then both sides are non-negative, so when $x \in [9,\ 11]$ we have

$$\sqrt{11-x} > x-9 \;\Leftrightarrow\; \left(\sqrt{11-x}\right)^2 > (x-9)^2 \;\Leftrightarrow$$

$$\Leftrightarrow\; 11 - x > x^2 - 18x + 81 \;\Leftrightarrow\; x^2 - 17x + 70 < 0$$

$$\Leftrightarrow\; \left|\ \Delta = 289 - 280 = 9,\quad \sqrt{\Delta} = 3,\ x_1 = \frac{17-3}{2} = 7, x_2 = \frac{17+3}{2} = 10\ \right|,$$

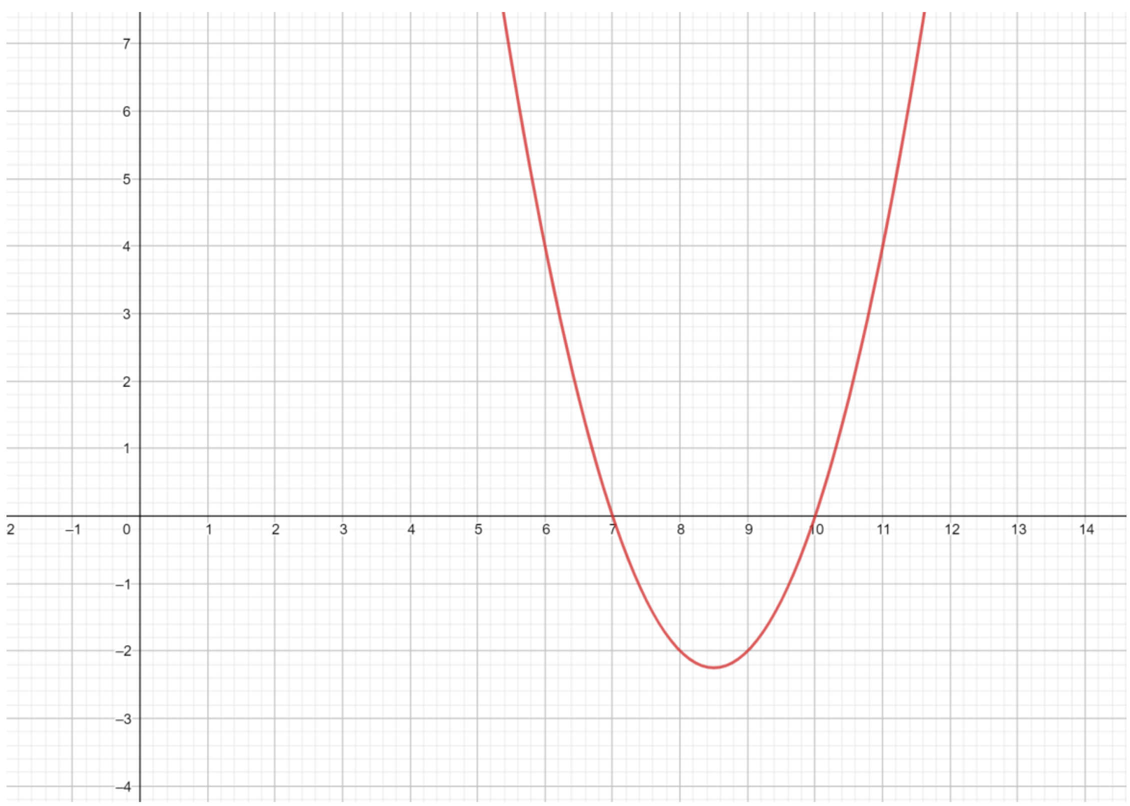

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Figure 5.40

hence as we see in *Figure 5.40*. $x \in (7,\ 10)$. Regarding the assumption $x \in [9,\ 11]$ we finally obtain $x \in [9, 10)$.

Considering both cases we get $x \in (-\infty,\ 9) \cup [9, 10) = (-\infty,\ 10)$.

e. Assume

$$3 - 2x - x^2 \geq 0 \quad \Leftrightarrow \quad x^2 + 2x - 3 \leq 0 \Leftrightarrow \quad x^2 + 2x + 1 \leq 4 \quad \Leftrightarrow$$

$$\Leftrightarrow \ (x+1)^2 \leq 4 \quad \Leftrightarrow \quad \sqrt{(x+1)^2} \leq \sqrt{4} \ \Leftrightarrow \ |x+1| \leq 2 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -2 \leq x + 1 \leq 2 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -3 \leq x \leq 1 \ \text{ or } \ x \in [-3,\ 1].$$

Notice that the right side of the inequality $\sqrt{3 - 2x - x^2} < 2x^2 + 4x - 3$ can

expressed by the trinomial $3 - 2x - x^2$ i.e.

$$2x^2 + 4x - 3 = 2(x^2 + 2x) - 3 = -2(-2x - x^2) - 3 =$$

$$-2(3 - 2x - x^2) + 3.$$

Then we can substitute $\sqrt{3 - 2x - x^2} = t$,

hence $3 - 2x - x^2 = t^2$ and

$$2x^2 + 4x - 3 = -2(3 - 2x - x^2) + 3 = -2t^2 + 3.$$

Now the inequality takes the following form:

$$t < -2t^2 + 3 \ \Leftrightarrow \ 2t^2 + t - 3 < 0.$$

After computing $\Delta = 1 + 24 = 25$, $\sqrt{\Delta} = 5$, we get $t_1 = -\frac{3}{2}, t_2 = 1$.

From *Figure 5.41* we conclude $-\frac{3}{2} < t < 1$. Therefore for $x \in [-3, 1]$ we have:

$$-\frac{3}{2} < \sqrt{3 - 2x - x^2} < 1 \quad \Leftrightarrow \quad \sqrt{3 - 2x - x^2} < 1 \ \Leftrightarrow \ \left(\sqrt{3 - 2x - x^2}\right)^2 < 1 \quad \Leftrightarrow$$

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$$\Leftrightarrow \; 3-2x-x^2<1 \quad \Leftrightarrow \quad x^2+2x+1>3 \quad \Leftrightarrow \quad (x+1)^2>3 \; \Leftrightarrow$$

$$\Leftrightarrow \; |x+1|>\sqrt{3} \quad \Leftrightarrow \; x+1<-\sqrt{3} \;\; \vee \;\; x+1>\sqrt{3} \;\; \Leftrightarrow$$

$$\Leftrightarrow \; x<-1-\sqrt{3} \;\; \vee \;\; x>-1+\sqrt{3} \;.$$

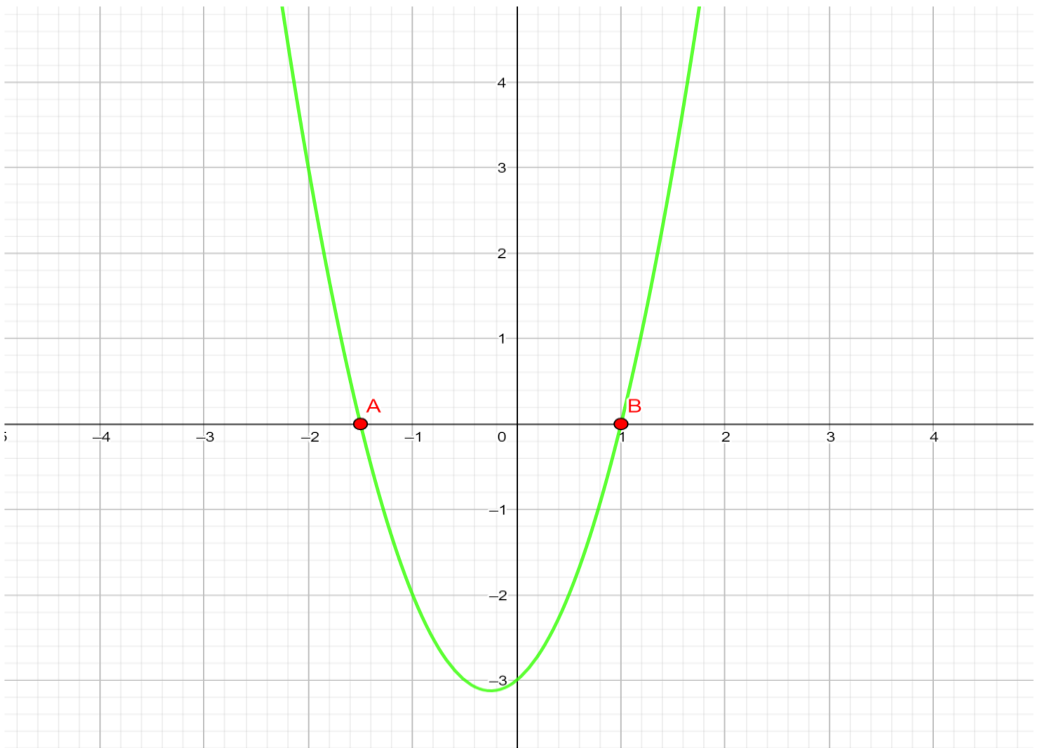

Figure 5.41

Finally, included the assumption $[-3,1]$ the solution is

$$x \in [-3,\; -1-\sqrt{3}) \cup \left(-1+\sqrt{3},\;\; 1\right].$$

3. Solve the equation:

$$\sqrt{11x+3}+\sqrt{3x-6}=\sqrt{4-2x}+\sqrt{12x+1}.$$

<u>Solution</u>

Assume that $11x+3 \geq 0,\;\; 3x-6 \geq 0,\;\; 4-2x \geq 0,\;\; 12x+1 \geq 0.$

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Solving all those inequalities at the same time we notice that the equation all adds up only for $x = 2$. It is easy to verify that $x = 2$ satisfies the equation, so it is its solution.

4. Solve the inequality:

$$\sqrt{2x-1} + \sqrt{3x-2} < \sqrt{4x-3} + \sqrt{5x-4}.$$

Solution

Assume that $2x - 1 \geq 0$ and $3x - 2 \geq 0$ and $4x - 3 \geq 0$ and $5x - 4 \geq 0$. Hence

$$x \geq \frac{4}{5}.$$

Let us transform the inequality, namely:

$$\left(\sqrt{4x-3} - \sqrt{2x-1}\right) + \left(\sqrt{5x-4} - \sqrt{3x-2}\right) > 0$$

$$\left| \quad \sqrt{a} - \sqrt{b} = \frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}} \quad \right|$$

$$\frac{(4x-3)-(2x-1)}{\sqrt{4x-3}+\sqrt{2x-1}} + \frac{(5x-4)-(3x-2)}{\sqrt{5x-4}+\sqrt{3x-2}} > 0$$

$$2(x-1)\left(\frac{1}{\sqrt{4x-3}+\sqrt{2x-1}} + \frac{1}{\sqrt{5x-4}+\sqrt{3x-2}}\right) > 0.$$

As the expression in parentheses is positive, so the last inequality is satisfied for $x - 1 > 0$. Each solution $x > 1$ satisfies the assumption $x \geq \frac{4}{5}$. Finally we obtain $x \in (1, \infty)$.

5. At what temperature, the veocity distribution function for the oxygen molecules will have maximum value at the speed $400\,\frac{m}{s}$?

Solution

The maximum speed for any gas occurs when it is at most probable temerature

$v_{mp} = \sqrt{\frac{2RT}{m}}$, where R is the gas constant, T is the absolute temperature, m is the molar mass of the gass. (Maxwell-Boltzman Distribution).

Given $v_{mp} = 400\,\frac{m}{s}$, $m = 32 \cdot 10^{-3} kg$ for 1 mole of oxygen molecules, $R = 8{,}31$.

Then

$$v_{mp}^2 = \frac{2RT}{m},$$

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$$T = \frac{m \cdot v_{mp}^2}{2R}$$
$$T = \frac{400^2 \cdot 32 \cdot 10^{-3}}{2 \cdot 8{,}31} \approx 308°\text{C}.$$

6. The temperature of the gas is raised from $27°\text{C}$ to $927°\text{C}$**.** What is the root mean square velocity?

Solution

The **root-mean-square velocity** is the measure of the of particles in a gas, defined as the **square root** of the **average velocity-squared** of the molecules in a gas. The **root-mean-square velocity** takes into account both molecular weight and temperature, two factors that directly affect the kinetic energy of a material.

T in Kelvin= $°\text{C} + 273$

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

Change $°\text{C}$ into Kelvin:

$$T_1 = 27°\text{C} + 273 = 300$$
$$T_2 = 927°\text{C} + 273 = 1200$$
$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \quad => \quad v_2 = v_1\sqrt{\frac{T_2}{T_1}}$$

and $v_2 = v_1\sqrt{\frac{1200}{300}} = 2v_1$

therefore root mean square velocity will be doubled.

7. Boat builders share an old rule of thumb for sailboats. The maximum speed K in knots is 1.35 times the square root of length L in feet of the boat's waterline. A customer is planning to order a sailboat with a maximum speed of 8 knots. How long should the waterline be?

Solution

The *knot* (kn) is ***a*** unit of speed equal to *one* nautical mile per hour, exactly 1.852 km/h The feet (ft') is a unit of length in the British imperial and United State customary systems of measurement, exactly $0{,}3048$ m.

$$K = 1.35\sqrt{L}$$
$$8 = 1.35\sqrt{L}$$

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$\sqrt{L} = \frac{8}{1.35} \approx 5.926 \quad => \quad L = 35.12$

The waterline should be 35.12 feet long.

Exercises

Task 5.32 Solve the equations.

a. $(x^2 - 4)\sqrt{1 - x} = 0.$

b. $x - \sqrt{x + 1} = 5.$

c. $x + \sqrt{10x + 6} = 9.$

d. $\sqrt{4 + 2x - x^2} = x - 2.$

e. $\sqrt{2x - 3} + \sqrt{4x + 1} = 4.$

f. $x = 15 + \sqrt{9 + 8x - x^2}.$

Task 5.33 Solve the inequalities.

a. $\sqrt{5 - x} < -2.$

b. $\sqrt{x + 3} > -23.$

c. $\sqrt{2x + 3} > x + 2.$

d. $\sqrt{x + 3} + \sqrt{3x - 2} \leq 7.$

e. $\sqrt{x - 2} + x > 4.$

f. $\sqrt{8 - x} > x - 6.$

Task 5.34 The circular velosity, v , in miles per hour, of a satellite orbiting Earth is given by the formula $v = \sqrt{\frac{1.24 \cdot 10^{12}}{r}}$, where r is a distance in miles from the satellite to the center of the Earth. How much greater is the velocity of a satellite orbitting at an altitude of 100 mi than one orbitting at 300 mi?

(Radius of a the Earth is 3950 mi, 1 mi $=1{,}609344$ km)

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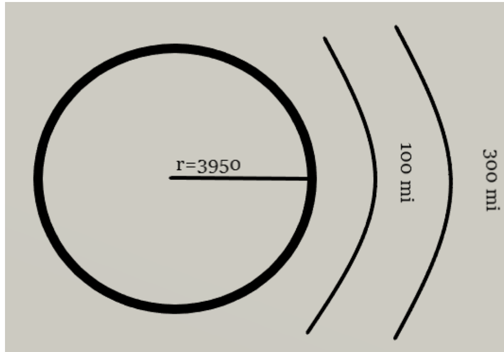

Figure 5.42

Answers

5.33

a. $x_1 = -2,\ \ x_2 = 1.$
b. $x = 8.$
c. $x = 3.$
d. $x = 3.$
e. $x = 2.$
f. No solutions.

5.34

a. No solutions.
b. $x \in [-3, \infty)$
c. No solutions.
d. $x \in \left[\frac{2}{3}, 6\right]$.
e. $x \in (3,\ \infty)$.
f. $x \in (-\infty, 7)$.

5.35 The velocity of a satellite orbitting at an altitude of $100\ \ \mathrm{mi}$ is 1.024 times greater than one orbitting at $300\ \ \mathrm{mi}$.

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Sample chapter exam

1. Determine in which set the functions are equal:

$$f(x) = \sqrt{(x-1)(x-5)} \quad \text{and} \quad g(x) = \sqrt{(x-1)} \cdot \sqrt{x-5}.$$

2. Solve the equations.

   a. $\sqrt{x+3} + \sqrt{x} = 3.$

   b. $3 - \sqrt{x-1} = \sqrt{3x-2}.$

   c. $(x^2 + x - 6)^{0.5} = \frac{1}{2}x - 1.$

3. Solve the inequalities.

   a. $x - 1 < \sqrt{7-x}$.
   b. $\sqrt{(x-6)(1-x)} < 2x + 3.$
   c. $\sqrt{1 + 10x + 5x^2} \geq 7 - 2x - x^2.$

4.* Find the domain of the given function $f(x) = \log_{17}\left(x + \sqrt{x^2+1}\right)$.

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5.5. Functions of form $\frac{1}{x^p}$

Aims:

1) Students know that $\frac{1}{x^p}$ is special case of the power function $y = kx^p$
2) Students know what special property's function $\frac{1}{x^p}$ m ay exhibit when *p* is odd and when *p* is even.
3) Apply these properties in graphing function $\frac{1}{x^p}$.

Definition of power function

Definition:

A power function is a single – term function that contains a variable as its base and a constant for its exponent.

In case of function $y = \frac{1}{x^p}$ we speak about power functions where *k* = 1 and *p* < 0.

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Power functions where p = - 1, - 3, - 5, …

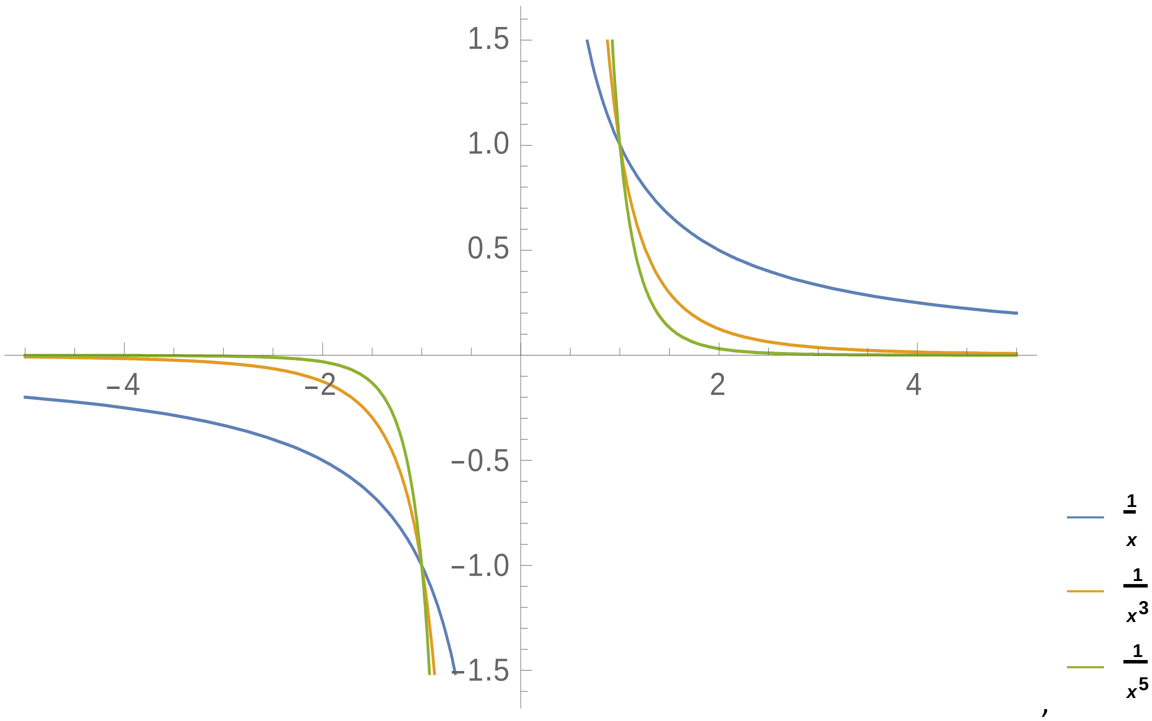

Figure 5.43 Graph 1

Properties:

- Domain: $X \neq 0$
- Domain of variation: $Y \neq 0$
- Eveness and odness: $y(-x) = -y(x)$ → odd
- Monotomy: monotonus
- Extremes: none
- Intersection points with coordinate axes: none
- Domain of convexity: $\check{X} = (-\infty; 0)$; $\hat{X} = (0; +\infty)$
- Inflection point: none
- Inverse functions:
  - If $p = -1$, then $x = \frac{1}{y}$
  - If $p < -2$, then $x = \frac{1}{\sqrt[|p|]{y}}$

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Power functions where p = - 2, - 4, -6, …

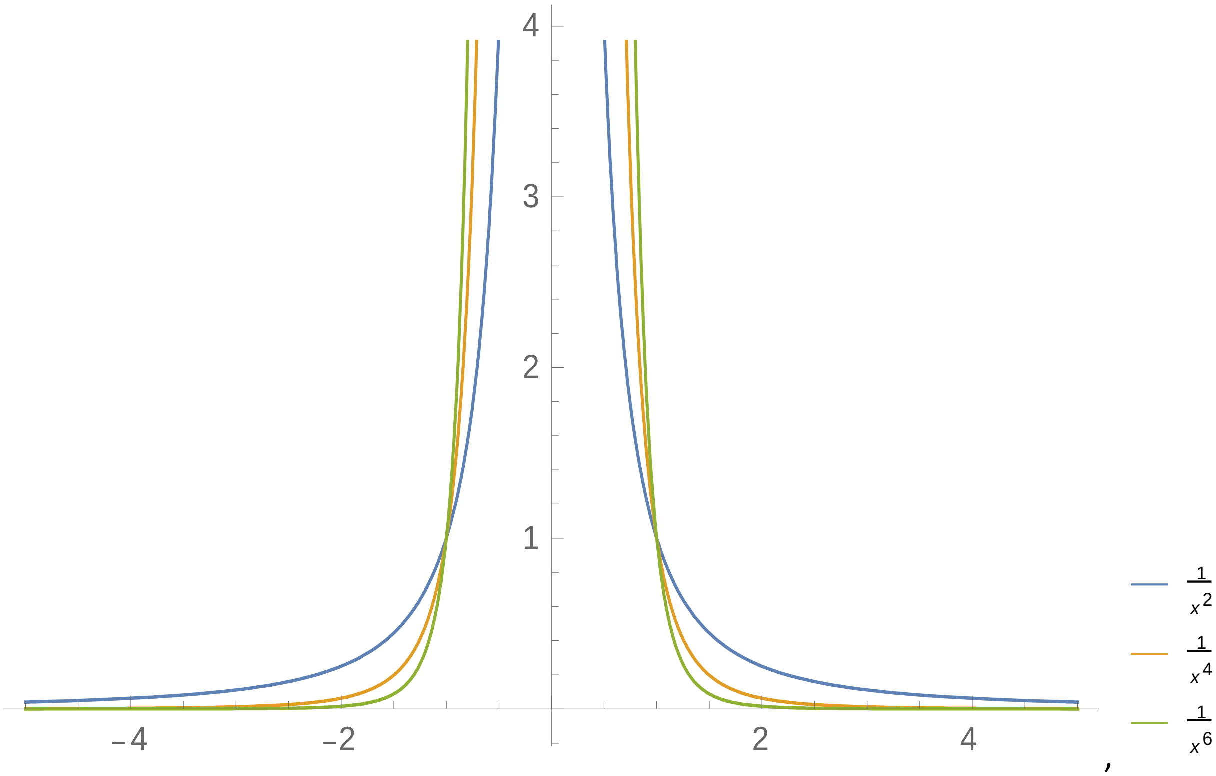

Figure 5.44 Graph 2

Properties:

- Domain: $X \neq 0$
- Domain of variation: $Y > 0$
- Eveness and odness: $y(-x) = -y(x)$ → odd
- Monotomy: $X \uparrow = (-\infty; 0); X \downarrow = (0; +\infty)$
- Extremes: none
- Intersection points with coordinate axes: none
- Domain of convexity: $\hat{X} = (-\infty; 0)$
- Inflection point: none
- Inverse functions:
    - If p = - 2, then $x = \frac{1}{\sqrt{y}}$
    - If p < -2, then $x = \frac{1}{\sqrt[|p|]{y}}$

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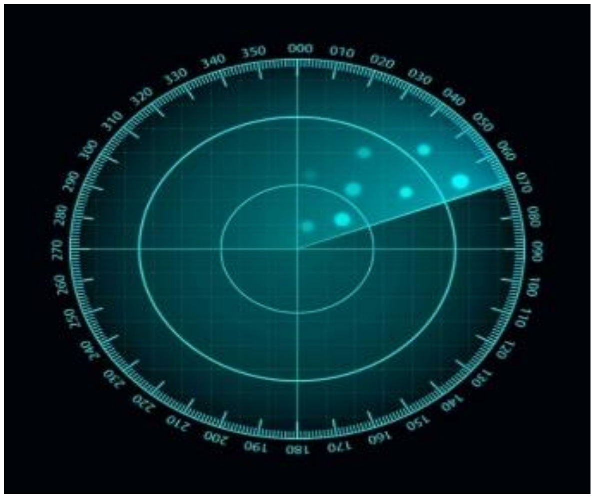
5.6. TRIGONOMETRY



1) Determine the course (azimuth) of the ship if it wants to reach the harbour

ABSTRACT:

In this chapter basic trigonometric concepts will be introduced, including the link between degrees and radians, conversion of angles from one measure to another, and calculating coterminal angles. The trigonometric functions will firstly be introduced as ratios in a right triangle, and their definitions will then be expanded to onto the number line. Techniques for algebraic and graphical solving of equations and inequalities will be provided in the final part of this chapter. Finally, specific usage of trigonometry in navigation will be provided in the end of this chapter. Exercises are also included with solutions at the end of each task together with GeoGebra files.

AIM: **To successfully convert between radian and degree measure, to calculate the trigonometric functions in a right triangle, to solve trigonometric equations and inequalities, and to use trigonometric functions and concepts in everyday tasks.**

Learning Outcomes:

1. To convert between radian and degree measure (including coterminal angles)
2. To use trigonometric functions in solving a right triangle
3. To graph different types of trigonometric functions, and determine their properties from the function equation
4. To solve simple trigonometric equations and inequalities
5. To apply trigonometric concepts in everyday tasks linked to maritime navigation

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Prior Knowledge: functions, coordinate geometry, basic geometry, Pythagorean theorem, ratios.

Relationship to real maritime problems: polar coordinates, navigation using polar coordinates, constraints problems (limited docking space), ...

Contents:

5.10.1. Types, graphs, important limits
5.10.2. CONNECTIONS AND APPLICATIONS
5.10.3. Equations
5.10.4. Inequalities
5.10.5. CONNECTIONS AND APPLICATIONS

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5.6.1. Types, graphs, important limits

Trigonometric functions are nowadays used in almost every branch of applied science, from optics, acoustics, electricity to maritime problems (calculating the height of waves and tides) and of course astronomy. Because of their relation to right angle geometry, trigonometry was reseach extensively even in ancient Greece. The first trigonometric tables were complied by Hipparchus of Niecaea (180-125 BC), who is now known as „the father of trigonometry". The discovery and rapid development of trigonometry was closely related to the discoveries in marine navigation.

To understand trigonometry, it is first crutial to understand the link between the degree and radian measure of an angle.

Definition: Radian

A radian, denoted rad is a unit for measuring angles. It represents the length of an arc on a unit circle corresponding to an angle formed between the point on the unit circle and the positive part of the x axis.

Example 5.43 The following sketch illustrates the definition:

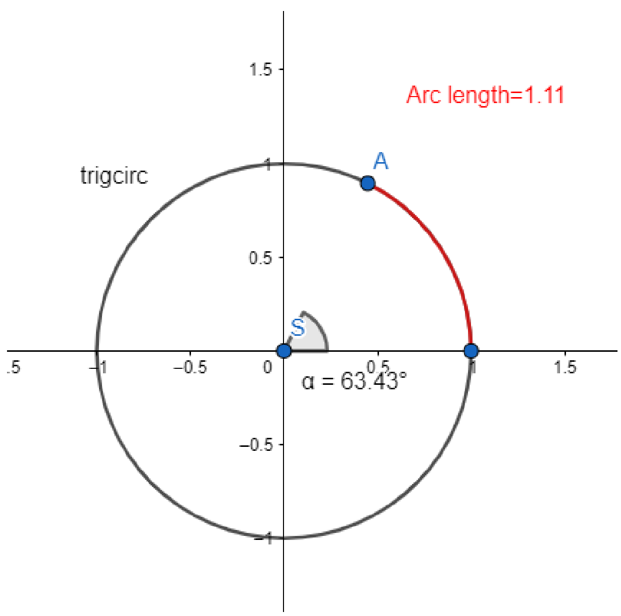

Figure 5.45

The arc length in red is the radian measure corresponding to the angle formed by the positive ray of the x axis, the centre of the circle S, and the point A.

The link below can be used to determine the radian measure of angles less than 360°

https://www.geogebra.org/calculator/mk7cqdxx

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Note that for 360° the radian measure is equal to the **circumference** of the circle, so the radian measure of **360°** is $\boldsymbol{2\pi}$.

Definition: Coterminal angles

For an angle greater than 360° or less than 0°, the **coterminal angle** is an angle in standard form between 0° and 360°. The coterminal angle can be calculated using the following formula:

$$coterminal(\alpha) = \alpha - \left\lfloor \frac{\alpha}{360°} \right\rfloor \cdot 360°$$

where $\lfloor \cdot \rfloor$ is the floor function.

Example 5.44 Determining the coterminal angle

$$coterminal(730°) = 730° - \mathbf{2} \cdot 360° = 730° - 720° = 10°$$

The number **2** is obtained by dividing 730° by 360° and rounding to the nearest **lower** integer.

$$coterminal(1081°) = 1081° - \mathbf{3} \cdot 360° = 1081° - 1080° = 1°$$

The number **3** is obtained by dividing 1080° by 360° and rounding to the nearest **lower** integer.

$$coterminal(-120°) = -120° - (-\mathbf{1}) \cdot 360° = -120° + 360° = 240°$$

The number **(-1)** is obtained by dividing -120° by 360° and rounding to the nearest **lower** integer.

If the angle is greater than 360°, then the radian measure of that angle can be calculated using the coterminal angle.

Example 5.45 Conversion between radians and degrees

For a central angle of 360°, we know that the corresponding radian measure is 2π, because the radian measure is the **circumference.**

For a central angle of 180° (half of 360°) the radian measure would be half of the circumference, so π.

Note that a central angle of 120° would close an arc of exactly 1/3 of the total circumference, so the values change **proportionally**. The arc length in relation to the circumference is equal to the angle measure in degrees in relation to 360°.

The following ratio can be used to determine the radian and degree measure of an angle:

$$\alpha° : 360° = \alpha_{rad} : 2\pi$$

or in fractional notation as:

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$$\frac{\alpha^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Example 5.46 Determining the radian measure

$$\alpha^\circ = 45^\circ, \alpha_{rad} = ?$$

Using the equation provided above:

$$\frac{\alpha^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Substituting $\alpha^\circ = 45^\circ$ into the formula:

$$\frac{45^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Reducing the left side by 45°:

$$\frac{1}{8} = \frac{\alpha_{rad}}{2\pi}$$

Cross-multiplying:

$$2\pi = 8\,\alpha_{rad}$$

Solving:

$$\alpha_{rad} = \frac{2\pi}{8} = \frac{\pi}{4}$$

We defined the radian measure of an angle using the **unit circle** (radius equal to 1).

Each point on the circle also has an x and y coordinate.

One such illustrative point is shown in the picture below.

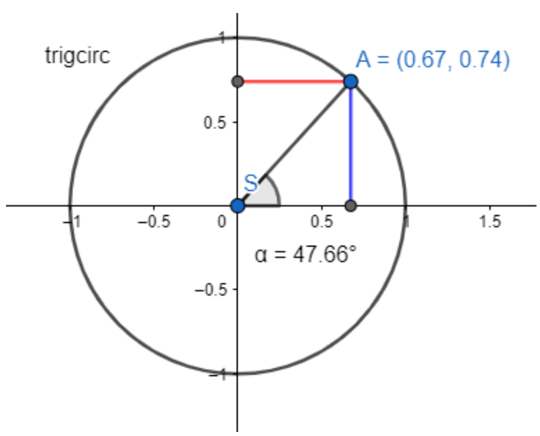

Figure 5.46

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The x coordinate, shown in red, can be read using the x axis.

The y coordinate, shown in blue, can be read using the y axis.

Let us also note that the length of the line segment AS is equal to the radius, so $|AS| = 1$ because the circle is a unit circle.

Definition: *The sine function*

The **sine function** associates each central angle to the y coordinate of the corresponding point on a unit circle.

For example, $\sin(0°) = 0$, because for 0° the corresponding point on the unit circle is (1,0). The y coordinate of that point is 0.

Since we can determine the radian measure for each degree measure, the argument of the sine function can also be a **radian measure**.

Example 5.47 Determining the sine value of angles

Let us determine the value of $\sin(\frac{\pi}{2})$.

As an exercise, try to prove that the radian measure of $\frac{\pi}{2}$ is equal to an central angle of $90°$. Now, we can determine the point on the trigonometric circle for which the corresponding angle is 90°:

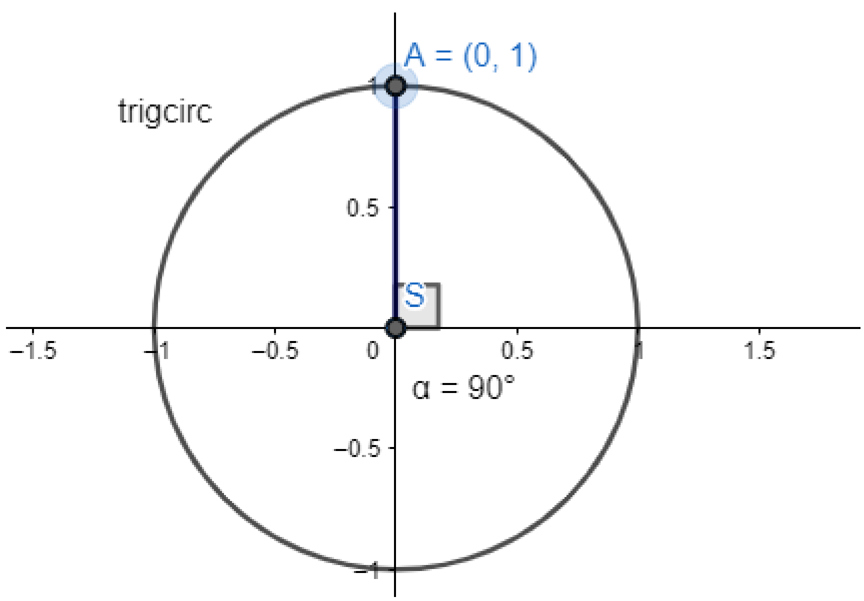

Figure 5.47

The y coordinate of the point represents the sine value, so $\sin\left(\frac{\pi}{2}\right) = 1$.

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Definition: *The cosine function*

The **cosine function** associates each central angle to the x coordinate of the corresponding point on a unit circle.

Example 5.48 Determining the cosine values of angles

Let us determine the value of $\cos(\frac{\pi}{2})$.

Since the radian measure of $\frac{\pi}{2}$ is equal to $90°$, the corresponding point is given in the figure from the previous example.

The x coordinate of the point represents the cosine value, so $\cos\left(\frac{\pi}{2}\right) = 1.$

Definition: *The tangent function*

For every central angle, the tangent function is the quotient of the sine and cosine functions of that angle, $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$.

Definition: The cotangent function

For every central angle, the cotangent function is the reciprocal of the tangent function, $\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{1}{\frac{\sin(\alpha)}{\cos(\alpha)}} = \frac{\cos\,(\alpha)}{\sin\,(\alpha)}$.

Trigonometric functions were historically discovered and defined using the right triangle.

In a right triangle, the trigonometric functions are defined as the following ratio:

$$\sin(\text{angle}) = \frac{\text{length of the opposite leg}}{\text{length of the hypotenuse}}.$$

$$\cos(\text{angle}) = \frac{\text{length of the adjacent leg}}{\text{length of the hypotenuse}}.$$

$$\tan(\text{angle}) = \frac{\text{length of the opposite leg}}{\text{length of the adjacent leg}}.$$

Example 5.49 Determining the sine and cosine values in a right triangle

Let us try to demonstrate that these definitions fit in with the unit circle definitions given above. The image below represents a part of the unit circle, with a point A chosen.

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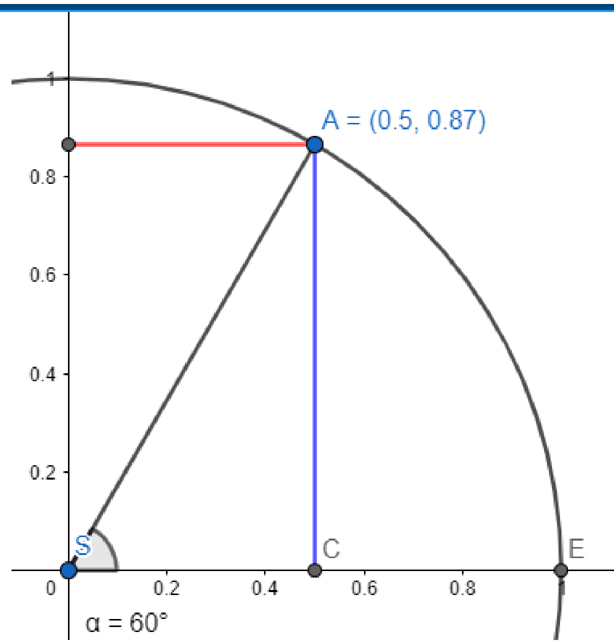

Figure 5.48

Firstly, note that the radius of the circle is 1, so $|SA| = |SE| = 1$.

Second, the triangle SAC is a right triangle, the right angle being in the vertex C.

It is easy to prove now that the triangle ASE is equilateral.

Therefore, the altitude AC bisects the side SE, so $|SC| = \frac{1}{2}$.

Using the right triangle definition, we can calculate the cosine of the angle in vertex A:

$$\cos(\alpha) = \frac{\text{length of the adjacent leg}}{\text{length of the hypotenuse}} = \frac{|SC|}{|SA|} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

By inspecting the $\mathbf{x}$ coordinate of point A, we see that the results are the same. Let us now try to calculate $\sin(\alpha)$. Using the Pythagorean theorem, we can calculate the remaining side, that is the altitude length from point A.

$$1^2 = \left(\frac{1}{2}\right)^2 + a^2, \qquad \text{where } a \text{ is the altitude length}$$

The expression is simplified into:

$$a^2 = \frac{3}{4}$$

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Since the side length cannot be negative, by applying the square root the solution is:

$$a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

Therefore, $|SA| = \frac{\sqrt{3}}{2}$.

Using the right triangle definition, we can now calculate the sine of the angle in vertex A:

$$\cos(\alpha) = \frac{\text{length of the opposite leg}}{\text{length of the hypotenuse}} = \frac{|SA|}{|SA|} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

By inspecting the y coordinate of point A, we see that the value given in the figure is 0.87. This is because the value is only a two decimal approximation of the true value of the point. Hence, $\frac{\sqrt{3}}{2} \approx 0.87$.

Finally, note that the point corresponding to the angle greater than 360° is equivalent to the point represented by the coterminal angle.

This fact allows us to calculate the sine and cosine value of **every** angle, by converting it to a coterminal angle, and then calculating the value of the coterminal angle.

For instance:

$$\cos(450°) = \cos(360° + 90°) = [\text{coterminal angle of } 450° \text{ is } 90°] = \cos(90°) = 0\,.$$

The sine and cosine values of specific points are shown in the following figure:

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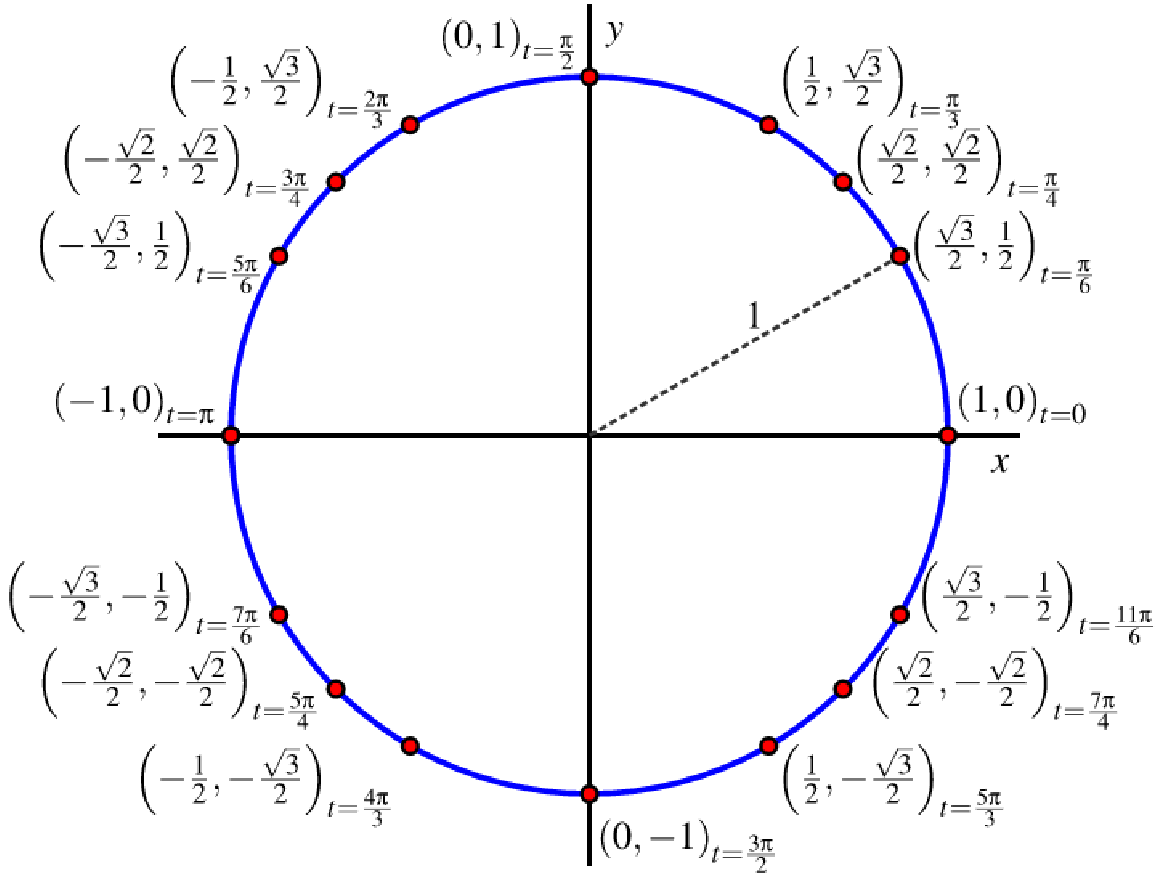

Figure 5.49

Let us rewrite some of the most important values in a table:

α	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin(\alpha)$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
$\cos(\alpha)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\tan(\alpha)$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$UNDEFINED$	0

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Example 5.50 Trigonometric functions in the xy-plane

Since the sine and cosine functions are defined for every angle, and therefore for every radian measure, it is possible to calculate their values for any number.

In the following link, the graphs of the sine and cosine functions are determined and drawn for x values in the interval $[0,12.5]$.

https://www.geogebra.org/m/sfzxqfzp

To see the animation, click on the Play button on the Num slider, or move it manually left/right.

The tangent function can also be shown in green. To show/hide the tangent function click on the green TAN point in the left side panel (scroll down, it is the last point in the panel).

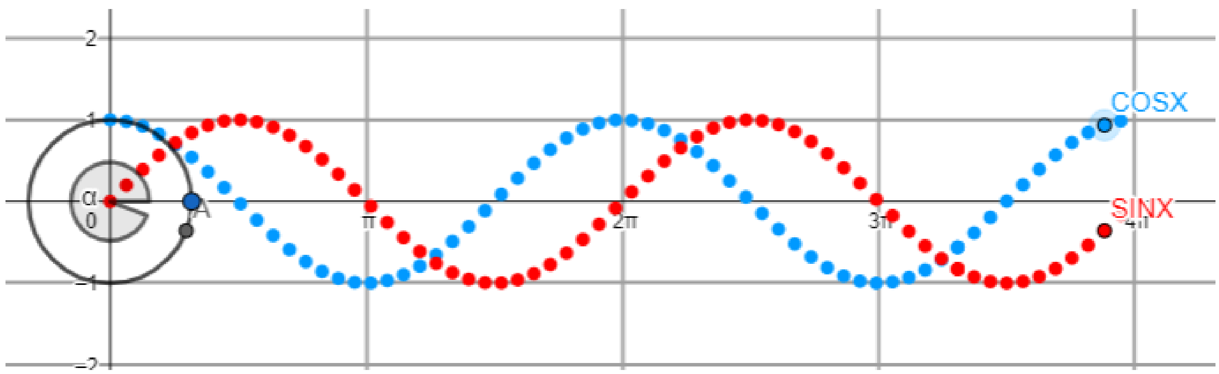

Figure 5.50

We see from the graph that the sine and cosine functions are wave-like functions.

The maximum value of both functions is 1, and the minimum value is -1.

The wave-like property is called **periodicity**, meaning that the sine and cosine functions repeat after a certain number.

In the above graph, it is easy to see that that number is 2π, so the sine and cosine function have a **period of 2π**.

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Excercise 1.

Convert from degree to radian measure or vice versa, and fill in the table:

$\alpha°$	α_{rad}
$60°$	
	$\frac{3\pi}{2}$
$150°$	
	3π
$450°$	
	$\frac{19\pi}{4}$
$-130°$	
	$-\frac{\pi}{6}$
$1000°$	

Solution:

$\alpha°$	α_{rad}
$60°$	$\frac{\pi}{3}$
$270°$	$\frac{3\pi}{2}$
$150°$	$\frac{5\pi}{6}$
$540°$	3π
$450°$	$\frac{5\pi}{2}$
$855°$	$\frac{19\pi}{4}$
$-120°$	$\frac{-\pi}{3}$
$-30°$	$-\frac{\pi}{6}$
$1000°$	$\frac{50\pi}{9}$

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Exercise 2.

Find the coterminal angles and fill in the table:

$\alpha°$	$coterminal(\alpha)$
$390°$	
$450°$	
$1000°$	
$-200°$	
$-1000°$	
$-721°$	

Solution:

$\alpha°$	$coterminal(\alpha)$
$390°$	$30°$
$450°$	$90°$
$1000°$	$280°$
$-200°$	$120°$
$-1000°$	$80°$
$-721°$	$359°$

Exercise 3

Using the following link, fill in the table with approximate values:

https://www.geogebra.org/calculator/tazfcyed

$\alpha°$	α_{rad}	$\sin(\alpha)$	$\cos(\alpha)$
$30°$			
	$\frac{3\pi}{2}$		
		0	1
$135°$			
	$\frac{7\pi}{6}$		

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		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

Solution:

$\alpha°$	α_{rad}	$\sin(\alpha)$	$\cos(\alpha)$
$30°$	$\frac{\pi}{6}$	0.5	0.87
$270°$	$\frac{3\pi}{2}$	-1	0
$0°$	0	0	1
$135°$	$\frac{3\pi}{4}$	0.71	-0.71
$210°$	$\frac{7\pi}{6}$	-0.5	-0.87
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

Exercise 4.

α	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin(\alpha)$	0	½	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
$\cos(\alpha)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\tan(\alpha)$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$UNDEFINED$	0

Using the trigonometric function table and coterminal angles, calculate the values:

α_{rad}	$coterminal(\alpha)$	$\sin(\alpha)$	$\cos(\alpha)$	$\tan(\alpha)$
$\frac{13\pi}{6}$				
6π				
$\frac{109\pi}{4}$				
$\frac{49\pi}{2}$				
$\frac{43\pi}{3}$				

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Solution:

α_{rad}	$coterminal(\alpha)$	$\sin(\alpha)$	$\cos(\alpha)$	$\tan(\alpha)$
$\frac{13\pi}{6}$	$\frac{\pi}{6}$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
6π	0	0	1	0
$\frac{109\pi}{4}$	$\frac{\pi}{4}$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\frac{49\pi}{2}$	$\frac{\pi}{2}$	1	0	$UNDEFINED$
$\frac{43\pi}{3}$	$\frac{\pi}{3}$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$

Exercise 5.

Using the following link, inspect the graph and explain.

https://www.geogebra.org/calculator/baxuhgth

a) Explain the property of parameter A by moving the slider left/right and observing the difference between the two functions? (parameter A is called the **amplitude**) Return the slider to the value A=1.
b) Explain the property of parameter C by moving the slider left/right and observing the difference between the two functions?
(parameter C is called the phase) Return the slider to the value C=0.
c) Explain the property of parameter B by moving the slider left/right and observing the difference between the two functions?
(the number $2\pi/B$ is called the **wavelength**)

Solution:

a) Parameter A determines the maximum and minimum value of the sine function. Therefore, it determines the **amplitude** of the sine function. b) Parameter C determines the left/right **shift** of the sine function. In physics, this shift is called the **phase**. c) Parameter B determines the shrinking/stretching of the sine function. If B=2, the function makes 2 sine waves compared to one sine wave made by the regular sine function. Therefore, the greater the parameter B, the **shorter** is the sine wave, so the wavelength shortens. In physics, the shorter wavelength corresponds to a greater frequency. This is very important in optics, acoustics, etc.

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CONNECTIONS AND APPLICATIONS

Exercise 6.

How to read a radar (why trigonometry matters)?

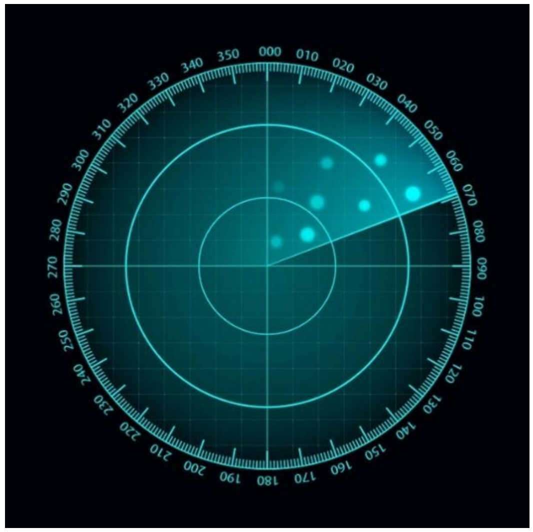

Figure 5.51

The current location of the object is always in the centre of the radar. But what are the funny markings on the side of the radar? In aeronautics, **directions** are determined using the **azimuth.** Azimuth is the angle between the object direction and true north. For example, the azimuth of direction North is 0°, the azimuth of direction West is 270°. Instead of writing angles, the directions on a radar are sets of 3 numbers. For example, the direction of 0° (north) is 000, the direction of East (90°) is 090 (read zero-nine-zero).

What about the circles? Well, all points on the same circle are equally distant from the circle centre, regardless of the direction.

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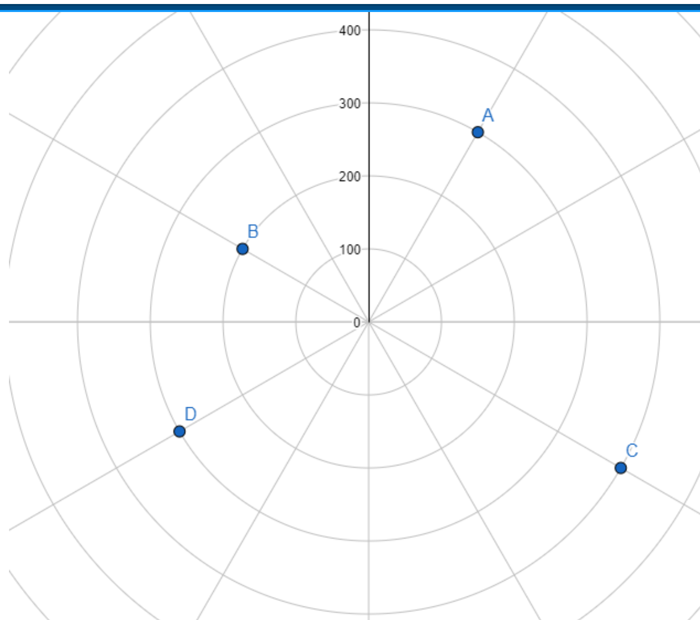

Figure 5.52

The points A and D lie on the same circle, so the distance to the centre is the same.

If a coordinate plane were to be modified so that circles would be drawn from the origin, it would be called a **polar coordinate** system.

Each point in a polar coordinate system has 2 parameters:

1) the distance from the origin (the circle it lies on)

2) the direction (angle) formed by the point and **the positive part of the x axis.**

So, for mathematicians, the direction 0 corresponds to East, the direction North is 90°, the direction West is 180° and the direction South is 270°.

Exercise 7.

A submarine accompanies a ship on its voyage. The position of the submarine and ship are given in the following Geogebra file:

https://www.geogebra.org/calculator/xsmccneq
At each point in time, the ships distance is 200 meters in the direction 060.

a) The submarine captain spots a dangerous reef located directly north from the submarines' location, 200 m away.
   1) Plot the point locating the reef in Geogebra

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2) What is the distance from the ship to the reef?
3) What would be the reefs direction when observed from the ship?

b) The submarine captain spots a potential threat from direction 120, 200 m away from the submarine.
1) Plot the threat location
2) Measure or calculate the threats distance from the ship.

c) The ships safe harbour is located 400 m away, in the direction 330 from the current submarine location.
1) Plot the harbour location
2) Determine the course of the boat if it wants to reach the harbour
3) Determine the distance from the boat to the harbour

Solution:

The complete soliton is shown in the following Geogebra app.

https://www.geogebra.org/m/w7yspzpn

a) To solve this part of the task, plot the reef as a point with coordinates (0,200).
In the app, click on the circle next to the Reef object.
To calculate the distance from the ship, in the menu select distance between points and measure the distance from the ship too the reef.

If you are working with the app, click on the point next to the text object indicating the distance from Ship to Reef.
To calculate the direction of the reef from the ship, draw a line through the points indicating the ship and reef. Then, plot a point directly north from the ship (the x coordinate must remain fixed), and label it North. Lastly, using the Angle tool in Geogebra, measure the angle measure with the first endpoint in point North, the vertex located in the ships position, and the endpoint is the reef.

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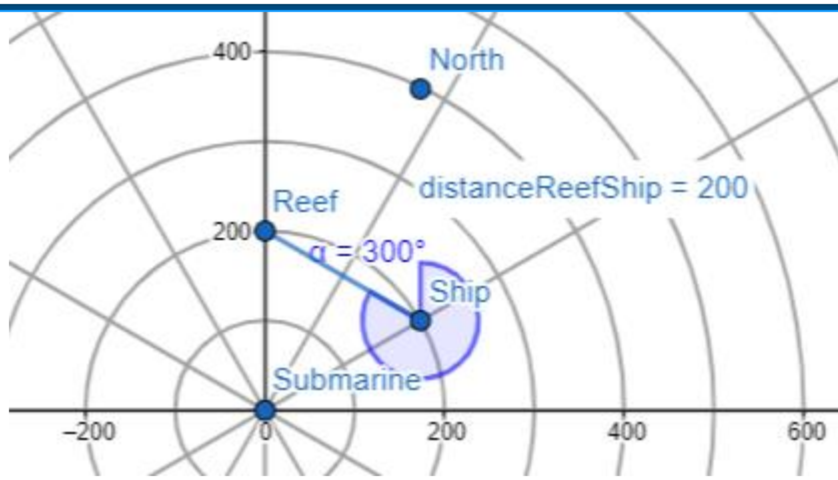

Figure 5.53

If you are using the solution, click on the corresponding objects to make them visible. The objects are the line segment f, the point North and angle α.
The required angle is 300°.

b) To plot the threat location, follow the line corresponding to the angle of 120° (the first line in the lower right quadrant of the coordinate plane. Then, using the circle indicating 200 m, plot the threat as a point in the intersection of the line and the circle, like shown in the picture below.

Lastly, measure the distance between the Submarine and threat using the measuring tool in Geogebra. The solution should look like this:

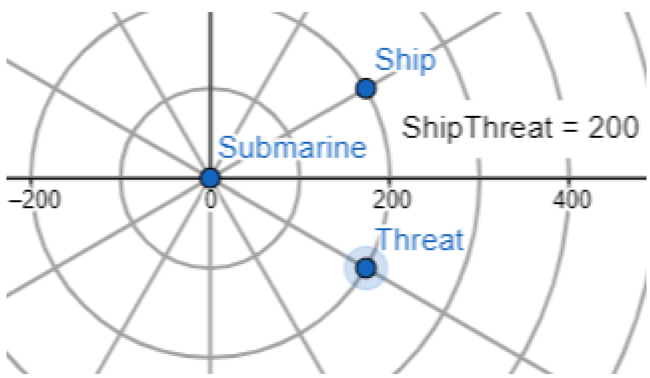

Figure 5.54

If you are using the provided solution, just activate the corresponding objects (point Threat and the text showing the distance)

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c) To indicate the harbour, use the line corresponding to the direction of 330, which is the first line in the upper left quadrant of the coordinate plane. Plot the harbour at the intersection of the line and the circle indicating the distance of 400 m. Using a measuring tool in Geogebra, measure the distance from the ship to the harbour. Using the angle tool in Geogebra, measure the angle from the point North through the vertex Ship and the endpoint Harbour.
The solution is shown below:

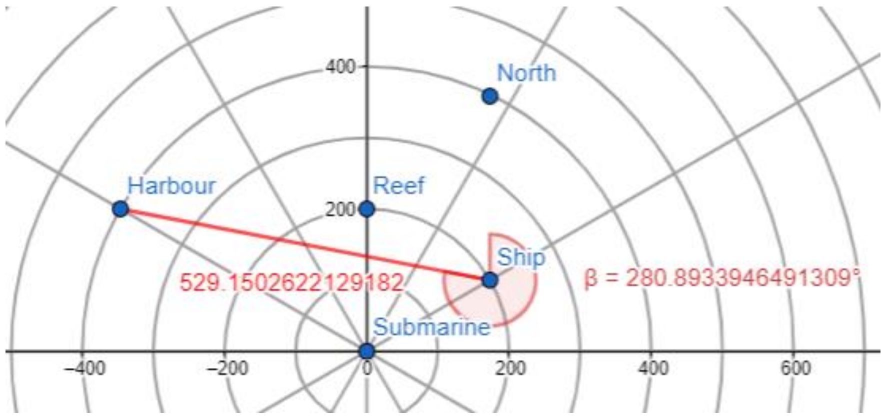

Figure 5.55

If you are using the solution app, just highlight the appropriate objects in the left pane.

Exercise 8.

The following link shows part of the Split sea area, with islands which are close by. https://www.geogebra.org/calculator/bznfax2e

Your ship is located in the Rogac port on the island of Solta.

a) Determine the direction and distance from the Rogac port to Split
b) Determine the direction and distance from the Rogac port to Supetar (look to the west).
c) c A distress signal is sent from point Signal. Using the Geogebra measuring tool, inspect which of the three harbours Split, Rogac or Supetar is the nearest to the location Signal.

Solution:

The complete solution is shown in the following app:

https://www.geogebra.org/calculator/qdbagzsw

a) To determine the direction, first choose a point lying north of port Rogac. The point must have the same x coordinate as port Rogac. To find the distance from Rogac to Split, use the measuring tool provided in Geogebra. To find the direction, use the angle tool in Geogebra,

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with the start point being Split, the vertex being Rogac and the endpoint being North. The final solution should look like this:

b)

Figure 5.56

If the solution app is used, click on the corresponding elements in the left pane to hide/show them on the map. The elements for the first part of the solution are point North, distance from Rogac to Split, and angle α. The angle of 44° corresponds to a direction of 044.

c) The direction of the Supetar port is almost 090 since it is west of port Rogac. To find the exact direction, plot the port Supetar with a point, using the point Supetar as the starting point, the point Rogac as the vertex, and the point North as the endpoint, draw an angle using the angle tool in Geogebra. The angle is 92.7°, so the direction is approximately 093. To determine the distance, draw a line segment from port Rogac to port Supetar, and measure its distance. The distance from Rogac to Supetar should be approximately 19.5 km. Another way to approximate the point Supetar is to use the radar circles. The point Supetar is between the circle indicating the distance of 18 km and the circle indicating the distance of 20 km, so the true distance must be between 18 and 20 km.

The final solution should look like this:

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Figure 5.57

If you are using the solution app, just highlight the point Supetar, the line segment with the distance, and angle β.

d) To find the shortest route from the port to point Signal, make a line segment and measure the distance from the port to Signal for all three ports using the Geogebra measuring tool. The solution should look like this:

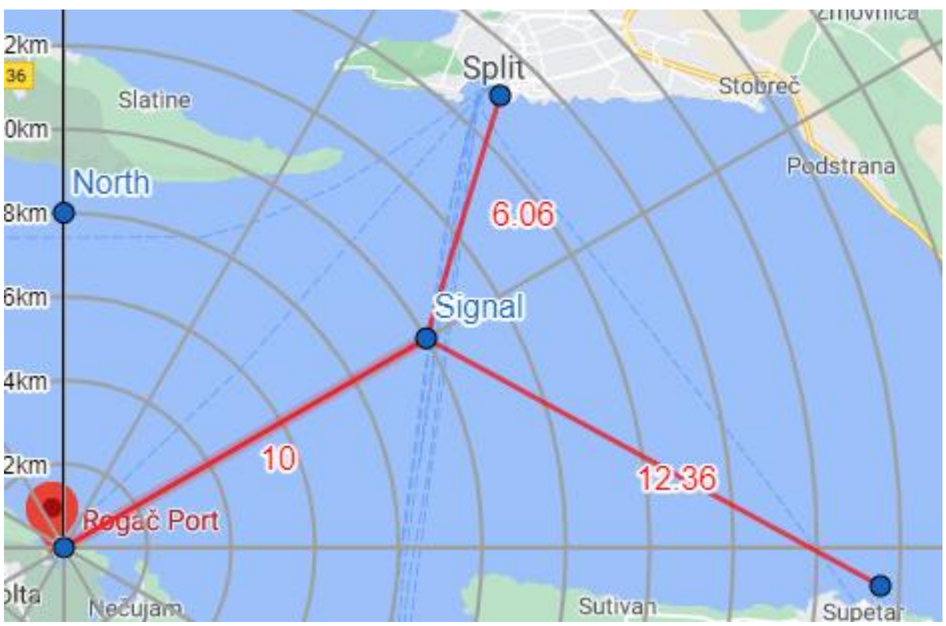

Figure 5.58

So, the nearest port is Split, so the rescue ship should be sent from Split. If you are using the solution app, show the corresponding elements.

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5.6.2. Equations

Basic equations

Suppose we want to know for which angles the sine function is equal to 0.5, one would write the equation $\sin(x) = 0.5$.

This represents a trigonometric equation.

There are two ways to solve a trigonometric equation, algebraically and graphically. First, let us demonstrate the algebraic way:

$$\sin(x) = 0.5$$

When calculating the angle, the **inverse trigonometric functions** or **arcus functions** are used. Without going into too much detail, an arcus function is also called an inverse because it eliminates the trigonometric term on the left hand side. We will apply the inverse sine function ($arcsin$, or $\sin^{-1}$) on both sides of the equation:

$$\arcsin\,(\sin(x)) = \arcsin\,(0.5)$$

The inverse trigonometric function cancels out the trigonometric function, so the equation is now:

$$x = \arcsin(0.5)$$

The value of $\arcsin(0.5)$ can be calculated using a calculator, which will provide the solution:

$$x = 30°$$

But is this the only solution?

Let us next examine the graphical method:

We will graph each side of the equation separately and look for intersections of the graphs.

Let us denote $f(x) = \sin(x)$ and $g(x) = 0.5$

Using Geogebra, we can easily plot the functions, which results in the following graph:

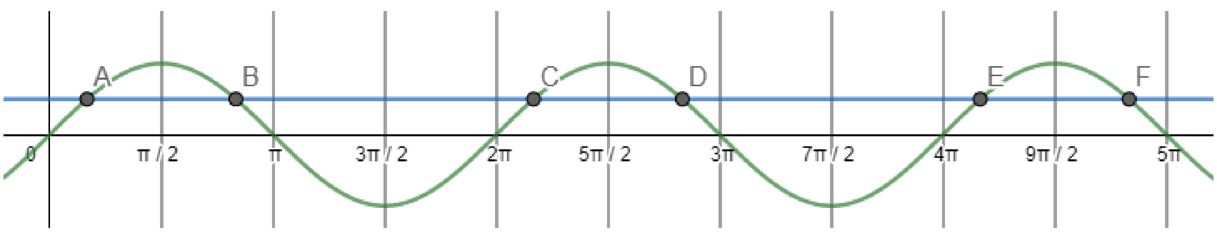

Figure 5.59

By observing the graph, we see that there are at least 6 different intersection points between the graphs, so there must be at least 6 solutions of the trigonometric equation.

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Furthermore, we see that the solutions of the equation are repeating periodically in every sine "wave".

Let us also recall that the sine of a number is the y coordinate on the point on the unit circle.

Using the trigonometric unit circle, we can plot the line $y = 0.5$ and obtain all points that intersect the circle, like in the picture below:

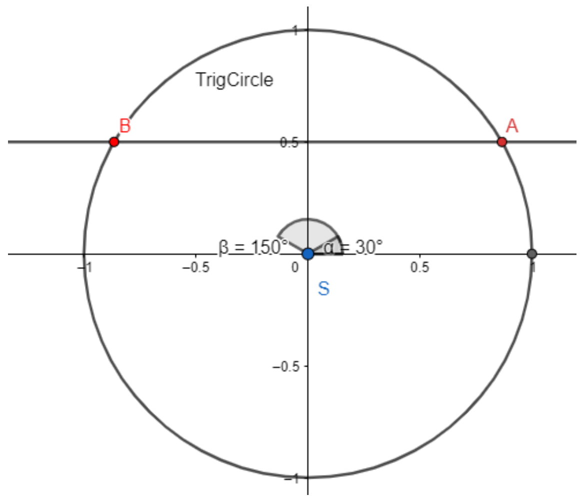

Figure 5.60

Hence, we see that the equation $\sin(x) = 0.5$ has two solutions in the interval from 0 to 2π. One solution is our calculated value $x = 30°$, but another solution is $x = 150°$. All other solutions repeat periodically, with each new "wave" of the sine function. So, the solutions of the equation are also $x = 390°, x = 510°, \text{etc.}$
All solutions of the equation can be obtained by adding a multiple of $360°$ to the two calculated solution ($30°$ and $150°$).

The general solutions can be written as $x = 30° + k \cdot 360°$, or $x = 150° + k \cdot 360°$, where k represents any integer.

The solutions of trigonometric equations are generally preferred in radians. We leave the conversion as an exercise for the students, and write the solutions in radian form:

$$x_1 = \frac{\pi}{6} + 2k\pi, \quad x_2 = \frac{5\pi}{6} + 2k\pi$$

The procedure for solving simple trigonometric equations can be summarized as follows:

1) Find the first value by calculating the arcus function
2) Graph the trigonometric circle and read the second solution or
   Use the following identities:

$$\sin(\pi - x) = \sin(x)$$
$$\cos(2\pi - x) = \cos(x)$$
$$\tan(\pi + x) = \tan(x)$$

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to obtain the second solution

3) Write all remaining solutions using the period of sine/cosine (2π) or tangent (π).

Quadratic trigonometric equations:

Let us briefly explain how to solve the quadratic trigonometric equations in the following form:

$$2\,(\sin(x))^2 + 3\sin(x) - 2 = 0$$

When solving trigonometric equations including quadratic terms, first try substituting the term with tIn this case, $t = \sin(x)$.

The equation is then transformed into:

$$2t^2 + 3t - 2 = 0$$

We can now solve this quadratic equation by factoring:

$$2\left(t - \frac{1}{2}\right)(t + 2) = 0$$

The solutions of the equation above are:

$$t_1 = \frac{1}{2}, t_2 = -2$$

Returning the substituted value, we obtain:

$$\sin(x) = \frac{1}{2}, \qquad \sin(x) = -2$$

Since the sine function values are always between -1 and 1, the equation $\sin(x) = -2$ has no solution.

Therefore, the solutions of the starting equation are solutions of the equation $\sin(x) = \frac{1}{2}$. This is a simple trigonometric equation solved above.

The final solutions are $x_1 = \frac{\pi}{6} + 2k\pi, \quad x_2 = \frac{5\pi}{6} + 2k\pi, k \in Z$.

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Complex trigonometric equations:

Some trigonometric equations cannot be solved algebraically, but the number of solutions can be obtained graphically.

One such equation is $\sin(x) = 0.5x$

To count the number of solutions, graph each side of the equation above separately using Geogebra:

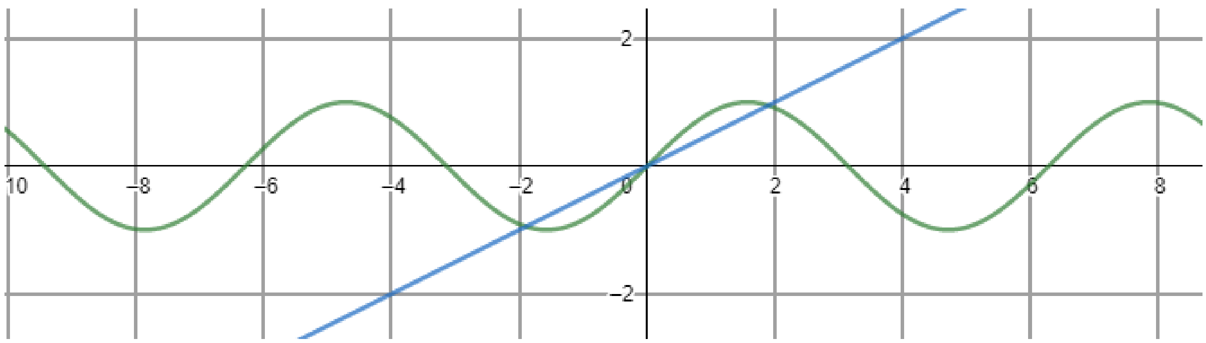

Figure 5.61

The left side is the sine function represented by the green graph, and the right side is the blue line.

From the graph it is obvious that there are 3 solutions of the equation.

Furthermore, in Geogebra, we can find the solutions by using the Intersection tool from the left side panel. This procedure gives us the approximate coordinates of the solutions.

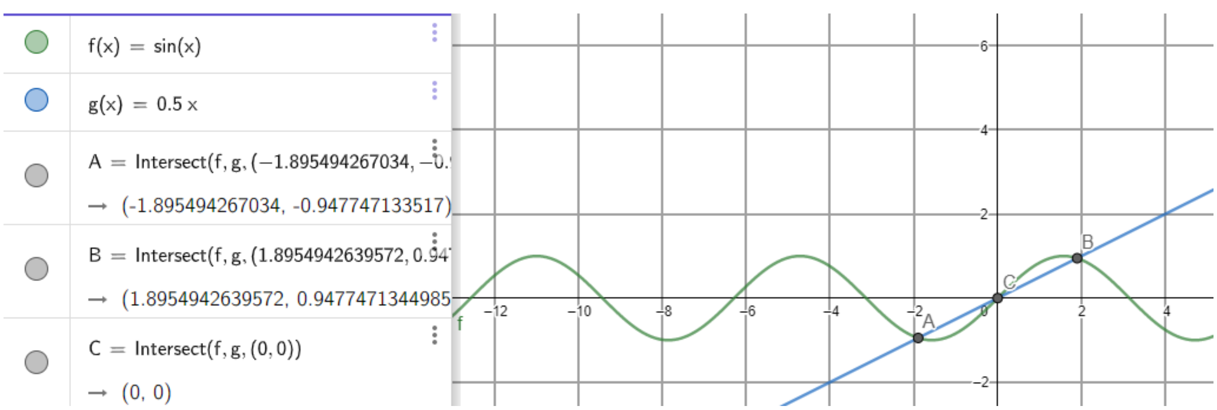

Figure 5.62

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Task 5.35 Calculate all solutions of the following equations:

a) $\cos(x) = -\frac{\sqrt{3}}{2}$
b) $\sin(x) = 0$
c) $\cos(x) = -1$
d) $\tan(x) = 1$

Solution:

a) $x_1 = \frac{5\pi}{6} + 2k\pi,\ x_2 = \frac{7\pi}{6} + 2k\pi, k \in Z$
b) $x_1 = 0 + 2k\pi, x_2 = \pi + 2k\pi, k \in Z$
c) $x = \pi + 2k\pi, k \in Z$
d) $x = \frac{\pi}{4} + k\pi,\ k \in Z$

Task 5.36 .

Calculate all solutions of the following equations:

a) $\sin(x)^2 - 7x + 3 = 0$
b) $(\cos(x) - 5)(\cos(x) + 1) = 0$
c) $(\tan(x) - 1)(\tan(x)) = 0$

Solution:

a) $x_1 = \frac{5\pi}{6} + 2k\pi,\ x_2 = \frac{7\pi}{6} + 2k\pi, k \in Z$
b) $x_1 = 0 + 2k\pi, x_2 = \pi + 2k\pi, k \in Z$
c) $x = \pi + 2k\pi, k \in Z$

Task 5.37 Graph the equation, and determine the number of solutions:

a) $\sin(x) = x - 3$
b) $\cos(\mathrm{x}) = \mathrm{x}^2$
c) $\sin(x) = \frac{x}{12}$

Solution:

a) 1 solution
b) 2 solutions
c) 6 solutions

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5.6.3. Inequalities

First, remember that the solution set of an inequality is almost always an interval. Let us demonstrate the technique using the following inequality:

$$\sin(x) \geq \frac{1}{2}$$

The first step in solving trigonometric inequality is to solve the corresponding equation:

$$\sin(x) = \frac{1}{2}$$

The solving process of that equation was shown in the previous chapter.

The two found solutions were $x_1 = \frac{\pi}{6} + 2k\pi$, and $x_2 = \frac{5\pi}{6} + 2k\pi,\ k \in Z$. To find all the remaining points which satisfy the inequality, we draw the trigonometric circle, and find all points such that the y coordinate is greater than 0.5.

The red interval in the figure below represents the solution set:

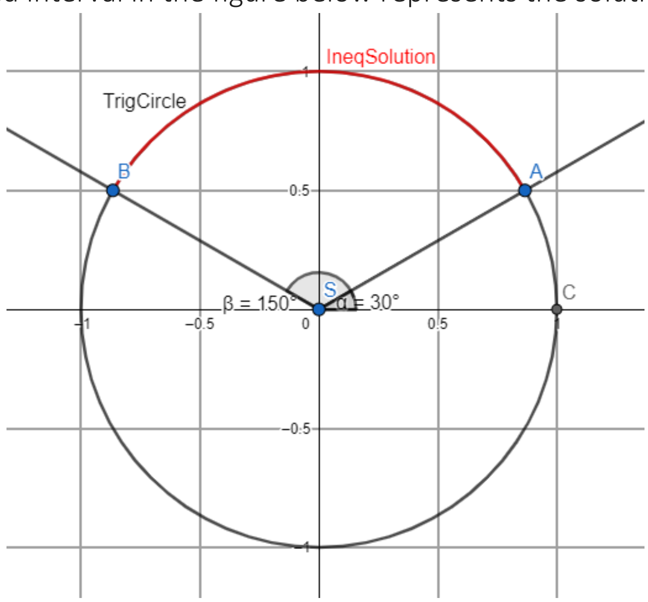

Figure 5.63

From the figure, we can see that all points between $\frac{\pi}{6}$, and $\frac{5\pi}{6}$ are included in the interval, including the endpoints. All the other solutions set are periodical in nature, repeating after 2π. Therefore, the solution of the inequality is:

$$x \in \left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right], k \in Z$$

Trigonometric equations and inequalities appear in many everyday tasks, some of which we will demonstrate in the following section.

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5.6.4. CONNECTIONS AND APPLICATIONS

Example 1:

The wave height in a bay t hours after midnight is given by the formula $f(t) = 1.7\sin\left(\frac{t}{1.9} - 3\right) + 2$. The function is graphed in the following link:

https://www.geogebra.org/calculator/cxdvxwcf

The ship can set sail only if the wave height is below 0.6 m.

a) Determine all time intervals in which the ship can exit the bay. (Help: plot the line representing the height required, then read the intersections).

b) Determine all moments in time for which the height is the largest.

c) Determine all moments when the wave height is 2 m.

Solution:

The complete solution is in the following app:

https://www.geogebra.org/calculator/uj4kcmq4

a) The line $h(x) = 0.6$ represents the required wave height of 0.6 m. To find all intervals in which the ship can exit the bay, find all the intersection points of the graph and the line.
The solutions should look like this:

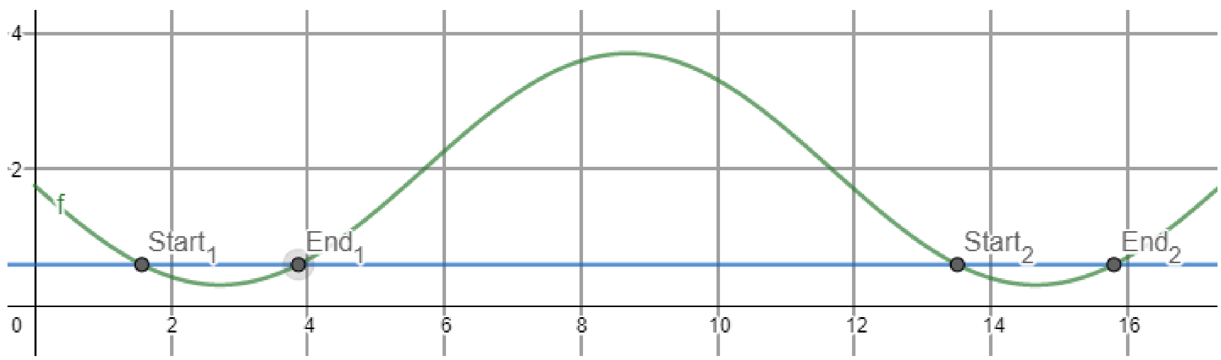

Figure 5.64

The corresponding intervals are from Start$_1$ to End$_1$, and from Start$_2$ to End$_2$, approximately from 1:40 to 4:00 and 13:45 – 15:50. If you are using the solution app, just show the corresponding elements (Start and end points, and the line).

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b) To determine the maximum height, use the Extremum tool in Geogebra from the left side panel. The solutions are:

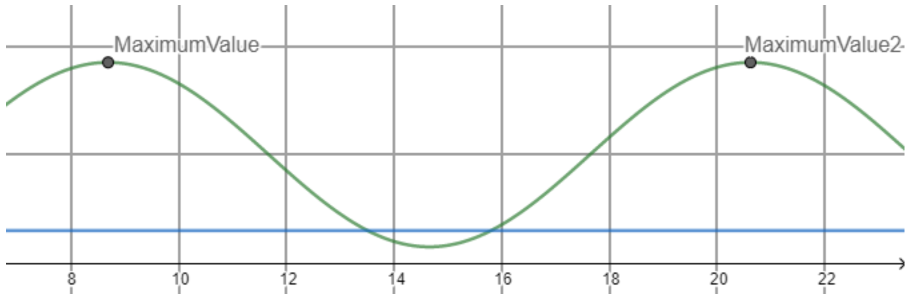

Figure 5.65

To find the exact value, read the x coordinate of the points. The maximum height occurs around 8:20 and 20:20. If you are using the solution app, just show the points named MaximumValue.

c) To find the required points, draw the line $g(x) = 2$ and find the intersection of the function and line using the Geogebra Intersection tool. The solution should look like this:

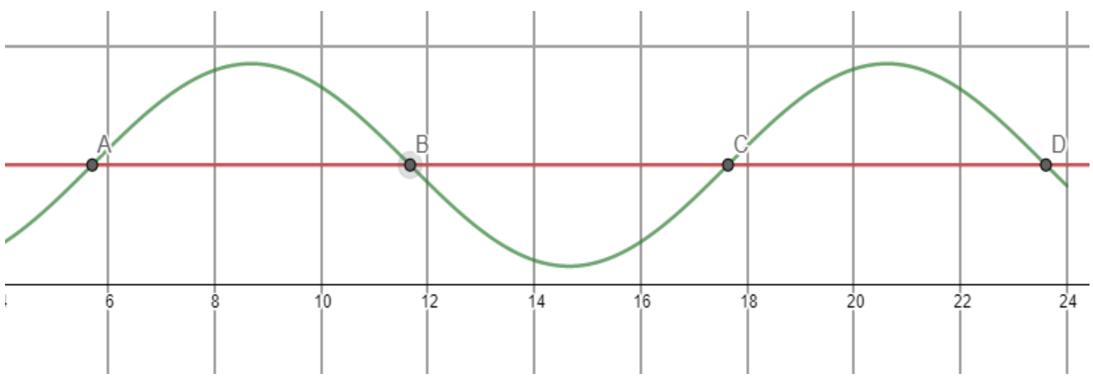

Figure 5.66

If you are using the solution app, highlight the corresponding elements (the line and the intersection point named A,B,C and D).

Example 2:

A ship is nearing a harbor. The ship's captain has been informed that only 25% of the docking surface is free. The distance of the boat to the harbor is 1000 m. The harbor has a surface of 1000 m. The following link models the situation:

https://www.geogebra.org/calculator/mutqk7p4

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a) Determine the course interval to successfully dock the ship

b) On the docking surface, there are 4 berths. Find the ships course for it to dock the fourth berth.

Solution:

The full solution can be viewed in the following app:

https://www.geogebra.org/calculator/qpf4hcqp

a) To calculate the appropriate course, the start point and the endpoint of the free space on the docking surface must be plotted. Since only 25% of the docking surface is free, the start point has coordinates (250,1000), and the endpoint has coordinates (500,1000). To find the direction an auxiliary object, point North, must be plotted. The point must be located directly north from the ship position. To find minimal valid direction, measure the angle from the starting point of the free part of the dock (point C), through the vertex at point Ship and the end point at point North. To find the maximal valid direction, measure the angle using the end point of the dock (point B) as the starting point, point Ship as the vertex and the end point at point North.

The solution should look like this:

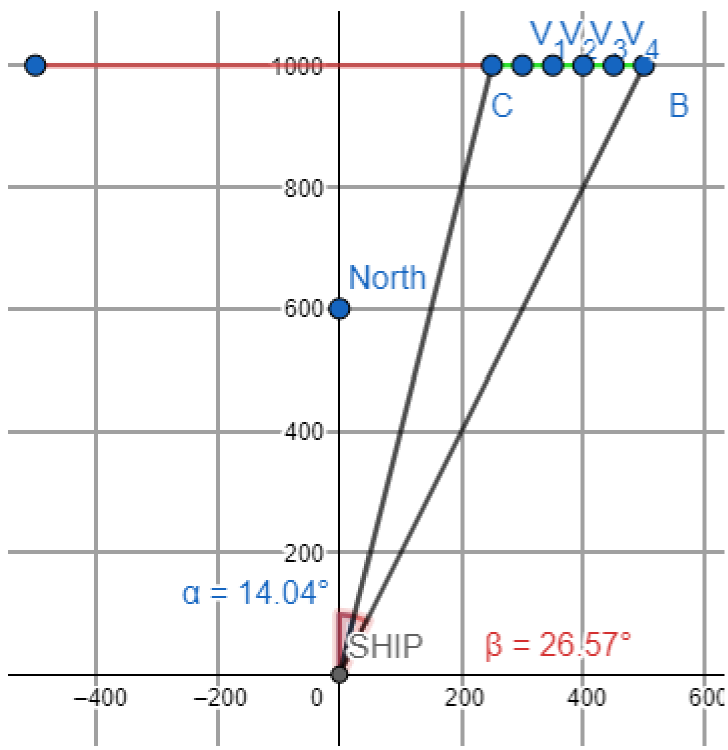

Figure 5.67

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The minimal required direction is 014, and the maximal is 026 (or 027).

If you are using the solution app, show the corresponding elements, line segments between point ship and points B and C, and the corresponding angles.

b) The fourth berth is located at point V_4. To find the direction, plot the line segment from point Ship to point V_4 and find the angle measure using the point V_4 as the starting point, the point Ship as the vertex, and the point North as the endpoint.
The solution should look like this:

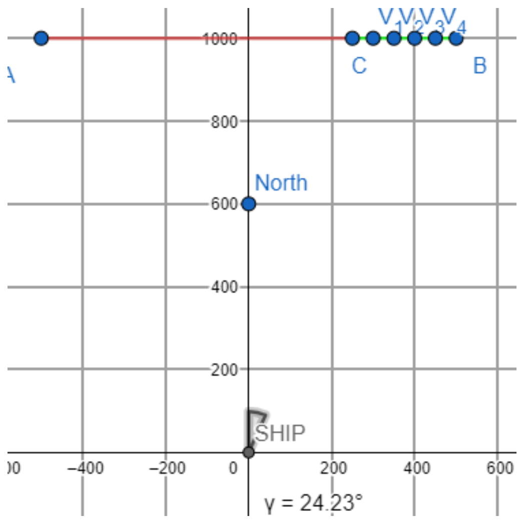

Figure 5.68

The required direction is 024.

If you are using the solution app, highlight the corresponding elements, angle γ.

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5.7. INVERSE TRIGONOMETRIC FUNCTIONS

It is possible to calculate the angle of the list of the ship using trigonometry; also used in marine navigation astronomy and GPS navigation systems, electrical engineering ...

DETAILED DESCRIPTION:

Inverse trigonometric functions are used in solving trigonometric equations that arise in finding the angles and sides of triangle. The inverse of any function is important - it provides a way to "get back."

AIM:

The students will learn how to interpret and graph an inverse trig. Function and will also learn to solve for an equation with an inverse function.

Learning Outcomes:

1. Understand and use the inverse sine function.
2. Understand and use the inverse cosine function.
3. Understand and use the inverse tangent function.
4. Use a calculator to evaluate inverse trigonometric functions.
5. Find exact values of composite functions with inverse trigonometric functions

Prior knowledge:

If no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function.

If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of the inverse function, denoted f^{-1}. The graph of f^{-1} is a reflection of the graph of about the line y = x.

Relationship to real maritime problems:

Contents:

The Inverse Sine Function

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The Inverse Cosine Function

The Inverse Tangent Function

Composition of Functions Involving Inverse Trigonometric Functions

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5.7.1. The inverse sine function

Figure 5.69 shows the graph of $y = \sin x$. Can we see that every horizontal line that can be drawn between -1 and 1 intersects the graph infinitely many times? Thus, the sine function is not one-to-one and has no inverse function.

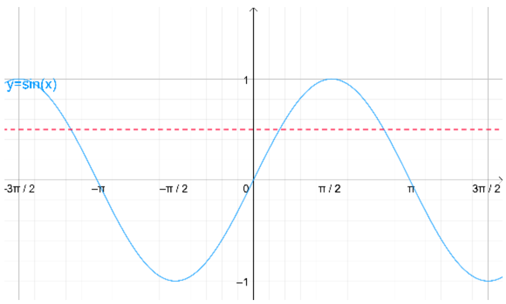

Figure 5.69 The horizontal line test shows that the sine function is not one-to-one and has no inverse function.

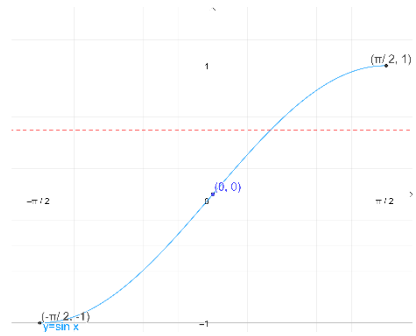

Figure 5.70 The restricted sine function passes the horizontal line test. It is one-to-one and has an inverse function.

In ***Figure 5.70***, we have taken a portion of the sine curve, restricting the domain of the sine function to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. With this restricted domain, every horizontal line that can be drawn between -1 and 1 intersects the graph exactly once. Thus, the restricted function passes the horizontal line test and is one-to-one.

On the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, y = \sin x$ has an inverse function.
The inverse of the restricted sine function is called the inverse sine function. Two notations are commonly used to denote the inverse sine function:

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

We will use $y = \arcsin x$.

Definition:

The inverse sine function, denoted by $\arcsin x$, is the inverse of the restricted sine function $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Thus,

$$y = \arcsin x \quad \text{means} \quad \sin y = x,$$

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-1 \leq x \leq 1$. We read $y = \arcsin x$ as "y equals the inverse sine at x."

One way to graph $y = \arcsin x$ is to take points on the graph of the restricted sine function and reverse the order of the coordinates. For example ***Figure 5.71*** shows that $\left(-\frac{\pi}{2}; -1\right)$, $(0; 0)$ and $\left(\frac{\pi}{2}; 1\right)$ are on the graph of the restricted sine function. Reversing the order of the cordinates gives $\left(1;\ -\frac{\pi}{2}\right)$,

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$(0;0)$ and $\left(1;\frac{\pi}{2}\right)$. We now use these three point to sketch the inverse sine function. The graph of $y=\arcsin x$ is shown in ***Figure 5.72***.

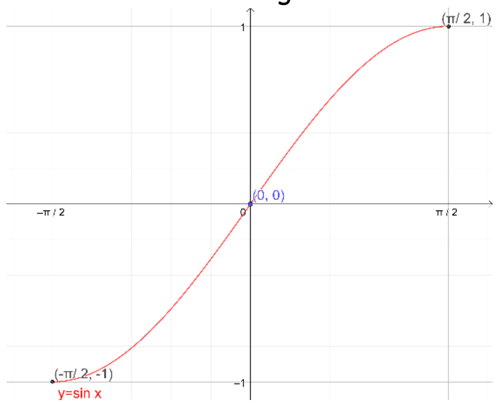

Figure 5.71 The restricted sine function, Domain: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$*, Range:* $[-\mathbf{1},\mathbf{1}]$

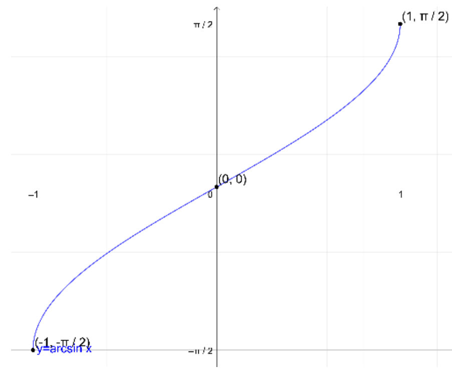

Figure 5.72 The graph of the inverse sine function, Domain $[-\mathbf{1},\mathbf{1}]$*, Range:* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

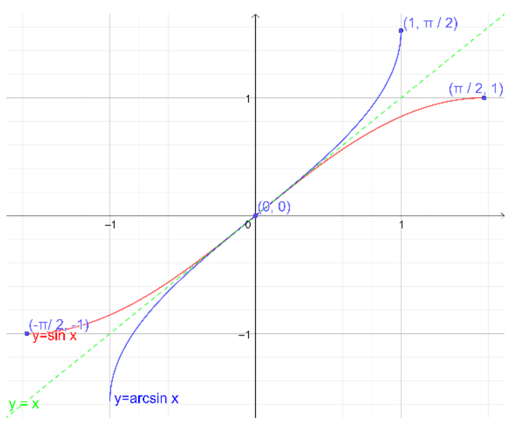

Figure 5.73 Using a reflection to obtain the graph of the inverse sine function.

Another way to obtain the graph o $y=\arcsin x$f is to reflect the graph of the restricted sine function about the line $y=x$, shown in Figure 5.The red graph is the restricted sine function and the blue graph is the graph of $y=\arcsin x$. Exact values of $\arcsin x$ can be found by thinking of $\arcsin x$ as the angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is x. For example, we can use the two points on the blue graph of the inverse sine function in Figure 5.73 to write

$$\arcsin(-1)=-\frac{\pi}{2};\ \arcsin(1)=\frac{\pi}{2}.$$

Because we are thinking of $\arcsin x$ in terms of an angle, we will represent such an angle by φ.

FINDING EXACT VALUES OF $\boldsymbol{ARCSIN\ x}$

1. Let $\boldsymbol{\arcsin x=\varphi}$.
2. Rewrite $\boldsymbol{\arcsin x=\varphi}$ as $\boldsymbol{\sin\varphi=x}$, where $\boldsymbol{\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]}$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\boldsymbol{\sin\varphi=x}$.

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φ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\varphi$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1 Exact values for $\sin\varphi$, $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$

Example 5.51 Finding the exact value of an inverse sine function

Find the exact value of $\arcsin\frac{\sqrt{2}}{2}$.

Solution:

1. Let $\boldsymbol{\arcsin x=\varphi}$. Thus, $\arcsin\frac{\sqrt{2}}{2}=\varphi$. We must find the angle $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, whose sine equals $\frac{\sqrt{2}}{2}$.
2. Rewrite $\boldsymbol{\arcsin x=\varphi}$ as $\boldsymbol{\sin\varphi=x}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Using the definition of the inverse sine function, we rewrite $\arcsin\frac{\sqrt{2}}{2}=\varphi$ as $\sin\varphi=\frac{\sqrt{2}}{2}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\boldsymbol{\sin\varphi=x}$. Table 1 shows that the only angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\varphi=\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Thus, $\varphi=\frac{\pi}{4}$. Because φ in step 1, represents $\arcsin\frac{\sqrt{2}}{2}$, we conclude that
$$\arcsin\frac{\sqrt{2}}{2}=\frac{\pi}{4}.$$

Example 5.52 Finding the exact value of an inverse sine function

Find the exact value of $\arcsin\left(-\frac{1}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arcsin x=\varphi}$. Thus, $\arcsin\left(-\frac{1}{2}\right)=\varphi$. We must find the angle $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, whose sine equals $\left(-\frac{1}{2}\right)$.
2. Rewrite $\boldsymbol{\arcsin x=\varphi}$ as $\boldsymbol{\sin\varphi=x}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Using the definition of the inverse sine function, we rewrite $\arcsin\left(-\frac{1}{2}\right)=\varphi$ as $\sin\varphi=-\frac{1}{2}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\boldsymbol{\sin\varphi=x}$. Table 1 shows that the only angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\varphi=-\frac{1}{2}$ is $\left(-\frac{\pi}{6}\right)$. Thus, $\varphi=-\frac{\pi}{6}$. Because φ in step 1, represents $\arcsin\left(-\frac{1}{2}\right)$, we conclude that
$$\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}.$$

NB! Some inverse sine expressions cannot be evaluated. Because the domain of the inverse sine function is $[-1;1]$ it is only possible to evaluate for values of x in this domain. Thus, $\arcsin(3)$ cannot be evaluated. There is no angle whose sine is 3.

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5.7.2. The inverse cosine function

Figure 5.74 shows how we restrict the domain of the cosine function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $[0;\pi]$, shown by the light green graph. Over this interval, the restricted cosine function passes the horizontal line test and has an inverse function.

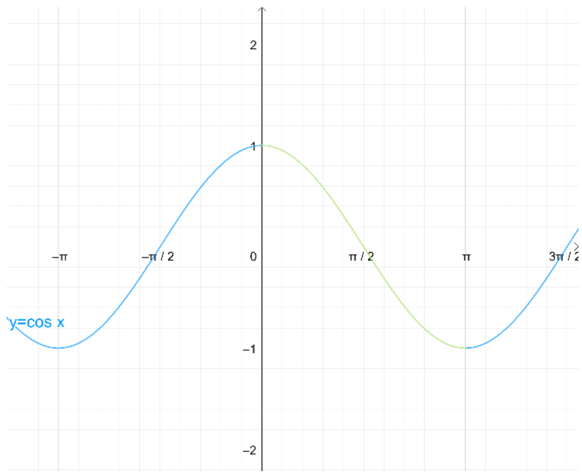

Figure 5.74 y=cos x is one-to one on interval $(\mathbf{0};\ \boldsymbol{\pi})$

Definition: The inverse cosine function, denoted by $\arccos x$, is the inverse of the restricted cosine function $y=\cos x$, $0\le x\le \pi$. Thus,

$$y=\arccos x \quad \text{means} \quad \cos y = x,$$

where $0\le y\le \pi$ and $-1\le x\le 1$.

One way to graph $y=\arccos x$ is to take points on the graph of the restricted cosine function and reverse the order of the coordinates. For example, *Figure 5.75* shows that $(0,1)$, $\left(\frac{\pi}{2},0\right)$ and $(\pi,-1)$ are on the graph of the restricted cosine function. Reversing the order of the coordinates gives $(1,0)$, $\left(0,\frac{\pi}{2}\right)$ and $(-1,\pi)$.

We now use these three points to sketch the inverse cosine function. The graph of $y=\arccos x$ is shown in *Figure 5.76*. You can also obtain this graph by reflecting the graph of the restricted cosine function about the line $y = x$.

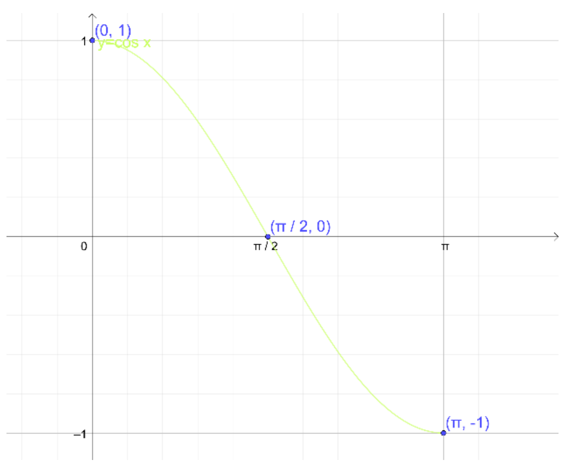

Figure 5.75 The restricted cosine function, Domain: $[\mathbf{0};\ \boldsymbol{\pi}]$*, Range:* $[-\mathbf{1},\mathbf{1}]$.

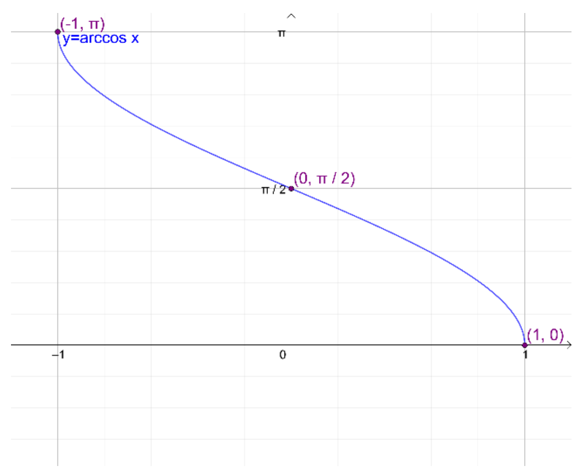

Figure 5.76 The graph of the inverse cosine function

Exact values of $\boldsymbol{y}=\mathbf{arccos}\,\boldsymbol{x}$ can be found by thinking of $\mathbf{arccos}\,\boldsymbol{x}$ as the angle in the interval $[\mathbf{0},\boldsymbol{\pi}]$ whose cosine is $\boldsymbol{x}$.

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FINDING EXACT VALUES OF $\boldsymbol{ARCCOS\,x}$

1. Let $\mathbf{arccos}\,\boldsymbol{x} = \boldsymbol{\varphi}$.
2. Rewrite $\mathbf{arccos}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{cos}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\boldsymbol{\varphi\epsilon[0;\pi]}$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[\mathbf{0};\boldsymbol{\pi}]$ that satisfies $\mathbf{cos}\,\boldsymbol{\varphi} = \boldsymbol{x}$.

$\boldsymbol{\varphi}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\mathbf{cos}\,\boldsymbol{\varphi}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 2 Exact values for $\cos\varphi$, $\varphi\epsilon[0;\pi]$.

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Example 5.53 Finding the exact value of an inverse cosine function

Find the exact value of $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arccos x = \varphi}$. Thus, $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \varphi$. We must find the angle $\varphi\epsilon[0;\pi]$, whose cosine equals $-\frac{\sqrt{3}}{2}$.
2. Rewrite $\boldsymbol{\arccos x = \varphi}$ as $\boldsymbol{\cos\varphi = x}$, where $\varphi\epsilon[0;\pi]$. Using the definition of the inverse cosine function, we rewrite $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \varphi$ as $\cos\varphi = -\frac{\sqrt{3}}{2}$, where $\varphi\epsilon[0;\pi]$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[0;\pi]$ that satisfies $\boldsymbol{\cos\varphi = x}$. Table 2 shows that the only angle in the interval $[0;\pi]$ that satisfies $\cos\varphi = -\frac{\sqrt{3}}{2}$ is $\frac{5\pi}{6}$. Thus, $\varphi = \frac{5\pi}{6}$. Because φ in step 1, represents $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, we conclude that
$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Example 5.54 Finding the exact value of an inverse cosine function

Find the exact value of $\arccos\left(-\frac{1}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arccos x = \varphi}$. Thus, $\arccos\left(-\frac{1}{2}\right) = \varphi$. We must find the angle $\varphi\epsilon[0;\pi]$, whose cosine equals $-\frac{1}{2}$.
2. Rewrite $\boldsymbol{\arccos x = \varphi}$ as $\boldsymbol{\cos\varphi = x}$, where $\varphi\epsilon[0;\pi]$. Using the definition of the inverse cosine function, we rewrite $\arccos\left(-\frac{1}{2}\right) = \varphi$ as $\cos\varphi = -\frac{1}{2}$, where $\varphi\epsilon[0;\pi]$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[0;\pi]$ that satisfies $\boldsymbol{\cos\varphi = x}$. Table 2 shows that the only angle in the interval $[0;\pi]$ that satisfies $\cos\varphi = -\frac{1}{2}$ is $\frac{2\pi}{3}$. Thus, $\varphi = \frac{2\pi}{3}$. Because φ in step 1, represents $\arccos\left(-\frac{1}{2}\right)$, we conclude that
$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

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5.7.3. The inverse tangent function

Figure 5.77 shows how we restrict the domain of the tangent function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ shown by the solid blue graph. Over this interval, the restricted tangent function passes the horizontal line test and has an inverse function.

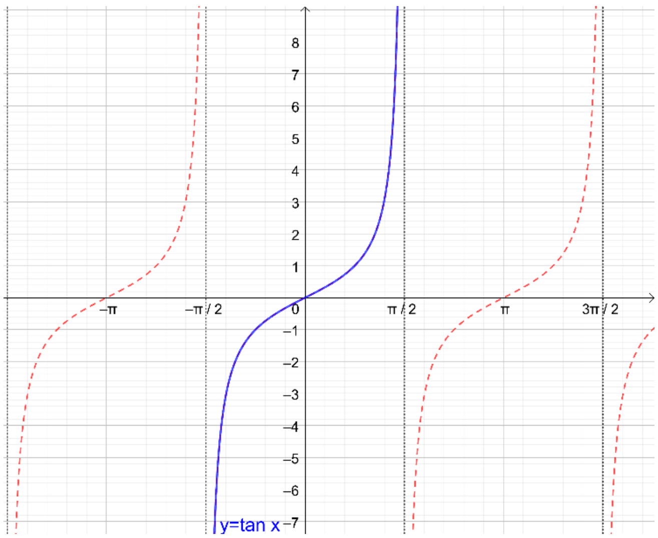

Figure 5.77 $\boldsymbol{y = \tan x}$ *is one-to-one on the interval* $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

Definition:

The inverse sine function, denoted by $\arctan x$, is the inverse of the restricted tangent function $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Thus,

$$y = \arctan x \quad \text{means} \quad \tan y = x,$$

where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $-\infty < x < \infty$. We read $y = \arctan x$ as "y equals the inverse tangent at x."

We graph $y = \arctan x$ by taking points on the graph of the restricted function and reversing the order of the coordinates. *Figure 5.78* shows that $\left(-\frac{\pi}{4};-1\right)$, $(0,0)$, $\left(\frac{\pi}{4};1\right)$ and are on the graph of the restricted tangent function. Reversing the order gives $\left(-1,-\frac{\pi}{4}\right)$, $(0,0)$ and $\left(1,\frac{\pi}{4}\right)$. We now use these three points to graph the inverse tangent function. The graph of $y = \arctan x$ is shown in *Figure 5.79*. Notice that the vertical asymptotes become horizontal asymptotes for the graph of the inverse function.

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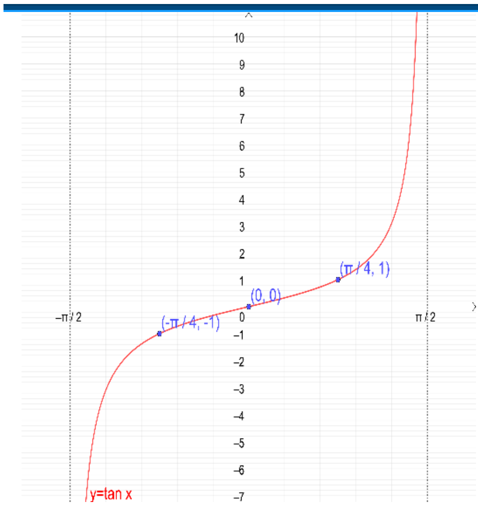

Figure 5.78 The restricted tangent function,
Domain: $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, *Range:* $(-\infty;\infty)$.

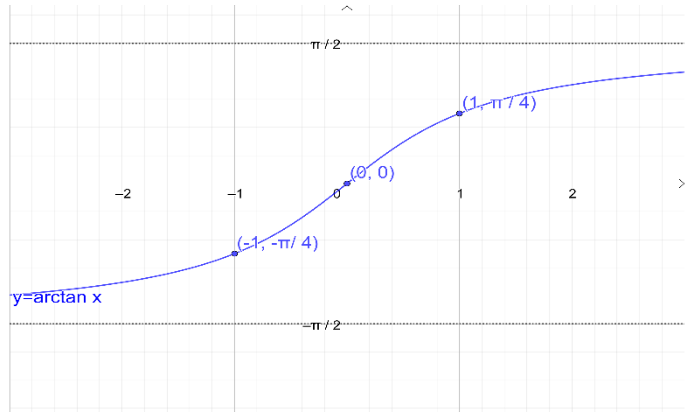

Figure 5.79 The graph of the inverse tangent function
Domain: $(-\infty;\infty)$, *Range:* $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

Exact values of $y = \arctan x$ can be found by thinking of $\arctan x$ as **the angle in the interval** $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ **whose tangent is** x.

FINDING EXACT VALUES OF $ARCTAN\ x$

1. Let $\arctan x = \varphi$.
2. Rewrite
3. $\arctan x = \varphi$ as $\tan\varphi = x$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.
4. Use the exact values in Table 2 to find the value of φ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi = x$.

φ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan\varphi$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

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Example 5.55 Finding the exact value of an inverse tangent function	*Example 5.56 Finding the exact value of an inverse tangent function*
Find the exact value of $\arctan(-\sqrt{3})$. **Solution:** **1.** Let $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$. Thus, $\arctan(-\sqrt{3}) = \varphi$. We must find the angle $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose tangent equals $-\sqrt{3}$. **2.** Rewrite $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. Using the definition of the inverse tangent function, we rewrite $\arctan(-\sqrt{3}) = \varphi$ as $\tan\varphi = -\sqrt{3}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. **3.** Use the exact values in Table 3 to find the value of $\boldsymbol{\varphi}$ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$. Table 3 shows that the only angle in the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi = -\sqrt{3}$ is $-\frac{\pi}{3}$. Thus, $\varphi = -\frac{\pi}{3}$. Because φ in step 1, represents $\arctan(-\sqrt{3})$, we conclude that $$\arctan(-\sqrt{3}) = -\frac{\pi}{3}.$$	Find the exact value of $\arctan(-1)$. **Solution:** **1.** Let $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$. Thus, $\arctan(-1) = \varphi$. We must find the angle $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose cosine equals -1. **2.** Rewrite $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. Using the definition of the inverse tangent function, we rewrite $\arctan(-1) = \varphi$ as $\tan\varphi = -1$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. **3.** Use the exact values in Table 3 to find the value of $\boldsymbol{\varphi}$ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$. Table 3 shows that the only angle in the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi = -1$ is $-\frac{\pi}{4}$. Thus, $\varphi = -\frac{\pi}{4}$. Because φ in step 1, represents $\arctan(-1)$, we conclude that $$\arctan(-1) = -\frac{\pi}{4}.$$

Table 3 Exact values for $\mathbf{tan}\,\boldsymbol{\varphi}$, $\boldsymbol{\varphi\epsilon}$ $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

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5.7.4. COMPOSITION OF FUNCTIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Inverse properties

The Sine Function and Its Inverse	**The Cosine Function and Its Inverse**	**The Tangent Function and Its Inverse**
• $\sin(\arcsin x) = x$, for every $x \in [-1,1]$ • $\arcsin(\sin x) = x$, for every $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$	• $\cos(\arccos x) = x$, for every $x \in [-1,1]$ • $\arccos(\cos x) = x$, for every $x \in [0; \pi]$	• $\tan(\arctan x) = x$, for every $x \in [-\infty, \infty]$ • $\arctan(\tan x) = x$, for every $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$

The restrictions on in the inverse properties are a bit tricky. For example,

$$\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

We know that $\arcsin(\sin x) = x$, for every $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. Observe that $\frac{\pi}{4}$ is in interval $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. But we cannot use $\arcsin(\sin x) = x$ to find the exact value of $\arcsin\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}$, because $\frac{5\pi}{4}$ is not in interval $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. Thus, to evaluate $\arcsin\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}$, we must first find $\sin\frac{5\pi}{4}$. Value $\frac{5\pi}{4}$ is in quadrant III, where the sine is negative.

$$\sin\frac{5\pi}{4} = \sin\left(2\pi - \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \text{ (the reference angle for } \frac{5\pi}{4} \text{ is } \frac{\pi}{4}\text{).}$$

We evaluate $\arcsin\left(\sin\frac{5\pi}{4}\right)$ as follows:

$$\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

To determine how to evaluate the composition of functions involving inverse trigonometric functions, first examine the value of sin(). You can use the inverse properties in the box only if x is in the specified interval.

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Example 5.57 Evaluating compositions of functions and their inverses	*Example 5.58 Evaluating compositions of functions and their inverses*
Find the exact value, if possible: $$\cos(\arccos 0.6)$$ **Solution:** The inverse property $\cos(\arccos x) = x$ applies for every $x \in [-1,1]$. To evaluate $\cos(\arccos 0.6)$, observe that $x = 0.6$. This value of x lies in $[-1,1]$, which is the domain of the inverse cosine function. This means that we can use the inverse property $\cos(\arccos x) = x$. Thus, $$\cos(\arccos 0.6) = 0.6.$$	Find the exact value, if possible: $$\arcsin\left(\sin\frac{3\pi}{2}\right)$$ **Solution:** The inverse property $\arcsin(\sin x) = x$ applies for every $x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. To evaluate $\arcsin\left(\sin\frac{3\pi}{2}\right)$, observe that $x = \frac{3\pi}{2}$. This value of x does not lie in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. To evaluate this expression, we first find $\sin\frac{3\pi}{2}$. $$\sin\frac{3\pi}{2} = \sin\left(2\pi - \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1.$$ The reference angle for $\frac{3\pi}{2}$ is $\frac{\pi}{2}$. $\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2}$. The angle in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whise sine is -1 is $-\frac{\pi}{2}$.

We can use points on terminal sides of angles in standard position to find exact values of expressions involving the composition of a function and a different inverse function.

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Example 5.59 Evaluating a composite trigonometric expression

Find the exact value, if possible:

$$\cos\left(\arctan\frac{5}{12}\right)$$

Solution:

The inner part of expression involves an angle.To evaluate such expression, we represent such angles by φ. We let φ represent the angle in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose tangent is $\frac{5}{12}$. Thus,

$$\varphi = \arctan\frac{5}{12}.$$

We are looking for the exact value of $\cos\left(\arctan\frac{5}{12}\right)$, with $\varphi = \arctan\frac{5}{12}$. Using the definition of the inverse tangent function, we can rewrite $\varphi = \arctan\frac{5}{12}$ as

$$\tan\varphi = \frac{5}{12}, where\ \varphi \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right).$$

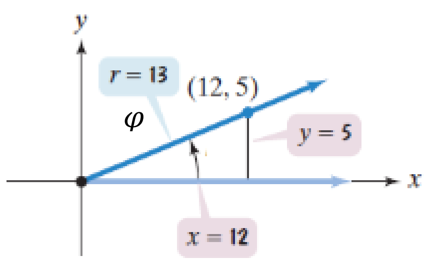

Figure 5.80 Representing $\tan\varphi = \frac{5}{12}$.

Because $\tan\varphi$ is positive, φ must be an angle in $\left(0;\frac{\pi}{2}\right)$. Thus, φ is a I quadrant angle. Figure 5.80 shows a right triangle in quadrant I w

$\tan\varphi = \frac{5}{12}$.

Side opposite φ, or y

Side adjacent to φ, or x

The hypotenuse of the triangle, r, or the distance from the origin to $(12,5)$, is found using $r = \sqrt{x^2+y^2}$.

$$r = \sqrt{x^2+y^2} = \sqrt{12^2+5^2} = \sqrt{169} = 13.$$

We use the values for x and r to find the exact value of $\cos\left(\arctan\frac{5}{12}\right)$.

Example 5.60 Evaluating a composite trigonometric expression

Find the exact value, if possible:

$$\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$$

Solution:

The inner part of expression involves an angle.To evaluate such expression, we represent such angles by φ. We let φ represent the angle in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is $-\frac{1}{3}$. Thus,

$$\varphi = \arcsin\left(-\frac{1}{3}\right).$$

We are looking for the exact value of $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$, with $\varphi = \arcsin\left(-\frac{1}{3}\right)$. Using the definition of the inverse sine function, we can rewrite $\varphi = \arcsin\left(-\frac{1}{3}\right)$ as

$$\sin\varphi = -\frac{1}{3}, where\ \varphi \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right].$$

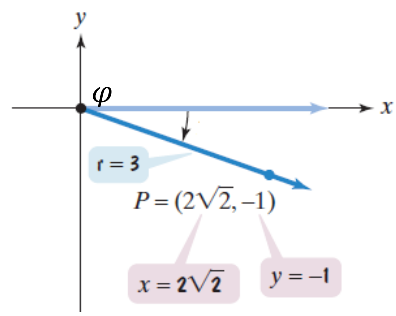

Figure 5.81 Representing $\sin\varphi = -\frac{1}{3}$.

Because $\sin\varphi$ is negative, φ must be an angle in $\left(-\frac{\pi}{2},0\right)$. Thus, φ is a IV quadrant angle. Figure 5.81 shows angle φ in quadrant IV with

$$\sin\varphi = -\frac{1}{3} = \frac{y}{r} = \frac{-1}{3}.$$

Thus, $y=-1$ and $r=3$. The value of x can be found using $r = \sqrt{x^2+y^2}$ or $r^2 = x^2+y^2$.

Use $r^2 = x^2+y^2$ with $y=-1$ and $r=3$

$$3^2 = x^2 + (-1)^2$$
$$x^2 + (-1)^2 = 3^2$$
$$x^2 + 1 = 9$$
$$x^2 = 8$$
$$x = \sqrt{8} = 2\sqrt{2}.$$

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$\cos\left(\arctan\frac{5}{12}\right)=\cos\varphi=\frac{side\ adjacent\ to\ \varphi,\ or\ x}{hypotenuse,\ or\ r}=\frac{12}{13}.$	We use $x=2\sqrt{2}$ and $y=-1$ to find the exact value of $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$. $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)=\cot\varphi=\frac{x}{y}=\frac{2\sqrt{2}}{-1}=-2\sqrt{2}.$

Some composite functions with inverse trigonometric functions can be simplified to algebraic expressions. To simplify such an expression, we represent the inverse trigonometric function in the expression by $\boldsymbol{\varphi}$. Then we use a right triangle.

Example 5.61 Simplifying an expression involving arcsin x

If $\boldsymbol{x}\in(\mathbf{0},\mathbf{1}]$, write $\mathbf{cos}(\mathbf{arcsin}\,\boldsymbol{x})$ as an algebraic expression in $\boldsymbol{x}$.

Solution:

We let $\boldsymbol{\varphi}$ represent the angle in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is $\boldsymbol{x}$. Thus,

$\boldsymbol{\varphi}=\mathbf{arcsin}\,\boldsymbol{x}$ and $\mathbf{sin}\,\boldsymbol{\varphi}=\boldsymbol{x}$, where $\boldsymbol{x}\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.

Because $\boldsymbol{x}\in(\mathbf{0},\mathbf{1}]$, $\mathbf{sin}\,\boldsymbol{\varphi}$ is positive. Thus, $\boldsymbol{\varphi}$ is the I-quadrant angle and can be represented as an acute angle of a right triangle.

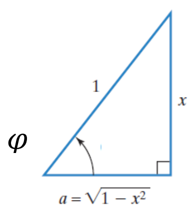

Figure 5.82 Representing $\sin\varphi = x$.

Figure 5.82 shows a right triangle with

$$\mathbf{sin}\,\boldsymbol{\varphi}=\boldsymbol{x}=\frac{\boldsymbol{x}}{\mathbf{1}}.$$

Side opposite φ

hypotenuse

The third side $\boldsymbol{a}$, in Figure 14, can be found using the Pythagorean Theorem.

$$\mathbf{1}^2=\boldsymbol{a}^2+\boldsymbol{x}^2$$

$$\boldsymbol{a}^2=\mathbf{1}^2-\boldsymbol{x}^2$$

$$\boldsymbol{a}=\sqrt{\mathbf{1}^2-\boldsymbol{x}^2}$$

We use the right triangle in *Figure 5.82* to write $\mathbf{cos}(\mathbf{arcsin}\,\boldsymbol{x})$ as an algebraic expression.

$$\cos(\arcsin x)=\cos\varphi=\frac{side\ adjacent\ \text{to}\ \varphi}{hypotenuse}=\frac{a}{hypotenuse}=\frac{\sqrt{1^2-x^2}}{1}=\sqrt{1^2-x^2}.$$

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5.7.5. APPLICATION EXAMPLES

Example 5.62 ARCHITECTURE

Figure 5.83

The support for a roof is shaped like two right triangles, as shown below. Find θ

Solution:

Figure 5.80 shows a right triangle with

$$\sin\theta = \frac{side\ opposite\ \theta}{hypotenuse} = \frac{9}{18} = \frac{1}{2}.$$

When $\sin\theta = \frac{1}{2}$. We must find the angle $\theta \epsilon \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, whose sine equals $\frac{1}{2}$. Use the exact values in Table 1 to find the value of θ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\theta = \frac{1}{2}$. Table 1 shows that the only angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\theta = \frac{1}{2}$ is $\frac{\pi}{6}$. Thus, $\theta = \frac{\pi}{6}$.

Example 5.63 RESCUE

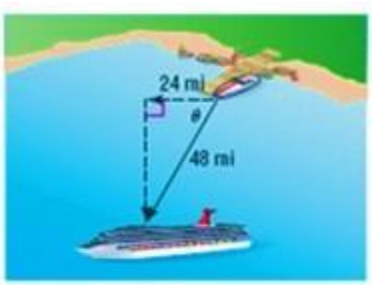

Figure 5.84

A cruise ship sailed due west 24 miles before turning south. When the cruise ship became disabled and the crew radioed for help, the rescue boat found that the fastest route covered a distance of 48 miles. Find the angle θ at which the rescue boat should travel to aid the cruise ship.

Solution:

Figure 16 shows a right triangle with

$$\cos\theta = \frac{side\ adjacent\ \text{to}\ \theta}{hypotenuse} = \frac{24}{48} = \frac{1}{2}.$$

When $\cos\theta = \frac{1}{2}$. We must find the angle $\theta \epsilon [0;\pi]$, whose cosine equals $\frac{1}{2}$. Use the exact values in Table 2 to find the value of θ in $[0;\pi]$ that satisfies $\cos\theta = \frac{1}{2}$. Table 2 shows that the only angle in the interval $[0;\pi]$ that satisfies $\cos\theta = \frac{1}{2}$ is $\frac{\pi}{3}$. Thus, $\theta = \frac{\pi}{3}$.

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Example 5.64 *DRAG RACE*

Figure 5.85

A television camera is filming a drag race. The camera rotates as the vehicles move past it. The camera is 30 meters away from the track. Consider θ and x as shown in the ***Figure 5.82.***

a. Find θ when $x = 6$ meters and $x = 14$ meters.

b. Write θ as a function of x.

Solution:

a. Figure 17 shows a right triangle with the relationship between θ and the sides is opposite and adjacent, so

$$\tan\theta = \frac{side\ opposite\ \theta}{side\ adjacent\ \text{to}\ \theta} = \frac{x}{30}.$$

If $x = 6$, then $\tan\theta = \frac{x}{30} = \frac{6}{30} = \frac{1}{5}$. We must find the angle $\theta \epsilon \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$, whose tangent equals $\frac{1}{5}$. We cannot use the exact values in Table 3 to find the value of θ in $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ that satisfies $\tan\theta = \frac{1}{5}$, because we have not such value in this table. But we can use a calculator -> $\tan^{-1}\frac{1}{5} \approx 11.3°$ This angle is in the interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ and satisfies $\tan\theta = \frac{1}{5}$.

If $x = 14$, then $\tan\theta = \frac{x}{30} = \frac{14}{30} = \frac{7}{15}$. We must find the angle $\theta \epsilon \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$, whose tangent equals $\frac{7}{15}$. We cannot use the exact values in Table 3 to find the value of θ in $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ that satisfies $\tan\theta = \frac{7}{15}$, because we have not such value in this table. But we can use a calculator -> $\tan^{-1}\frac{7}{15} \approx 25°$ This angle is in the interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ and satisfies $\tan\theta = \frac{7}{15}$.

b. In step 1 we find

$$\tan\theta = \frac{x}{30}$$
$$\mathit{take}\ \arctan\ \mathit{of\ both\ sides}$$

$$\arctan(\tan\theta) = \arctan\frac{x}{30}$$
$$\mathit{use\ inverse\ tangent\ proerties}\ \arctan(\tan x) = x, \text{for every } x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$$

Thus, θ as a function of x is

$$\theta = \arctan\frac{x}{30}.$$

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5.7.6. EXERCISES

Find the exact value of each expression

1. $\arcsin\frac{1}{2}$;
2. $\arcsin\frac{\sqrt{2}}{2}$;
3. $\arcsin\left(-\frac{1}{2}\right)$;
4. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$;
5. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$;
6. $\arctan\left(-\frac{\sqrt{2}}{2}\right)$;
7. $\arctan(-1)$;
8. $\arctan\sqrt{3}$;
9. $\arcsin 0$;
10. $\arcsin 1$.

Use a calculator to find the value of eachexpression rounded to two decimal places

1. $\arcsin(-20)$;
2. $\arcsin 0.3$;
3. $\arccos\frac{1}{8}$;
4. $\arcsin 0.47$;
5. $\arctan(-20)$;
6. $\arctan 30$;
7. $\arccos\frac{\sqrt{5}}{7}$;
8. $\arccos\frac{4}{9}$;
9. $\arctan(-\sqrt{5061})$;
10. $\arcsin(-0.625)$.

Find the exact value of each expression, if possible. Do not use a calculator

1. $\sin(\arcsin 0.9)$;
2. $\arcsin\left(\sin\frac{\pi}{3}\right)$;
3. $\arcsin\left(\sin\frac{5\pi}{6}\right)$;
4. $\tan(\arctan 125)$;
5. $\arctan\left(\tan\left[-\frac{\pi}{6}\right]\right)$;
6. $\arcsin(\sin\pi)$;
7. $\sin(\arcsin\pi)$;
8. $\cos(\arccos 0.57)$;
9. $\arccos\left(\cos\frac{4\pi}{3}\right)$;
10. $\arccos(\cos 2\pi)$;
11. $\arctan\left(\tan\left[-\frac{\pi}{3}\right]\right)$;
12. $\arctan\left(\tan\frac{3\pi}{4}\right)$.

Use a sketch to find the exact value of each expression

1. $\cos\left(\arcsin\frac{1}{2}\right)$;
2. $\tan\left(\arccos\frac{5}{13}\right)$;
3. $\tan\left(\arcsin\left[-\frac{3}{5}\right]\right)$;
4. $\sin\left(\arctan\frac{7}{24}\right)$;
5. $\cot\left(\arcsin\left(-\frac{4}{5}\right)\right)$;
6. $\cos\left(\arcsin\frac{5}{13}\right)$;
7. $\sin\left(\arccos\frac{\sqrt{2}}{2}\right)$;
8. $\sin\left(\arctan\left[-\frac{3}{4}\right]\right)$.

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Use a right triangle to write each expression as an algebraic expression. Assume that x is positive and that the given inverse trigonometric function is defined for the expression in x

1. $\tan(\arccos x)$;
2. $\cos(\arcsin 2x)$;
3. $\cos\left(\arcsin\frac{1}{x}\right)$;
4. $\cot\left(\arctan\frac{x}{\sqrt{3}}\right)$;
5. $\sin(\arctan x)$;
6. $\sin(\arccos 2x)$;
7. $\cot\left(\arctan\frac{x}{\sqrt{2}}\right)$;
8. $\cot\left(\arcsin\frac{\sqrt{x^2-9}}{x}\right)$.

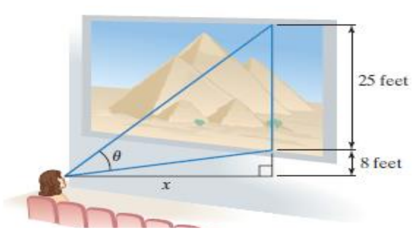
Use transformations (vertical shifts, horizontal shifts, reflections, stretching, or shrinking) of these graphs to graph each function. Then use interval notation to give the function's domain and range

1. $f(x) = \arcsin x + \frac{\pi}{2}$;
2. $f(x) = \arccos(x+1)$;
3. $g(x) = -2\arctan x$;
4. $f(x) = \arcsin(x-2) - \frac{\pi}{2}$;
5. $f(x) = \arccos x + \frac{\pi}{2}$;
6. $h(x) = -3\arctan x$;
7. $f(x) = \arccos(x-2) - \frac{\pi}{2}$;
8. $f(x) = \arcsin\frac{x}{2}$.

Determine the domain and the range of each function

1. $f(x) = \sin(\arcsin x)$;
2. $f(x) = \cos(\arccos x)$;
3. $f(x) = \arcsin(\cos x)$;
4. $f(x) = \arcsin(\sin x)$;
5. $f(x) = \cos(\arccos x)$;
6. $f(x) = \arccos(\sin x)$.

5.7.7. APPLICATION EXERCISES



1. Your neighborhood movie theater has a 25-foot-high screen located 8 feet above your eye level. If you sit too close to the screen, your viewing angle is too small, resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit feet back from the screen, your viewing angle θ, is given by

$$\theta = \arctan\frac{33}{x} - \arctan\frac{8}{x}.$$

Find the viewing angle in radians, at distance of 5 feet, 10 feet, 15 feet, 20 feet and 25 feet.

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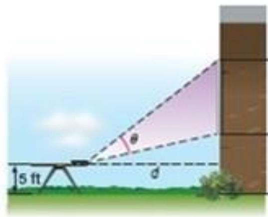
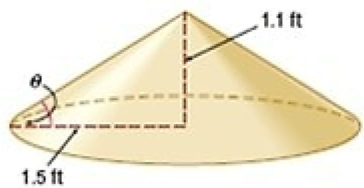


2. SPORTS. Steve and Ravi want to project a prosoccer game on the side of their apartment building. They have placed a projector on a table that stands 5 feet above the ground and have hung a 12-foot-tall screen 10 feet above the ground.

a. Write a function expressing θ in terms of distance d.

b. Use a graphing calculator to determine the distance for the maximum projecting angle.



3. SAND. When piling sand, the angle formed between the pile and the ground remains fairly consistent and is called the angle of repose. Suppose Jade creates a pile of sand at the beach that is 3 feet in diameter and 1.1 feet high.

a. What is the angle of repose?

b. If the angle of repose remains constant, how many feet in diameter would a pile need to be to reach a height of 4 feet?

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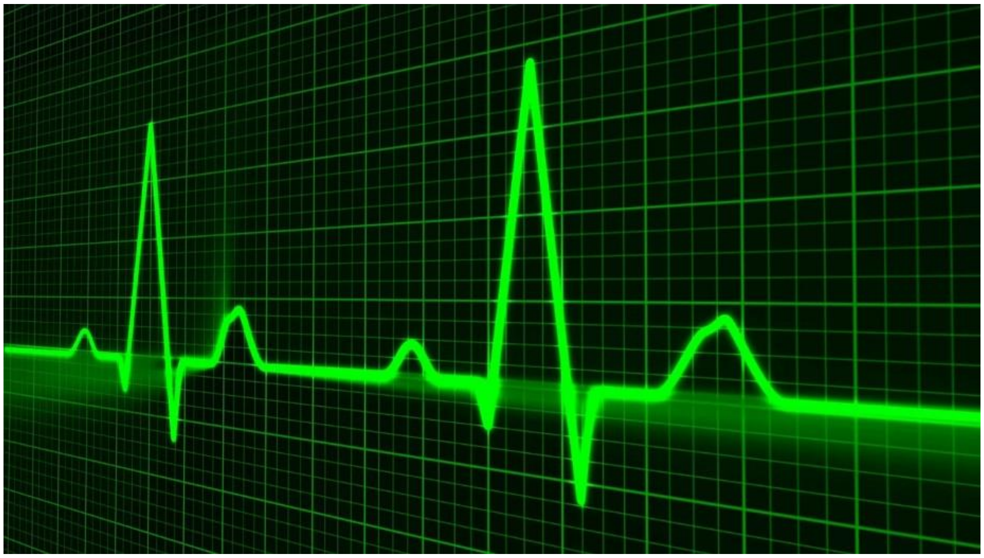
5.8. LIMITS OF FUNCTIONS



Limits are used in signal processing. Source: https://edutorij.e-skole.hr/share/proxy/alfresco-noauth/edutorij/api/proxy-guest/2fdc0f24-204f-41cb-8d7b-6239d7a7c598/img/mm-m4-03-a-01-2020_02_12_23_35_42-2020_05_11_13_11_26-jpg-1617297841916.jpg

ABSTRACT:

The concept of a limit is a fundamental concept of calculus and mathematical analysis. This chapter begins by describing how to find the limit of a function at a given point. Not all functions have limits at all points, and it is discussed how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

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Learning Outcomes

1. Describe the limit of a function.
2. Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
3. Apply a graph to estimate the limit of a function or to identify when the limit does not exist.
4. Determine one-sided limits and provide examples.
5. Identify the importance of limits

Contents

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5.8.1. LIMITS AND CONTINUITY OF FUNCTIONS

Limit at a point

A function has a limit L in a point a if the value of the function approaches the value L when the input x approaches the value a.

Whenever you see *limit*, you can think of *approaching*.

The notation is as follows

$$\lim_{x \to a} f(x) = L$$

And mathematical definition is:

If for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, then the function $f(x)$ has a limit L in a point a and we can write $\lim_{x \to a} f(x) = L$.

Limit is mostly calculated in points where the function is not defined.

Here is an example. The function $f(x) = \frac{\ln(x+1)}{\ln(x)}$ is not defined for $x < 0$. Function is presented in Figure 5.86.

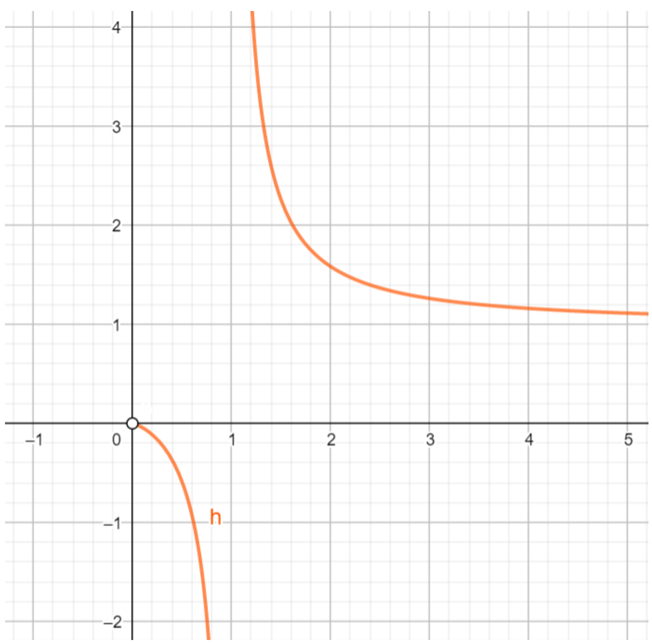

Figure 5.86

Hence, $a = 0$. If the input x approaches to zero, the value of the function approaches to $y = 0$ and we can write $\lim_{x \to 0} \frac{\ln(x+1)}{\ln(x)} = 0$

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Limit at infinity

If the value of a function approaches to the value L as x gets larger, then a function f has a limit L at infinity.

The notation is as follows

$$\lim_{x\to\infty} f(x) = L$$

Definition (limit):

If for every $\varepsilon > 0$, there exists $m > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x > m$, then the function $f(x)$ has a limit L in infinity and we can write $\lim_{x\to\infty} f(x) = L$.

Here is an example. The function $f(x) = \frac{1}{x}$.

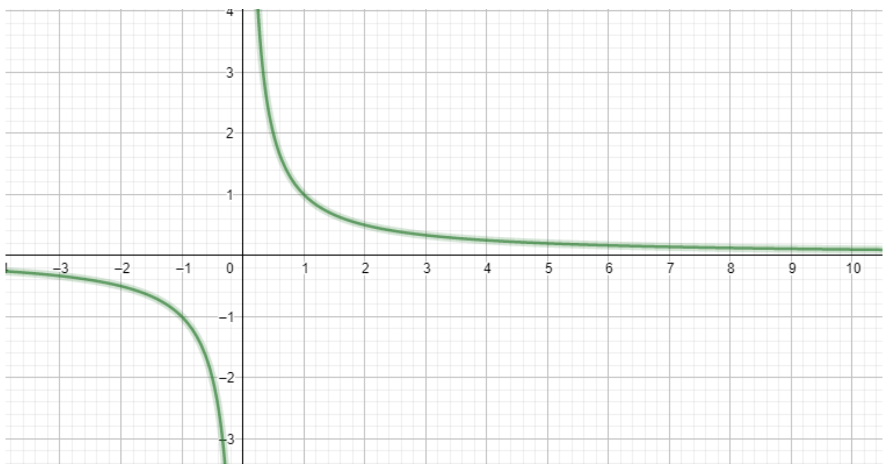

Figure 5.87

As the input value x gets larger, the value of a function approaches to zero.

x	$f(x)$
10	0,1
100	0,01
1000	0,001

Also, we can see that as the value of x gets smaller, the value of a function also approaches to zero.

x	$f(x)$
–10	–0,1
–100	–0,01
–1000	–0,001

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And we can conclude that $\lim_{x\to\pm\infty} f(x) = 0$.

Infinite limit

If the value of a function gets larger as x approaches to value a, then a function f has a limit infinity in a point a.

The notation is as follows

$$\lim_{x\to a} f(x) = \infty$$

Definition:

If for every $M > 0$, there exists $\delta > 0$, such that $f(x) > M$ whenever $0 < |x-a| < \delta$, then the function $f(x)$ has a limit infinity in a point a and we can write $\lim_{x\to a} f(x) = +\infty$.

If for every $M < 0$, there exists $\delta > 0$, such that $f(x) < M$ whenever $0 < |x-a| < \delta$, then the function $f(x)$ has a limit infinity in a point a and we can write $\lim_{x\to a} f(x) = -\infty$.

Sometimes if we are not concerned whether $f(x)$ approaches to $+\infty$ or $-\infty$ we use this notation $\lim_{x\to a} f(x) = \pm\infty$ or just $\lim_{x\to a} f(x) = \infty$.

If we look at the function $f(x) = \frac{1}{x}$ in Figure 3, we can notice that as input value x approaches to zero the value of function approaches to $+\infty$ or $-\infty$. We can conclude that $\lim_{x\to 0} f(x) = \infty$.

Properties of limits

Suppose that $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = N$.

Then

P1 $\lim_{x\to a}\big(f(x) \pm g(x)\big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = M \pm N$

Addition property - the limit of a sum/difference is equal to the sum/difference of limits

P2 $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = M \cdot N$

Multiplication property - the limit of a product is equal to the product of limits

Note that this implies the following:

P2* $\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x), c \in R$ I.e.

$\lim_{x\to 0} 2x = 2 \lim_{x\to 0} x$

P3 $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{M}{N}, N \neq 0$

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Division property - the limit of a quotient is equal to the quotient of limits

P4 $\lim_{x \to a} f(x)^k = M^k, M > 0$

Important limits are also called **common limits** and are usually listed in tables and/or known by heart.

The first common limit to remember is:

$$\lim_{x \to \pm\infty} \frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{0}$$

The second, more general limit is even more important:

$$\lim_{x \to \pm\infty} \frac{\mathbf{1}}{\boldsymbol{x^n}} = \mathbf{0}, \qquad \text{where } n \in N$$

Also, as a side note, the limit of a constant is always equal to that constant:

$$\lim_{x \to \pm\infty} \boldsymbol{c} = \boldsymbol{c}$$

Let us now show the calculation of a more complex limit:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x}$$

Note that the technique described below is suitable only when calculating the limit at infinity.

The most common technique when dealing with limits is to divide the numerator and denominator of the fraction by the highest order polynomial in the denominator.

The expression can be written as:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x} : \frac{x^3}{x^3}$$

Divide the numerator and denominator separately:

$$\lim_{x \to \pm\infty} \frac{(x^3 + 3x^2 + 3): x^3}{(x^3 - 7x): x^3}$$

Since the parenthesis is divided by x^3, we can divide each term in the parenthesis separately:

$$\lim_{x \to \pm\infty} \frac{x^3: x^3 + 3x^2: x^3 + 3: x^3}{x^3: x^3 - 7x: x^3}$$

Remember the division property of powers? $x^m: x^n = x^{m-n}$
Now let us apply that property to every term in the numerator and denominator:

$$\lim_{x \to \pm\infty} \frac{x^{3-3} + 3x^{2-3} + 3x^{-3}}{x^{3-3} - 7x^{1-3}}$$

Calculating the differences, we obtain:

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$$\lim_{x\to\pm\infty}\frac{x^0+3x^{-1}+3x^{-3}}{x^0-7x^{-2}}$$

Finally, let us calculate each term, and rewrite using the following power property: $ax^{-n}=\frac{a}{x^n}$.

Also, remember that $x^0=1$ if $\mathrm{x}\neq 0$.

$$\lim_{x\to\pm\infty}\frac{1+\frac{3}{x}+\frac{3}{x^3}}{1-\frac{7}{x^2}}$$

Only now are we able to use the property P.3. (the limit of a quotient is the quotient of limits)

$$\frac{\lim\limits_{x\to\pm\infty}\left(1+\frac{3}{x}+\frac{3}{x^3}\right)}{\lim\limits_{x\to\pm\infty}\left(1-\frac{7}{x^2}\right)}$$

After using the property P.1. (the limit of a sum is the sum of limits) the expression simplifies further:

$$\frac{\lim\limits_{x\to\pm\infty}1+\lim\limits_{x\to\pm\infty}\frac{3}{x}+\lim\limits_{x\to\pm\infty}\frac{3}{x^3}}{\lim\limits_{x\to\pm\infty}1-\lim\limits_{x\to\pm\infty}\frac{7}{x^2}}$$

We now use P.2*. (the constant can be written before the limit):

$$\frac{\lim\limits_{x\to\pm\infty}1+3\lim\limits_{x\to\pm\infty}\frac{1}{x}+3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty}1-7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}}$$

Finally, we can use the common limits written above:

$$\frac{\lim\limits_{x\to\pm\infty}1+3\lim\limits_{x\to\pm\infty}\frac{1}{x}+3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty}1-7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}}=\frac{1+3\cdot 0+3\cdot 0}{1-7\cdot 0}=\frac{1+0+0}{1-0}=\frac{1}{1}=1$$

Therefore, we conclude that:

$$\lim_{x\to\pm\infty}\frac{x^3+3x^2+3}{x^3-7x}=1$$

The procedure for calculating limits may seem complex right now, but we assure the reader that the procedure is quite straightforward:

1) Divide by the greatest order polynomial of the denominator
2) Simplify the expression in the numerator and denominator.

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This part was most of the work, but the work done has nothing to do with limits, just carefully dividing polynomials and using power properties. Here steps may be omitted if the student is confident with basic polynomial algebra.

3) Using the properties of limits, reduce the limit to a series of common limits, and calculate separately each common limit.

Let us once again stress that the following procedure is only useful for calculating limits when x approaches infinity.

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5.8.2. Exercises

Task 5.38 Find the limits

1. $\lim_{x \to 3}(x^2 - 2)$
2. $\lim_{x \to \infty} x^2$
3. $\lim_{x \to 0} \frac{1}{x^2}$
4. $\lim_{x \to \infty} \frac{2}{x^3}$
5. $\lim_{x \to \infty} \frac{x-6}{2x+1}$
6. $\lim_{x \to \infty} \frac{3x^2+2x-6}{4x^2-5}$
7. $\lim_{x \to \infty} \frac{5x^3+2}{3x^2+x-3}$
8. $\lim_{x \to 1} \frac{x-1}{x^2-1}$

Solutions

1. $\lim_{x \to 3}(x^2 - 2) = (3^2 - 2) = 7$

   This exercise was simple. We can just substitute 3 instead of x and calculate. Property P1 is used in this example

2. $\lim_{x \to \infty} x^2 = [\infty^2] = \infty$

   If x approaches infinity, we can assume that x is a huge number, e.g. 1000 or even greater and substitute 1000 instead of x in our expression. Since $1\,000^2 = 1\,000\,000$ is even higher, we can conclude that the result is infinity.

   Notice that we put ∞^2 in brackets $[\infty^2]$. That is because ∞^2 is not a regular mathematical expression.

3. $\lim_{x \to 0} \frac{1}{x^2} = \left[\frac{1}{0^2}\right] = \left[\frac{1}{0}\right] = \infty$

   Dividing by zero is not defined, so we can imagine that x is a very small number, e.g. x=0,001 or even smaller. In that case the value of expression $\frac{1}{x^2} = \frac{1}{0{,}001^2} = \frac{1}{0{,}000001} = 1\,000\,000$ which is pretty large number. We can conclude that as the x approaches to zero, the value of expression gets greater and approaches to infinity.

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4. $\lim_{x\to\infty}\frac{2}{x^3}=\left[\frac{2}{\infty}\right]=0$

   When we divide some number (e.g. 2) with some big number, e.g. 1000^3 or even greater the result is a very small number, close to zero.

5. $\lim_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]$

   As x gets greater, the values of denominator and numerator are approaching to infinity. If denominator and numerator are polynomials we can solve this problem by dividing all through by the highest power of x. In this exercise we divided by x.

$$\lim_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{1-\frac{6}{x}}{2+\frac{1}{x}}$$

Since $\lim_{x\to\infty}\frac{6}{x}=0$ and $\lim_{x\to\infty}\frac{1}{x}=0$ we get

$$\lim_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{1-\frac{6}{x}}{2+\frac{1}{x}}=\frac{1}{2}$$

6. $\lim_{x\to\infty}\frac{3x^2+2x-6}{4x^2-5}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{6}{x^2}}{\frac{4x^2}{x^2}-\frac{5}{x^2}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{6}{x^2}}{4-\frac{5}{x^2}}=\frac{3}{4}$

Since both denominator and numerator are polynomials, we divided all through by the x^2.

7. $\lim_{x\to\infty}\frac{5x^3+2}{3x^2+x-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{5+\frac{2}{x^3}}{\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}}=\left[\frac{5}{0}\right]=\infty$

8. $\lim_{x\to 1}\frac{x-1}{x^2-1}=\left[\frac{0}{0}\right]$

   If we divide denominator and numerator by x^2 it will not be helpful.

   In cases when the values of denominator and numerator both approach to zero, we can solve the problem by using algebra to simplify the expression.

   We know that $x^2-1=(x-1)(x+1)$

$$\lim_{x\to 1}\frac{x-1}{x^2-1}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

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5.8.3. CONNECTIONS AND APPLICATIONS

Example 5.65 Graph analysis

The graph depicts the sea level in the Bay of Fundy in a period of 36 hours.

https://www.geogebra.org/calculator/n2ueypx6

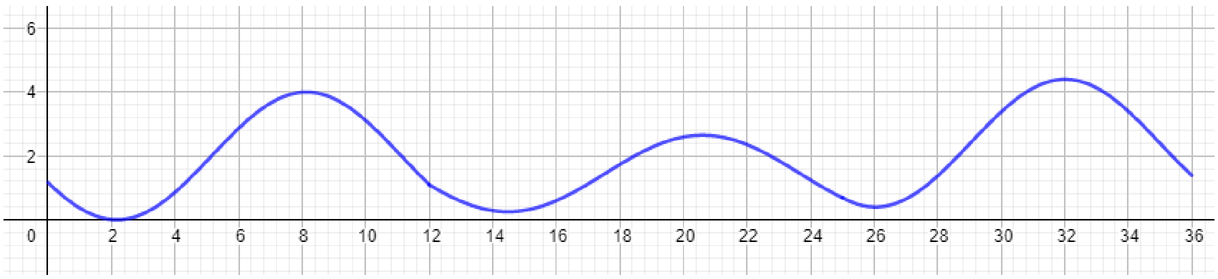

Figure 5.88

Determine:

a) The tides
b) The maximum sea level
c) The sea level at t=10 hours, t=18 hours, t=32 hours.
d) At what times was the sea level equal to 2 meters
e) All intervals when the sea level was increasing

Solution:

a) The increase of the sea level indicates flood, so flood occurs in the following approximate intervals $[2,8], [14,20.5], [26,32]$.

The decrease of the sea level indicates ebb, so ebb occurs in the following approximate intervals $[0,2], [8,14], [20.5,26], [32,36]$.

b) The maximum sea level is the highest point on the graph, which occurs at time $t = 32$. The corresponding sea level is 4.2 m.

c) To find the sea level at given points, read the y-coordinate of points on the graph at $t = 10, t = 18, t = 32$, like this:

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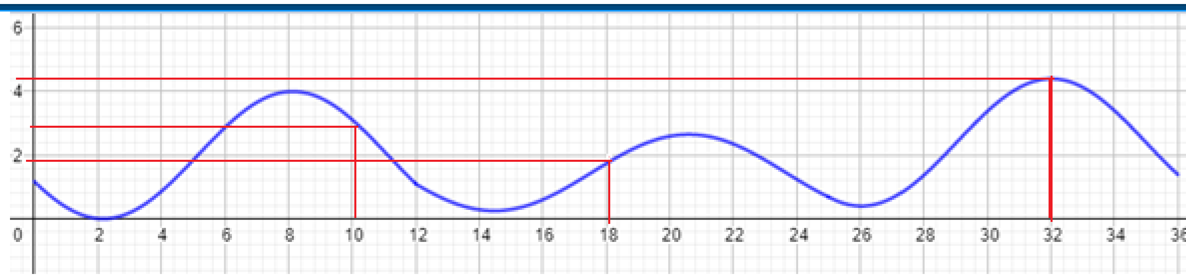

Figure 5.89 The corresponding points are approximately $(\mathbf{10}, \mathbf{2.8}), (\mathbf{18}, \mathbf{2}), (\mathbf{32}, \mathbf{4.4})$.

d) To find the times corresponding to the sea level of 2 meters, draw the line $y = 2$ and read the x-coordinates of all intersection points. The solutions are approximately: $t = 5, t = 11, t = 18, t = 23, t = 29, t = 35$.

e) The increasing sea level intervals are equivalent to flood, solved in part a).

Example 5.66

Find more limits applications in Google spreadsheets on link https://docs.google.com/spreadsheets/d/1ca0GpGKnrTfz2yHVk38PG-_abBrfGGZW4MBnbp6SDIw/edit?usp=sharing

Example 5.67

Hydrostatic pressure

https://phet.colorado.edu/sims/html/under-pressure/latest/under-pressure_en.html

Using the link above, we will examine some properties of the hydrostatic pressure formula.

Directions for the setup of the experiment.

i. Open the link
ii. In the lower left corner, pick the first model (one fossette and one pool).
iii. In the upper right corner tick the grid option (as to measure the water depth)
iv. Pour water into the pool to the brim
v. Drag and drop the barometer to measure the pressure above the pool and in the pool
vi. First try the air pressure above the pool:

1) When the depth increases, the pressure ___________.
2) What is the atmospheric pressure (the pressure at 0 m, in pool level)?
3) Fill out the table:

Water depth (X)	Water pressure (Y)
0	

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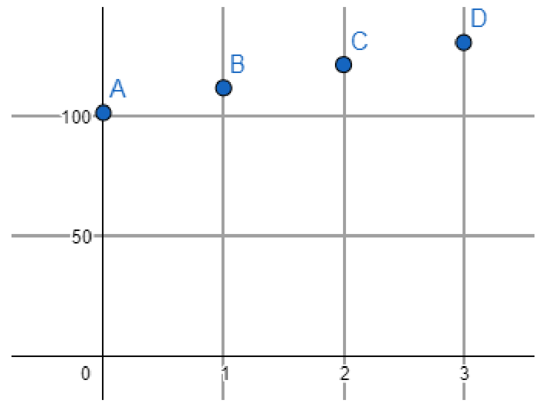
1	
2	
3	



4) The points are shown in Geogebra. What can you notice about the points? Is there a pattern? Explain your reasoning.

____-

The hydrostatic pressure formula is:

$$p_{hyd}(h) = p_{atm} + \rho g h$$

where ρ is the liquid density, g is the Earths gravitational constant (9.81 m/s^2), and h is the depth.

5) Given the salt water density is $\rho = 1025\frac{kg}{m^3}$, the atmospheric pressure is $p_{atm} = 101325\ Pa$ and $g = \frac{9.81m}{s^2}$, determine the hydrostatic pressure at:

a) 10 m
b) 15 m
c) 30 m?

6) The world record in deep sea diving (without injury) is 214 m. With conditions described above ($\rho = 1025\frac{kg}{m^3}, p_{atm} = 101325\ Pa, g = \frac{9.81m}{s^2}$) determine the pressure at the given depth.

7) What is the percent increase in water pressure at depth 214 m when compared with atmospheric pressure?

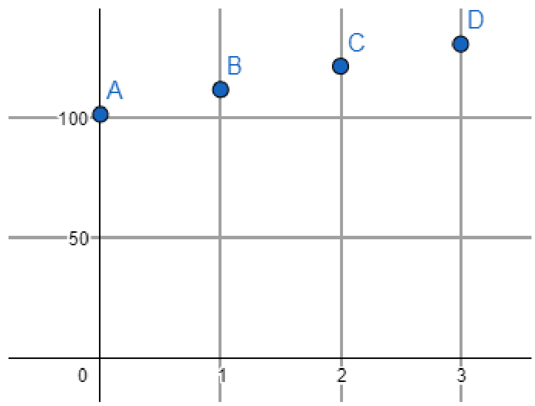
Solution:

1) When the depth increases, the pressure **increases**.

2) What is the atmospheric pressure is 101.326 kPa, so 101 326 Pa.

3) Fill out the table:

Water depth (X)	Water pressure (Y)
0	101.326
1	111.126
2	120.926
3	130.726



4) When the points from table in part 3) are graphed in a coordinate plane, a pattern can be seen.
All the points seem to lie on the same line. This means that the pressure at various depths can be determined from the graph, by extending the line.

5) Given the formula $p_{hyd}(h) = p_{atm} + \rho g h$ and all the known constants, plug in the numbers to calculate the pressure:
   a) h = 10 m, $p_{hyd}(10) = 101\,325 + 1025 \cdot 9{,}81 \cdot 10 = 201\,887\ Pa$
   b) h= 15 m, $p_{hyd}(15) = 101\,325 + 1025 \cdot 9{,}81 \cdot 15 = 252\,153\ Pa$
   c) h= 30 m, $p_{hyd}(30) = 101\,325 + 1025 \cdot 9{,}81 \cdot 30 = 402\,982\ Pa$

6) To calculate the pressure, plug in h = 214 m into the equation:

$$p_{hyd}(214) = 101\,325 + 1025 \cdot 9{,}81 \cdot 214 = 2\,253\,148\ Pa$$

7) To find the percent increase, calculate the quotient $\frac{p_{hyd}(214)}{p_{atm}} = 22.23$

So, the pressure at depth 214 m is 22.23 times greater than the atmospheric pressure. This is a percent increase of 2123%!

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5.8.4. Important limes

AIMS:

1) Students know definition of first important limit.
2) Students know how to apply first remarkable limit in solving tasks.
3) Students know different consequences of this limit.

Definition 1: First amazing limit is limit $\lim_{x\to 0}\frac{\sin x}{x}$.

Theorem 1: (Squeeze Theorem[1])

Suppose that for all x on [a, b] (except possibly at $x = c$) we have, $f(x) \le h(x) \le g(x)$. Also suppose that, $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$ for some $a \le c \le b$. Then, $\lim_{x\to c} h(x) = L$.

Theorem 2: **First remarkable limit equals 1.**

Proove:

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we will try to take it slow. Let us start by assuming that $0 \le \theta \le \frac{\pi}{2}$. Since we are proving a limit that has $\Theta \to 0$ it is okay to assume that Θ is not too large (*i.e.*, $\theta \le \frac{\pi}{2}$). Also, by assuming that Θ is positive we are going to first prove that the above limit is true if it is the right-hand limit. As you will see if we can prove this then the proof of the limit will be easy.

So, now that we have got our assumption on Θ taken care of let us start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.,

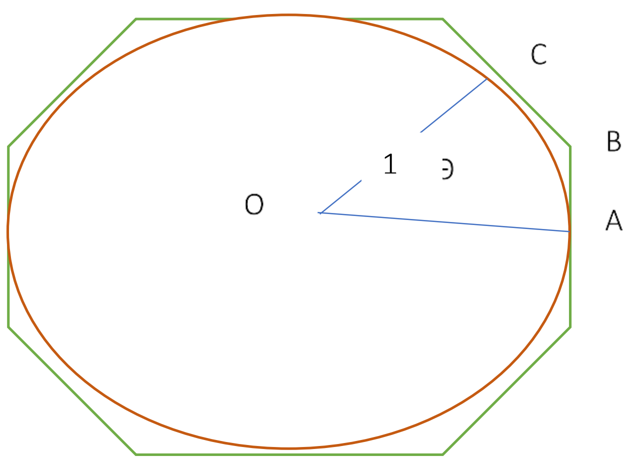

Figure 5.90

[1] Also called Two Soldiers Theorem, Sandwich Theorem

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Points A and C are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet B.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

Now denote the portion of the circle by arc AC and the lengths of the two portions of the octagon shown by $|AB|$ and $|BC|$. Then by the observation about lengths we made above we must have,

$$\text{arc } AC < |AB| + |BC| \qquad (1)$$

Next, extend the lines AB and OC as shown below and call the point that they meet D. The triangle now formed by AOD is a right triangle. All this is shown in the figure below.

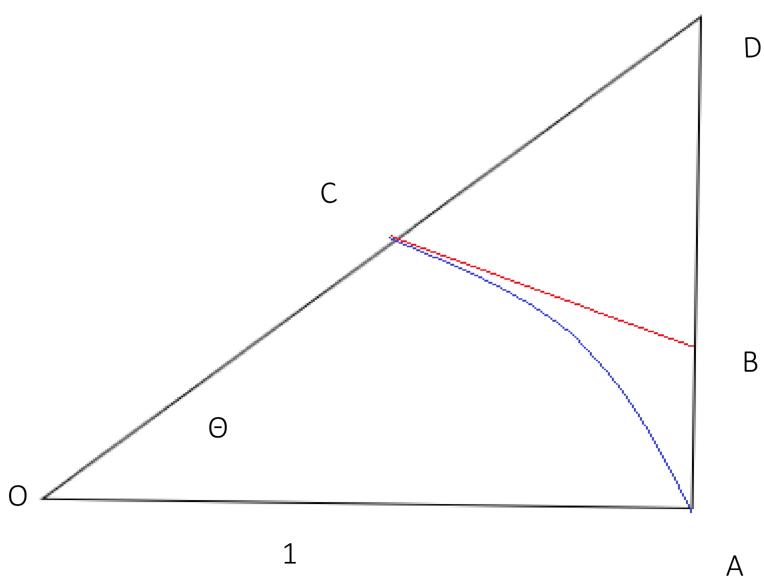

Figure 5.91

The triangle BCD is a right triangle with hypotenuse BD and so we know $|BC| < |BD|$. Also notice that $|AB| + |BD| = |AD|$. If we use these two facts in (1) we get,

$$\text{arc AC} < |\text{AB}| + |\text{BC}| < |\text{AB}| + |\text{BD}| = |\text{AD}| \qquad (2)$$

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As noted already the triangle *AOD* is a right triangle and so we can use a little right triangle trigonometry to write $|AD| = |AO| \tan \Theta$. Also note that $|AO| = 1$ since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$\text{arc } AC < |AD| < |AO| \tan \Theta = \tan \Theta. \quad (3)$$

Let's recall that the length of a portion of a circle is given by the radius of the circle times the angle (in radians) that traces out the portion of the circle we're trying to measure. For our portion this means that,

arc $AC = |AO|\Theta = \Theta$

So, putting this into (3) we see that

$$\theta = ark\ AC < \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ or } \cos\theta < \frac{\sin\theta}{\theta} \quad (4).$$

Let's connect *A* and *C* with a line and drop a line straight down from *C* until it intersects *AO* at a right angle and let's call the intersection point *E*.

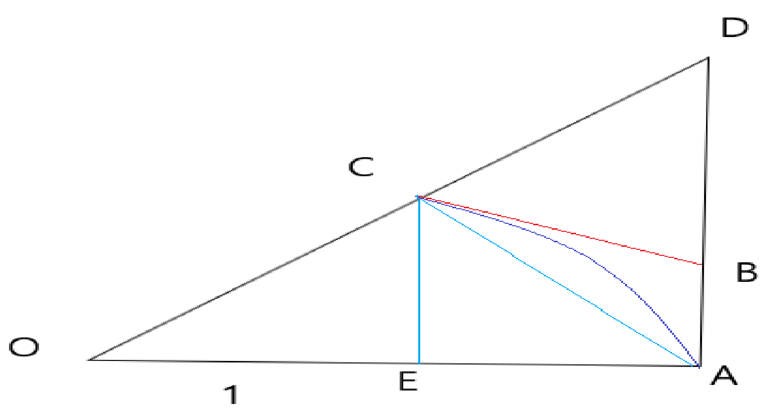

Figure 5.92

First thing to notice here is that,

$$|CE| < |AC| < \text{arc } AC. \quad (5)$$

Triangle *EOC* is a right triangle with a hypotenuse of $|CO| = 1$. Using some right triangles trig we can see that

$$|CE| = |CO| \sin \Theta = \sin \Theta.$$

Putting this into (5) and recalling that arc $AC = \Theta$ we get,

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sin Θ =|*CE*|< arc *AC* = *Θ*

and with a little rewriting we get

$$\frac{\sin\theta}{\theta} < 1. \qquad (6)$$

Putting (4) and (6) together we see that $\cos\theta < \frac{\sin\theta}{\theta} < 1$ provided $0 \le \frac{\sin\theta}{\theta} \le \frac{\pi}{2}$. Let's also note that,

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad \lim_{\theta\to 0} 1 = 1.$$

We are now set up to use the Squeeze Theorem. The only issue that we need to worry about is that we are staying to the right of θ =0

Squeeze Theorem will tell us that, $\lim\limits_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$

So, we know that the limit is true if we are only working with a right-hand limit. However, we know that sin Θ is an odd function and it means that,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

If we approach zero from the left (i.e., negative *Θ* 's) then we'll get the same values in the function as if we'd approached zero from the right (i.e., positive Θ's) and so,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1.$$

We have now shown that the two one-sided limits are the same and so we must also have,

$$\frac{\sin\theta}{\theta} = 1.$$

Example 5.68 Find the limit of the function $\lim\limits_{x\to 0}\frac{\sin 7x}{5x}$.

Solution: As you can see, the function under the limit is close to the first remarkable limit, but the limit of the function itself is not equal to one. In such tasks for the limits, it is necessary to select the variable in the denominator with the same coefficient that is contained in the variable under the sine. In this case, divide and multiply by 7.

$$\lim_{x\to 0}\frac{\sin 7x}{5x} = \lim_{x\to 0}\frac{7\sin 7x}{5\cdot 7x} = \frac{7}{5}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{5}.$$

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Example 5.69 : Find the limit of the function $\lim_{x\to 0}\frac{\sin 6x}{\tan 11x}$.

Solution: To understand the result, we write the function as $\frac{\sin 6x}{\tan 11x}=\frac{\sin 6x}{\frac{\sin 11x}{\cos 11x}}=\frac{\cos 11x\cdot\sin 6x}{\sin 11x}$.

To apply the wonderful limit rules, we multiply and divide by factors. Further, we write the limit of the product of functions in terms of the product of the limits.

$$\lim_{x\to 0}\frac{\sin 6x}{\tan 11x}=\frac{6}{11}\lim_{x\to 0}\cos 11x\cdot\frac{11x}{\sin 11x}\cdot\frac{\sin 6x}{6x}=\frac{6}{11}\cdot\lim_{x\to 0}\cos 11x\cdot\lim_{x\to 0}\frac{\sin 6x}{6x}\cdot\lim_{x\to 0}\frac{11x}{\sin 11x}=\frac{6}{11}\cdot 1\cdot 1\cdot 1=\frac{6}{11}$$

5.8.5. Consequences

1) $\lim_{x\to 0}\frac{\tan x}{x}=1$
2) $\lim_{x\to 0}\frac{\arcsin x}{x}=1$
3) $\lim_{x\to 0}\frac{\arctan x}{x}=1$
4) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

5.8.6. Exercises

1) $\lim_{x\to 0}\frac{\sin 2x+\sin 3x}{2x}$
2) $\lim_{x\to 0}\frac{\sin 2x-\sin x}{3x}$
3) $\lim_{x\to 0}\frac{\cos x-1}{x^2}$
4) $\lim_{x\to 0}\frac{10x-3\sin x}{x}$
5) $\lim_{x\to 0}\frac{\sin 2x}{x}$

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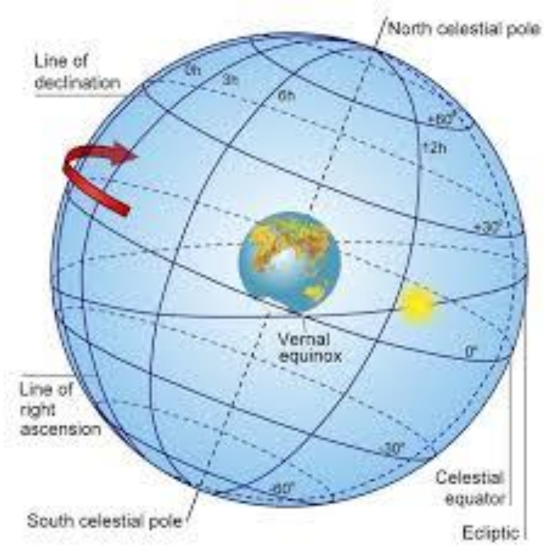
5.9. CURVES



Circles on the Celestial Sphere. Source: https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcQIjsr6eDKu3drpbHT50NRBz90kP62MTp_ZNQ&usqp=CAU

ABSTRACT:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. Figure 5.93 illustrates the four conic sections: the circle, the ellipse, the parabola, and the hyperbola.

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Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. Error! Reference source not found. illustrates the four conic sections: the circle, the ellipse, the parabola, and the hyperbola

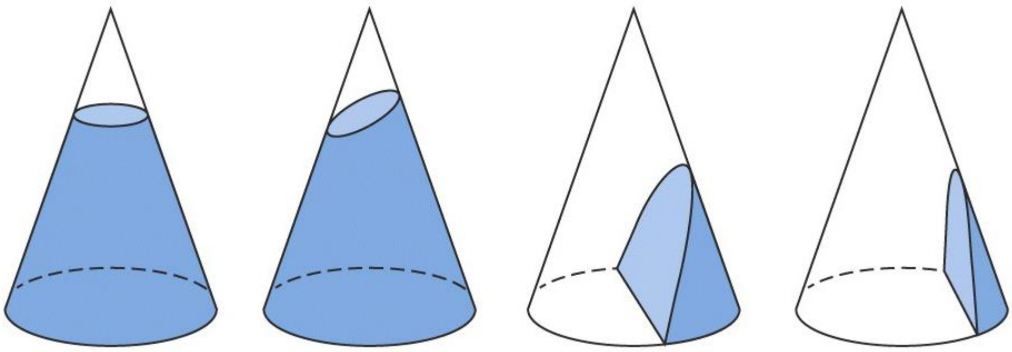

Figure 5.93 Obtaining the conic sections by intersecting a plane and a cone

5.9.1. The ellipse

OBJECTIVES AND OUTCOMES:

- Graph ellipses centred at the origin.
- Write equations of ellipses in standard form.
- Rewrite the equation of an ellipse in standard form.
- Graph an ellipse not centred at the origin.
- Solve application with ellipses.

CONTENTS

In this section, we study the symmetric oval-shaped curve known as the ellipse. In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. We will use a geometric definition for an ellipse to derive its equation.

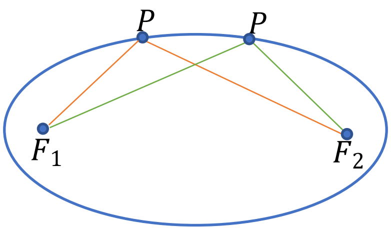

Figure 5.94 Ellipse

Definition:

An ellipse is the set of all points P, in a plane the sum of whose distances from two fixed points, F_1 and F_1 is constant (see ***Figure** 5.89*). These two fixed points are called the foci (plural of focus).

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The midpoint of the segment connecting the foci is the center of the ellipse.

Figure 5.90 illustrates that an ellipse can be elongated in any direction. In this section, we will limit our discussion to ellipses that are elongated horizontally or vertically. The line through the foci intersects the ellipse at two points, called the **vertices** (singular: **vertex**). The line segment that joins the vertices is the **major axis**. Notice that the midpoint of the major axis is the center of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis at the center is called the **minor axis** of the ellipse.

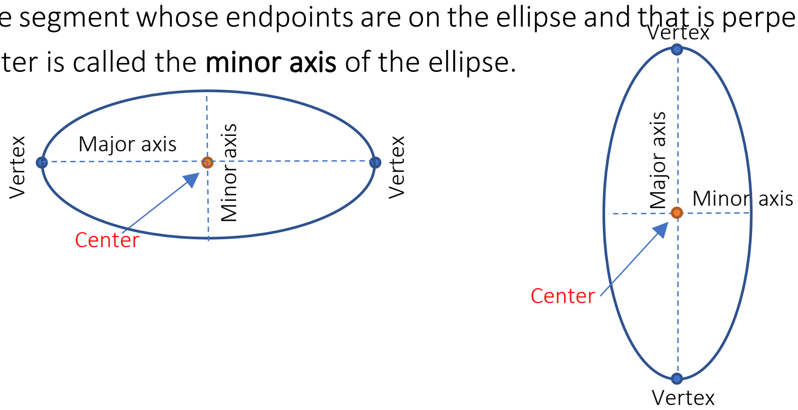

Figure 5.95 Horizontal and vertical elongations of an ellipse

Standard form of the equation of an ellipse

The rectangular coordinate system gives us a unique way of describing an ellipse. It enables us to translate an ellipse's geometric definition into an algebraic equation.

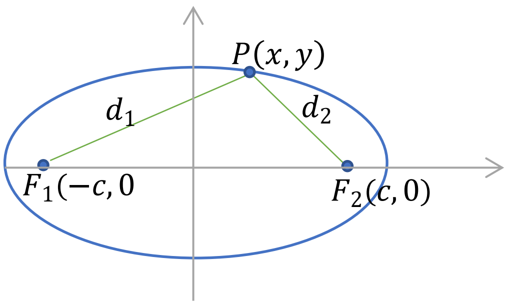

Figure 5.96 Ellipse

We start with **Figure 4** to obtain an ellipse's equation. We have placed an ellipse that is elongated horizontally into a rectangular coordinate system. The foci are on the $x - axis$ at $(-c, 0)$ and $(c, 0)$, as in **Figure 5.91.** In this way, the center of the ellipse is at the origin. We let represent the coordinates of any point on the ellipse. What does the definition of an ellipse tell us about the point in **Figure 5.91**?

For any point (x, y) on the ellipse, the sum of the distances to the two foci, $d_1 + d_2$, must be constant. As we shall see, it is convenient to denote this constant by $2a$. Thus, the point (x, y) is on the ellipse if and only if

$$d_1 + d_2, = 2a.$$

Use the distance formula

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

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After eliminating radicals and simplifying, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Look at the triangle in **Figure 5.91**. Notice that the distance from F_1 to F_2 is $2c$. Because the length of any side of a triangle is less than the sum of the lengths of the other two sides, $2c < d_1 + d_2$. Equivalently, $2c < 2a$ and $c < a$. Consequently, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Substituting b^2 for $a^2 - c^2$ in the preceding equation, we obtain

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}.$$

Then, simplifying, we get

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}.$$

This last equation is the **standard form of the equation of an ellipse centred at the origin**. There are two such equations, one for a horizontal major axis and one for a vertical major axis.

Standard forms of the equations of an ellipse

The **standard form of the equation of an ellipse** with center at the origin, and major and minor axes of lengths $2a$ and $2b$ (where and are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Figure 5.97 illustrates that the vertices are on the major axis, a units from the center. The foci are on the major axis, c units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$.

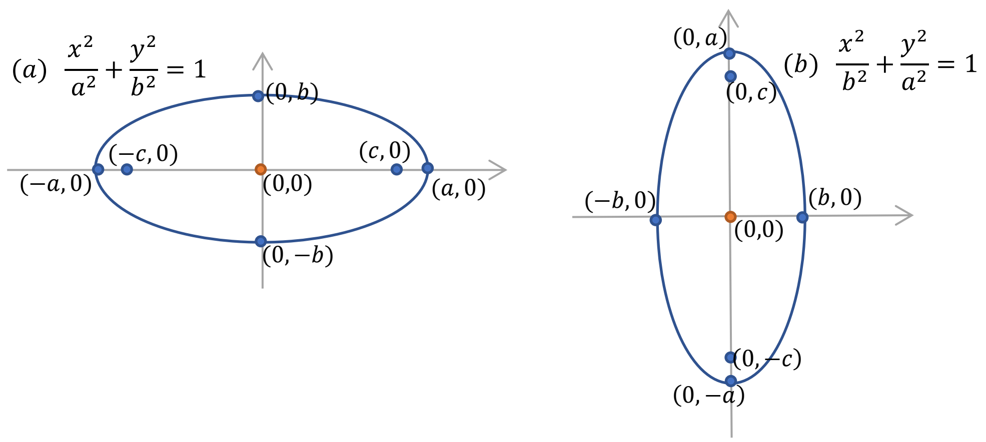

Figure 5.97 (a) Major axis is horizontal with length 2a; (b) Major axis is vertical with length 2a

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The intercepts shown in *Figure 5.97* (a) can be obtained algebraically. Let us do this for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

x –intercepts: Set $y=0$	y –intercepts: Set $x=0$
$\frac{x^2}{a^2}=1$ $x^2 = a^2$ $x=\pm a.$ x –intercepts are $-a$ and a.The graph passes through $(-a,0)$ and $(a,0)$, which are the vertices.	$\frac{y^2}{b^2}=1$ $y^2=b^2$ $y=\pm b.$ y –intercepts are $-b$ and b.The graph passes through $(0,-b)$ and $(0,\mathrm{b})$.

Using the standard form of the equation of an ellipse

We can use the standard form of an ellipse's equation to graph the ellipse. Although the definition of the ellipse is given in terms of its foci, the foci are not part of the graph. A complete graph of an ellipse can be obtained without graphing the foci.

Example 5.70 : Graphing an ellipse centered at the origin

Graph and locate the foci: $\frac{x^2}{9}+\frac{y^2}{4}=1.$

Solution: The given equation is the standard form of an ellipse's equation with $a^2=9$ (the larger of the two denominators) and $b^2=4$ (the smaller of the two denominators). Because the denominator of the x^2 –term is greater than the denominator of the y^2 –term, the major axis is horizontal.

- Based on the standard form of the equation, we know the vertices are $(-a,0)$ and $(a,0)$.
  Because $a^2=9, \mathrm{a}=3.$ Thus, the vertices are $(-3,0)$ and $(3,0)$, shown in Figure 6.
- Now let us find the endpoints of the vertical minor axis. According to the standard form of the equation, these endpoints are $(0,-b)$ and $(0,b)$. Because $b^2=4,\ \mathrm{b}=2.$ Thus, the endpoints of the minor axis are $(0,-2)$ and $(0,2)$. They are shown in ***Figure 5.98***.

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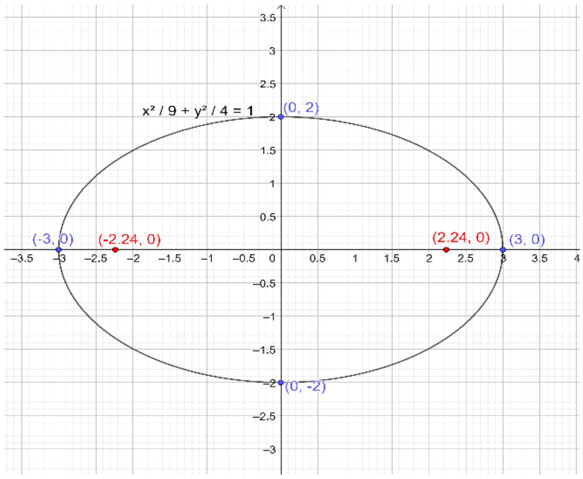

Figure 5.98 The graph of $\frac{x^2}{3^2}+\frac{y^2}{2^2}=\mathbf{1}$.

- Finally, we find the foci, which are located at $(-c,0)$ and $(c,0)$. We can use the formula $c^2=a^2-b^2$ to do so. We know that $a^2=9$ and $b^2=4$. Thus,

$$c^2=a^2-b^2=9-4=5.$$

- Because $c^2=5, c=\sqrt{5}$. The foci, $(-c,0)$ and $(c,0)$ are located at $(-\sqrt{5},0)$ and $(\sqrt{5},0)$. They are shown in ***Figure 5.98***.
- You can sketch the ellipse in ***Figure 5.98*** by locating endpoints on the major and minor axes.

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1.$$

Endpoints of the major axis are 3 units to the right and left of the center. Endpoints of the minor axis are 2 units up and down from the center.

Example 5.71 : Graphing an ellipse centered at the origin

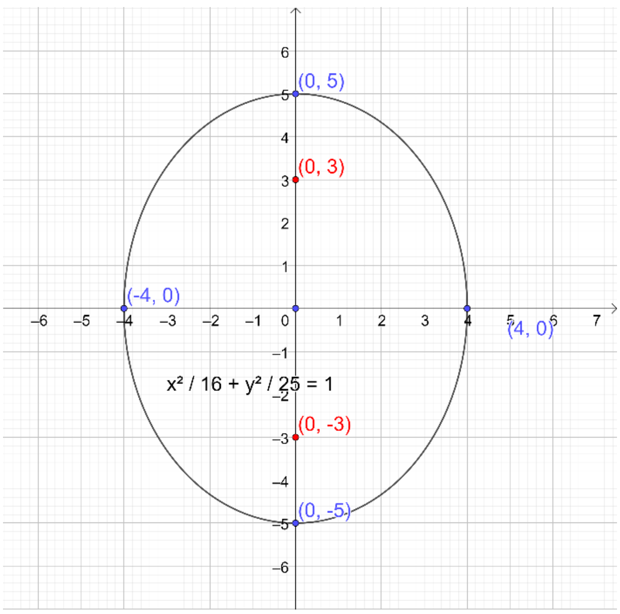

Figure 5.99 The graph of $\mathbf{25x^2+16y^2=400}$.

Graph and locate the foci: $25x^2+16y^2=400$.

<u>Solution:</u> We begin by expressing the equation in standard form. Because we want 1 on the right side, we divide both sides by 400.

$$\frac{25x^2}{400}+\frac{16y^2}{400}=\frac{400}{400};$$

$$\frac{x^2}{16}+\frac{y^2}{25}=1.$$

$b^2=16$ – this is the smaller of the two denominators; $a^2=25$ – this is the larger of the two denominators.

- The equation is the standard form of an ellipse's equation with $a^2=25$ and $b^2=16$. Because the denominator of the y^2- term is greater than the denominator of the x^2- term, the major axis is vertical. Based on the standard form of the equation, we know the vertices are $(0,-a)$ and $(0,a)$. Because $a^2=25$, $a=5$. Thus, the vertices are $(0,-5)$ and $(0,5)$, shown in ***Figure 5.94***.

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Now let us find the endpoints of the horizontal minor axis. According to the standard form of the equation, these endpoints are $(-b, 0)$ and $(b, 0)$. Because $b^2 = 16, \mathrm{b} = 4$. Thus, the endpoints of the minor axis are $(-4, 0)$ and $(4, 0)$. They are shown in ***Figure 5.94***.

- Finally, we find the foci, which are located at $(0, -c)$ and $(0, c)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 25$ and $b^2 = 16$. Thus,
- $c^2 = a^2 - b^2 = 25 - 16 = 9$.
- Because $c^2 = 9, c = 3$. The foci, $(0, -c)$ and $(0, c)$ are located at $(0, -3)$ and $(0,3)$. They are shown in ***Figure 5.94***.
- You can sketch the ellipse in Figure 7 by locating endpoints on the major and minor axes.

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1.$$

Endpoints of the minor axis are 4 units to the right and left of the center. Endpoints of the major axis are 5 units up and down from the center.

Example 5.72 : Finding the equation of an ellipse from its foci and vertices

Find the standard form of the equation of an ellipse with foci at $(-1,0)$ and $(1,0)$ and vertices $(-2,0)$ and $(2,0)$.

Solution: Because the foci are located at $(-1,0)$ and $(1, 0)$, on the x-axis, the major axis is horizontal. The center of the ellipse is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, ti either vertex, $(-2,0)$ or $(2,0)$,
is 2. Thus, $a = 2$.

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1 \quad or \quad \frac{x^2}{4} + \frac{y^2}{b^2} = 1.$$

We must still find b^2. The distance from the center, $(0, 0)$, to either foci $(-1,0)$ or $(1,0)$, is 1, so $c = 1$. Using $c^2 = a^2 - b^2$, we have

$$1^2 = 2^2 - b^2$$

and

$$b^2 = 2^2 - 1^2 = 3.$$

Substituting 3 for b^2 in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives us the standard form of the ellipse's equation

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Translations of ellipses

Horizontal and vertical translations can be used to graph ellipses that are not centered at the origin. ***Figure 5.95*** illustrates that the graphs of

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$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Have the same size and shape. However, the graph of the first equation is centered at (h,k) rather than at the origin.

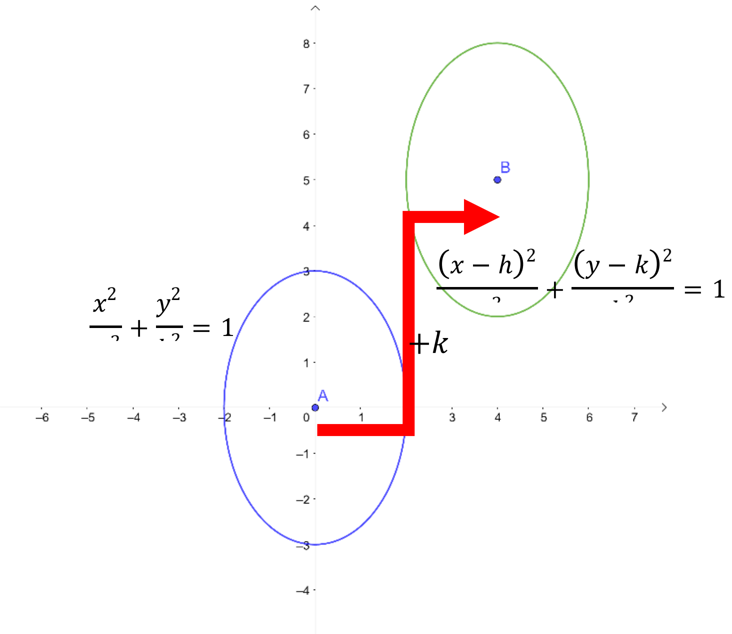
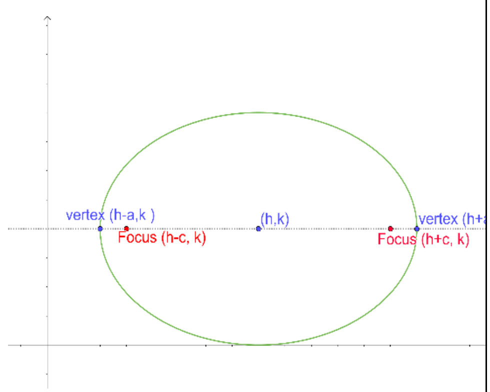


Figure 5.100 Translations an ellipse's graph

Table 1 gives the standard forms of equations of ellipses centered at $(h,k)$and shows their graphs.

EQUATION	CENTER	GRAPH
Equation: $$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ $$a^2>b^2$$ Foci are c units right and c units left of center, where $c^2=a^2-b^2$.	**Center**:(h,k) **Major axis**: Parallel to the x-axis, horizontal **Vertices**: $(h-a,k)$ $(h+a,k)$	

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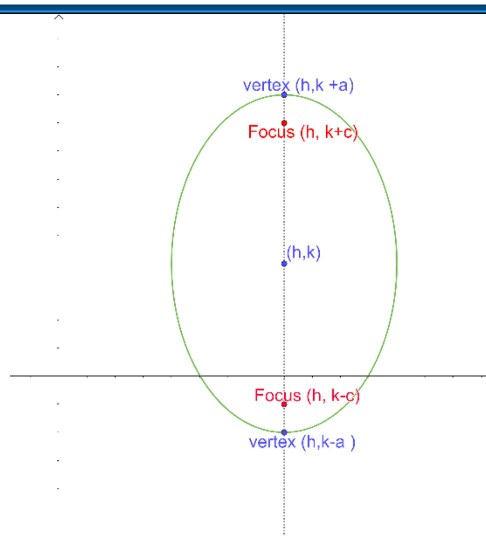
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ $$a^2>b^2$$ Foci are c units above and c units below of center, where $c^2 = a^2 - b^2$.	Center: (h,k) Major axis: Parallel to the y-axis, vertical Vertices: $(h,k-a)$ $(h,k+a)$	

Table 1 Standard Forms of Equations of Ellipses Centered at (h,k)

Example 5.73 : Graphing an ellipse centered at (h,k)

Graph:

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

Where are the foci located?

Solution: To graph the ellipse, we need to know its center, (h,k). In the standard forms of equations centered at (h,k) is the number subtracted from x and is the number subtracted from y.

$$\frac{(x-1)^2}{4}+\frac{\left(y-(-2)\right)^2}{9}=1.$$

We see that $h=1$ and $k=-2$. Thus, the center of the ellipse, (h,k) is $(1,-2)$. We can graph the ellipse by locating endpoints on the major and minor axes. To do this, we must identify a^2 and b^2.

$$b^2=4-\ \ the\ smaller\ of\ the\ two\ deniminators;$$

$$a^2=9-\ \ the\ larger\ of\ the\ two\ deniminators.$$

The larger number is under the expression involving y. This means that the major axis is vertical and parallel to the y −axis.

We can sketch the ellipse by locating endpoints on the major and minor axes.

$$\frac{(x-1)^2}{2^2}+\frac{(y+2)^2}{3^2}=1.$$

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We categorize the observations in the textboxes above as follows:

For a Vertical Major Axis with Center $(1,-2)$	
Vertices	Endpoints of Minor Axis
$(1,-2+3)=(1,1)$	$(1+2,-2)=(3,1)$
$(1,-2-3)=(1,-5)$	$(1-2,-2)=(-1,1)$

Using the center and these four points, we can sketch the ellipse shown in Figure 5.101. With $c^2 = a^2 - b^2$ we have $c^2 = 9 - 4 = 5.$ So the foci are located $\sqrt{5}$ units above and below the center, at $\left(1,-2+\sqrt{5}\right)$ and $\left(1,-2-\sqrt{5}\right)$.

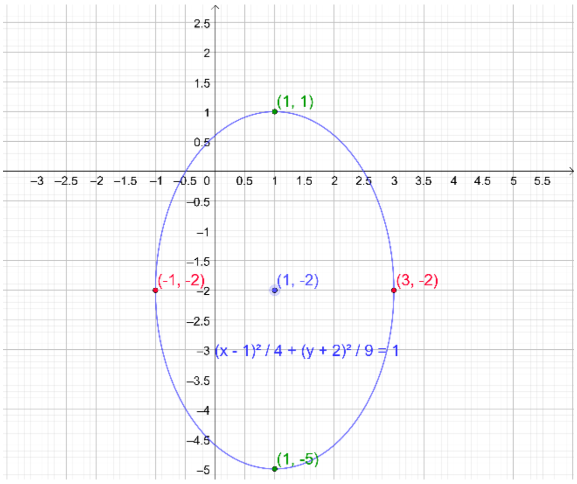

Figure 5.101 The graph of an ellipse centered at (1, -2)

Remark

In some cases, it is necessary to convert the equation of an ellipse to standard form by completing the square on x and y. For example, suppose that we wish to graph the ellipse whose equation is

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

Because we plan to complete the square on both on x and y, we need to rearrange terms so that

- x −terms are arranged in descending order;
- y −terms are arranged in descending order;
- The constant term appears on the right.

This is the given equation

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0$$

Group terms and add 11 to both sides

$$9x^2 - 18x + 4y^2 + 16y = 11$$

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To complete the square, coefficients of $x^2$and y^2 must be 1. Factor out 9 and 4, respectively

$$9(x^2 - 2x + ?) + 4(y^2 + 4y + ?) = 11.$$

Complete each square by adding the square of half the coefficient of x and y, respectively.

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16.$$

Factor

$$9(x-1)^2 + 4(y+2)^2 = 36.$$

Divide both sides by 36

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = \frac{36}{36}.$$

Simplifying

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1.$$

The equation is now in standard form. This is precisely the form of the equation that we graphed in Example 5.71.

APPLICATIONS

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. German scientist Johannes Kepler (1571–1630) showed that the planets in our solar system move in elliptical orbits, with the sun at a focus. Earth satellites also travel in elliptical orbits, with Earth at a focus. One intriguing aspect of the ellipse is that a ray of light or a sound wave emanating from one focus will be reflected from the ellipse to exactly the other focus. A whispering gallery is an elliptical room with an elliptical, dome-shaped ceiling. People standing at the foci can whisper and hear each other quite clearly, while persons in other locations in the room cannot hear them. Statuary Hall in the U.S. Capitol Building is elliptical. President John Quincy Adams, while a member of the House of Representatives, was aware of this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, easily eavesdropping on the *private conversations of other House members located near the other focus.*

The elliptical reflection principle is used in a procedure for disintegrating kidney stones. The patient is placed within a device that is elliptical in shape. The patient is placed so the kidney is centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone. The convergence of the ultrasound waves at the kidney stone causes vibrations that shatter it into fragments. The small pieces can then be passed painlessly through the patient's system. The patient recovers in days, as opposed to up to six weeks if surgery is used instead.

Ellipses are often used for supporting arches of bridges and in tunnel construction.

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Example 5.74 : An application involving an elliptical orbit

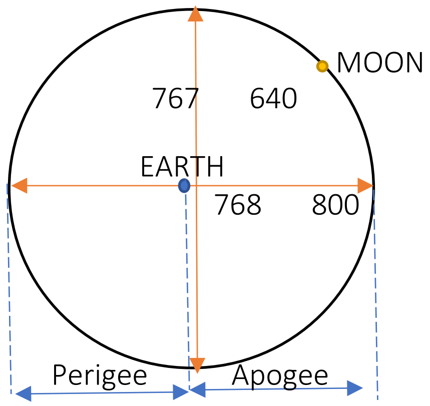

Figure 5.102

The Moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in **Figure 10**. The major and minor axes of the orbit have lengths of 768,800 kilometres and 767,640 kilometres, respectively. Find the greatest and smallest distances (the apogee and perigee), respectively from Earth's center to the Moon's center.

Solution: Note that Earth is not the center of the Moon's orbit.

Because $2a = 768\,800$ and $2b = 767\,640$, we have $a = 384\,400$ and $b = 383\,820$ which implies that

$$c^2 = a^2 - b^2 = (384\,400\,)^2 - (383\,820)^2$$
$$= 445\,567\,600.$$

Because $c^2 = 445\,567\,600$; $c = \sqrt{445\,567\,600} \approx 21\,108.$

So, the greatest distance between the center of Earth and the center on the Moon is

$$a + c = 384\,400 + 21\,108 = 405\,508\ kilometers.$$

And the smallest distance is

$$a - c = 384\,400 - 21\,108 = 363\,292\ kilometers.$$

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of eccentricity.

Definition:

The eccentricity of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

Note that $0 < e < 1$ for every ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

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For an ellipse that is nearly circular, the foci are close to the center and the ratio $\frac{c}{a}$ is small, as shown in ***Figure 5.103a***. On the other hand, for an elongated ellipse, the foci are close to the vertices, and the ratio $\frac{c}{a}$ is close to 1, as shown in ***Figure 5.103*b**.

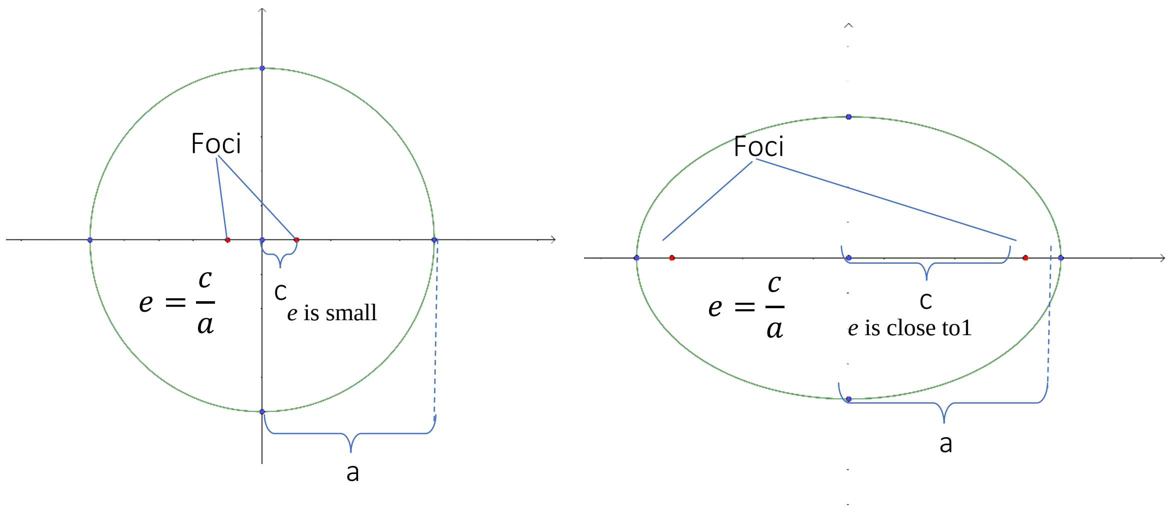


Figure 5.103 *a)* *b)*

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the nine planetary orbits are as follows.

Mercury: $e \approx 0.2056$, Saturn: $e \approx 0.0542$,

Venus: $e \approx 0.0068$, Uranus: $e \approx 0.0472$,

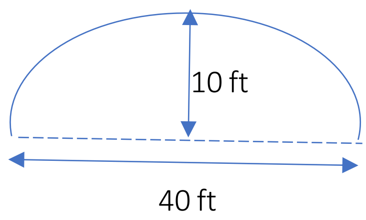
Earth: $e \approx 0.0167$, Neptune: $e \approx 0.0086$,

Mars: $e \approx 0.0934$, Pluto: $e \approx 0.2488$,

Jupiter: $e \approx 0.0484$.

Example 5.75 : An application involving an ellipe

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet (see **Figure 12**). Your truck has a width of 10 feet and a height of 9 feet. Will your truck clear the opening of the archway?



Solution: Because your truck's width is 10 feet, to determine the clearance, we must find the height of the archway 5 feet from the center. If that height is 9 feet or less, the truck will not clear the opening.

In **Figure 5.99**, we've constructed a coordinate system with the x −axis on the ground and the origin at the center of the archway. Also shown is the truck, whose height is 9 feet.

Figure 5.104 A semielliptical archway

Using the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we can experess the equation of the blue archway in **Figure 12** as

$$\frac{x^2}{20^2}+\frac{y^2}{10^2}=1$$

or

$$\frac{x^2}{400}+\frac{y^2}{100}=1.$$

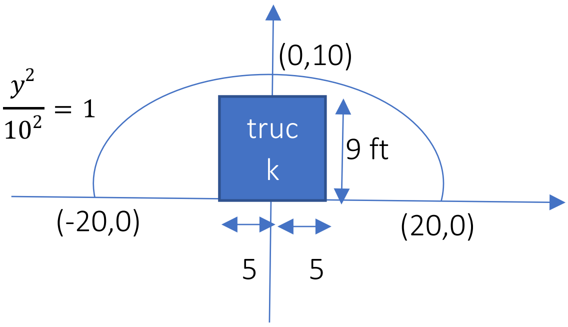


Figure 5.105

As shown in **Figure 16**, the edge of the 10-foot-wide truck corresponds to $x=5$. We find the height of the archway 5 feet from the center by substituting 5 for x and solving for y.

$$\frac{5^2}{400}+\frac{y^2}{100}=1.$$

Square 5

$$\frac{25}{400}+\frac{y^2}{100}=1.$$

Clear fractions by multiplying both sides by 400

$$400\left(\frac{25}{400}+\frac{y^2}{100}\right)=400\cdot 1.$$

Using the distributive property and simplifying we get

$$25+4y^2=400,$$

then subtracting 25 from both sides

$$4y^2=375.$$

Deviding both sides by 4 and taking onli the positive square root (the archway is above the x −axis, so y is nonnegative.

$$y^2=\frac{375}{4}; \qquad y=\sqrt{\frac{375}{4}}\approx 9.68.$$

Thus, the height of the archway 5 feet from the center is approximately 9.68 feet. Because truck's height is 9 feet, there is enough room for the truck to clear the archway.

PRACTICE EXERCISES

Graph each ellipse and locate the foci.

1. $\frac{x^2}{16}+\frac{y^2}{4}=1;$

2. $\frac{x^2}{9}+\frac{y^2}{36}=1;$

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3. $\frac{x^2}{25}+\frac{y^2}{64}=1;$
4. $\frac{x^2}{49}+\frac{y^2}{81}=1;$
5. $\frac{x^2}{\frac{9}{4}}+\frac{y^2}{\frac{25}{4}}=1;$
6. $x^2=1-4y^2;$
7. $25x^2+4y^2=100;$
8. $5x^2+16y^2=64;$
9. $7x^2=35-5y^2;$
10. $\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{25}{16}}=1.$

Find the standard form of the equation of each ellipse and give the location of its foci.

11. $.2x^2+9y^2+16x-90y+239=0;$
12. $5x^2+y^2-3x+40=0;$
13. $x^2+4y^2+10x-16y+25=0;$
14. $36x^2+4y^2-4y-44=0;$
15. $16x^2+100y^2+64x-300y-111=0;$
16. $2x^2+3y^2-4x-5y+1=0.$

Find the standard form of the equation of each ellipse satisfying the given conditions.

17. Foci: $(-5,0)$, $(5,0)$; vertices: $(-8,0)$, $(8,0)$;
18. Foci: $(-2,0)$, $(2,0)$; vertices: $(-6,0)$, $(6,0)$;
19. Foci: $(0,-4)$, $(0,4)$; vertices: $(0,-7)$, $(0,7)$;
20. Foci: $(0,-3)$, $(0,3)$; y –interscepts: -3 and 3;
21. Foci: $(0,-2)$, $(0,2)$; y –interscepts: -2 and 2;
22. Major axis horizontal with length 8; length of minor axis – 4; center: $(0,0)$;
23. Major axis horizontal with length 12; length of minor axis – 6; center: $(0,0)$;
24. Major axis vertical with length 10; length of minor axis – 4; center: $(-2,3)$;
25. Major axis vertical with length 20; length of minor axis – 10; center: $(2,-3)$;
26. Endpoints of major axis: $(7,9)$ and $(7,3)$; Endpoints of minor axis: $(5,6)$ and $(9,6)$;
27. Endpoints of major axis: $(2,2)$ and $(8,2)$; Endpoints of minor axis: $(5,3)$ and $(5,1)$.

Graph each ellipse and give the location of its foci

28. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1;$
29. $\frac{(x-1)^2}{16}+\frac{(y+2)^2}{9}=1;$
30. $(x+3)^2+4(y-2)^2=16;$
31. $(x-3)^2+9(y+2)^2=18;$
32. $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{25}=1;$
33. $\frac{(x-3)^2}{9}+\frac{(y+1)^2}{16}=1;$
34. $\frac{x^2}{25}+\frac{(y-2)^2}{36}=1;$
35. $\frac{(x-4)^2}{4}+\frac{y^2}{25}=1;$

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36. $9(x-1)^2 + 4(y+3)^2 = 36$;
37. $36(x+4)^2 + (y+3)^2 = 36$.

Convert each equation to standard form by completing the square on x and y. Then graph the ellipse and give the location of its foci.

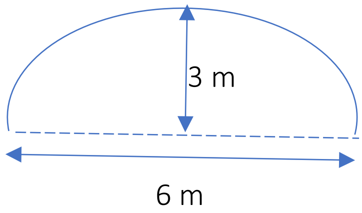
38. $9x^2 + 25y^2 - 36x + 50y - 164 = 0$;
39. $4x^2 + 9y^2 - 32x + 36y + 64 = 0$;
40. $9x^2 + 16y^2 - 18x + 64y - 71 = 0$;
41. $x^2 + 4y^2 + 10x - 8y + 13 = 0$;
42. $4x^2 + y^2 + 16x - 6y - 39 = 0$;
43. $4x^2 + 25y^2 - 24x + 100y + 36 = 0$.

Find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

44. $\begin{cases} x^2 + y^2 = 1 \\ x^2 + 9y^2 = 9 \end{cases}$;
45. $\begin{cases} x^2 + y^2 = 25 \\ 25x^2 + y^2 = 25 \end{cases}$;
46. $\begin{cases} x^2 + y^2 = 25 \\ 25x^2 + y^2 = 25 \end{cases}$;
47. $\begin{cases} \frac{x^2}{25} + \frac{y^2}{9} = 1 \\ y = 3 \end{cases}$;
48. $\begin{cases} 4x^2 + y^2 = 4 \\ 2x - y = 2 \end{cases}$;
49. $\begin{cases} \frac{x^2}{4} + \frac{y^2}{36} = 1 \\ x = -2 \end{cases}$;
50. $\begin{cases} 4x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$.

Application exercises

51. Will a truck that is 2.4 m wide carrying a load that reaches 2.1 m feet above the ground clear the semielliptical arch on the one-way road that passes under the bridge shown in the figure?



52. A semielliptic archway has a height of 6 m and a width of 15 m. Can a truck 4 m high and 3 m wide drive under the archway without going into the other lane?
53. If an elliptical whispering room has a height of 6 m and a width of 30 m, where should two people stand if they would like to whisper back and forth and be heard?
54. A semielliptical arch over a tunnel for a one-way road through a mountain has a major axis of 15 m and a height at the center of 3 m.
    - Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.
    - Find an equation of the semielliptical arch over the tunnel.
    - You are driving a moving truck that has a width of 2.4 m and a height of 2,7 m. Will the moving truck clear the opening of the arch?
55. Halley's comet has an elliptical orbit, with the sun at one focus. The eccentricity of the orbit is approximately 0.967. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.)
    - Find an equation of the orbit. Place the center of the orbit at the origin, and place the major axis on the x −axis.
    - Use a graphing utility to graph the equation of the orbit.
    - Find the greatest (aphelion) and smallest (perihelion) distances from the sun's center to the comet's center.
56. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 50 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

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5.9.2. The hyperbola

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane.

OBJECTIVES AND OUTCOMES:

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Rewrite the equation of a hyperbola in standard form.
- Solve applied problems involving hyperbolas.

a. CONTENTS

Figure 5.106 The hyperbolic shadows

In this section, we study the curve with two parts known as the hyperbola. In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called branches.

Figure 5.101 shows a cylindrical lampshade casting two shadows on a wall. These shadows indicate the distinguishing feature of hyperbolas. Although each branch might look like a parabola, its shape is actually quite different.

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Definition:

A hyperbola is the set of points in a plane the difference of whose distances from two fixed points, called foci, is constant.

Figure 5.107 illustrates the two branches of a hyperbola. The line through the foci intersects the hyperbola at two points, called the vertices. The line segment that joins the vertices is the transverse axis. The midpoint of the transverse axis is the center of the hyperbola. Notice that the center lies midway between the vertices, as *well as midway between the foci.*

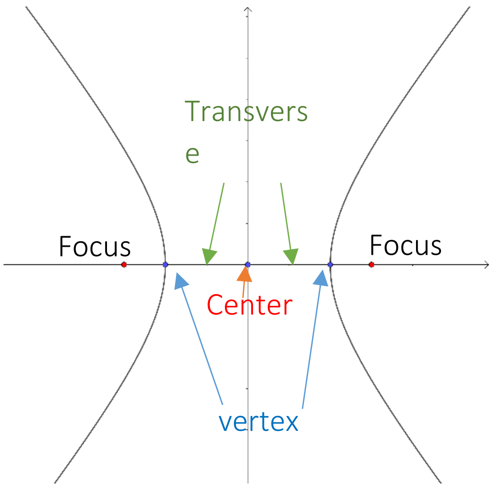

Figure 5.107 The two branches of a hyperbola

Standard form of the equation of a hyperbola

The rectangular coordinate system enables us to translate a hyperbola's geometric definition into an algebraic equation. **Figure 3** is our starting point for obtaining an equation. We place the foci, F_1 and F_2 on the x −axis at the points $(-c, 0)$ and $(c, 0)$. Note that the center of this hyperbola is at the origin. We let (x, y) represent the coordinates of any point, P, on the hyperbola.

For any point (x, y) on the hyperbola, the absolute value of the difference of the distances from the two foci, $|d_2 - d_1|$, must be constant. We denote this constant by $2a$ just as we did for the ellipse. Thus, the point (x, y) is on the hyperbola if and only if

$$|d_2 - d_1| = 2a.$$

Using distance formula

$$\left|\sqrt{(x+c)^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2}\right| = 2a.$$

After eliminating radicals and simplifying, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

For convenience, let $b^2 = c^2 - a^2$. Substituting b^2 for $c^2 - a^2$ in the preceding equation, we obtain

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$$b^2x^2 - a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2 give us

$$\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This last equation is called the standard form of the equation of a hyperbola centered at the origin. There are two such equations.

The first is for a hyperbola in which the transverse x −axis lies on the second is for a hyperbola in which the transverse axis lies on the y −axis.

Standard forms of the equations of a hyperbola

The standard form of the equation of a hyperbola with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad or \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Figure 5.103a illustrates that for the equation on the left, the transverse axis lies on the $x-$ axis. **Figure 5.103b** illustrates that for the equation on the right, the transverse axis lies on the $y-$ axis. The vertices are a units from the center and the foci are c units from the center. For both equations, $b^2 = c^2 - a^2$. Equivalently, $c^2 = a^2 + b^2$.

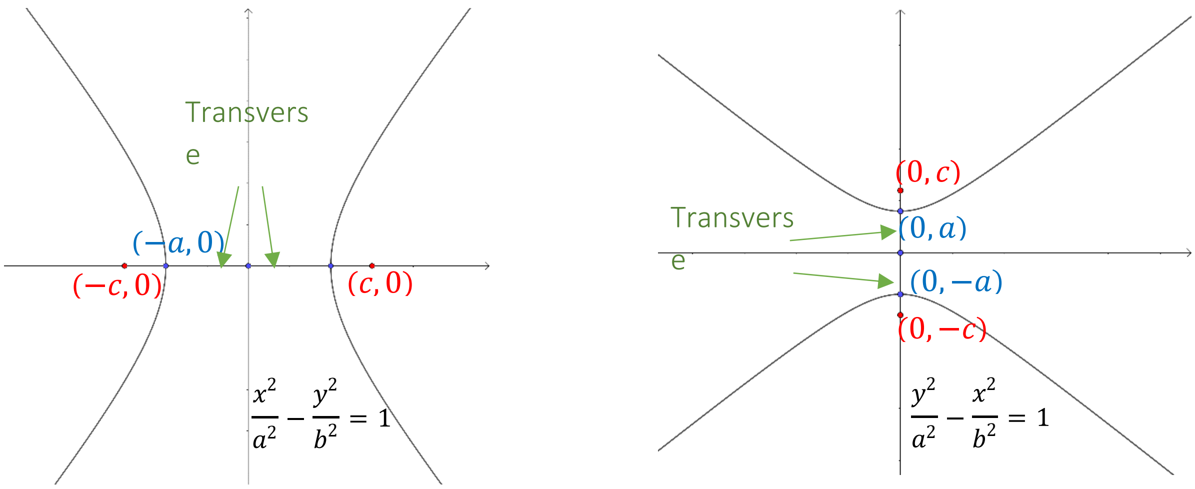

Figure 5.108 a) Transverse axis lies on the x-axis; b) Transverse axis lies on the y-axis

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Using the standard form of the equation of a hyperbola

We can use the standard form of the equation of a hyperbola to find its vertices and locate its foci. Because the vertices are a units from the center, begin by identifying a^2 in the equation. In the standard form of a hyperbola's equation, a^2 is the number under the variable whose term is preceded by a plus sign. If the x^2 −term is preceded by a plus sign, the transverse axis lies along the x −axis. Thus, the vertices are a units to the left and right of the origin. If the y^2 −term is preceded by a plus sign, the transverse axis lies along the y −axis. Thus, the vertices are a units above and below the origin.

We know that the foci are c units from the center. The substitution that is used to derive the hyperbola's equation, $c^2 = a^2 + b^2$, is needed to locate the foci when a^2 and b^2 are known.

Example 5.76 : FINDING VERTICES AND FOCI FROM A HYPERBOLA'S EQUATION

Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Solution: Both equations are in standard form. We begin by identifying a^2 and b^2 in each equation.

a. The first equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

b. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Because the x^2 −term is preceded by a plus sign, the transverse axis lies along the x −axis. Thus, the vertices are a units to the left and right of the origin. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 16, a = 4.$ Thus, the vertices are $(-4,0)$ and $(4, 0)$, shown in Figure 5.109**a**.

We use $c^2 = a^2 + b^2$, to find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We know that $a^2 = 16$ and $b^2 = 9$; we need to find c^2 in order to find c.

Because $c^2 = 25, c = 5.$ The foci are located at $(-5,0)$ and $(5, 0)$. They are shown in Figure 5.109**a**.

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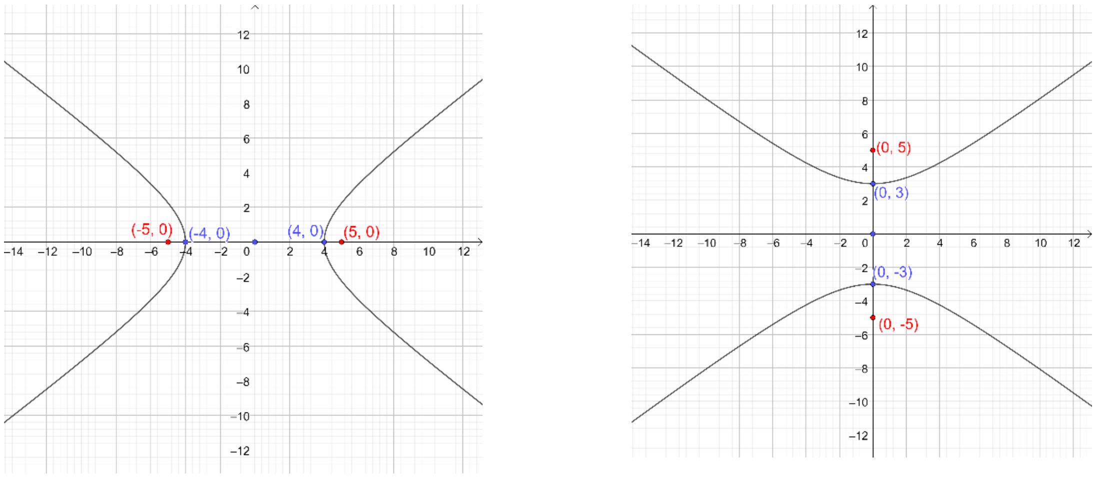

Figure 5.109 *a) The graph of* $\frac{x^2}{16}-\frac{y^2}{9}=1$ *b) The graph of* $\frac{y^2}{9}-\frac{x^2}{16}=1$

The second given equation is in the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

c. $\frac{y^2}{9}-\frac{x^2}{16}=1$

Because the y^2 −term is preceded by a plus sign, the transverse axis lies along the y −axis. Thus, the vertices are a units above and below the origin. Based on the standard form of the equation, we know the vertices are $(0,-a)$ and $(0,a)$. Because $a^2=9, a=3$. Thus, the vertices are $(0,-3)$ and $(0,3)$, shown in Figure 5.109**b**.

We use $c^2=a^2+b^2$, to find the foci, which are located at $(0,-c)$ and $(0,c)$. We know that $a^2=9$ and $b^2=16$; we need to find c^2 in order to find c.

Because $c^2=25, c=5$. The foci are located at $(0,-5)$ and $(0,5)$. They are shown in Figure 5.109**b.** .

In Example 5.76, we used equations of hyperbolas to find their foci and vertices. In the next example, we reverse this procedure.

Example 5.77 : FINDING THE EQUATION OF A HYPERBOLA FROM ITS FOCI AND VERTICES

Find the standard form of the equation of a hyperbola with foci at $(-3,0)$ and $(0,3)$ and vertices $(0,-2)$ and $(0,2)$, shown in **Figure 5.110**.

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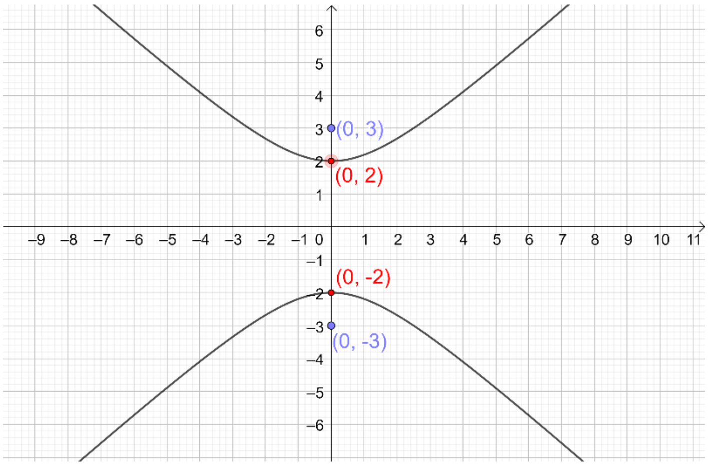

Figure 5.110

Solution: Because the foci are located at $(-3,0)$ and $(0,3)$, on the y –axis the transverse axis lies on the y –axis. The center of the hyperbola is midway between the foci, located at $(0,0)$.Thus, the form of the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0,0)$, to either vertex, $(0,-2)$ or $(0,2)$, is 2, so $a = 2$.

$$\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1 \quad or \quad \frac{y^2}{4} - \frac{x^2}{b^2} = 1.$$

We must still find b^2. The distance from the center, $(0,0)$, to either focus, $(-3,0)$ or $(0,3)$, is 3. Thus, $c = 3$. Using $c^2 = a^2 + b^2$, we have

$$3^2 = 2^2 + b^2$$

and

$$b^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting 5 for b^2 in $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ gives us the standard form of the hyperbola's equation. The equation is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1.$$

The asymptotes of a hyperbola

As and get larger, the two branches of the graph of a hyperbola approach a pair of intersecting straight lines, called asymptotes. The asymptotes pass through the center of the hyperbola and are helpful in graphing hyperbolas.

Figure 5.111 shows the asymptotes for the graphs of hyperbolas centered at the origin. The asymptotes pass through the corners of a rectangle. Note that the dimensions of this rectangle are $2a$ by $2b$. The line segment of length $2b$ is the conjugate axis of the hyperbola and is perpendicular to the transverse axis through the center of the hyperbola.

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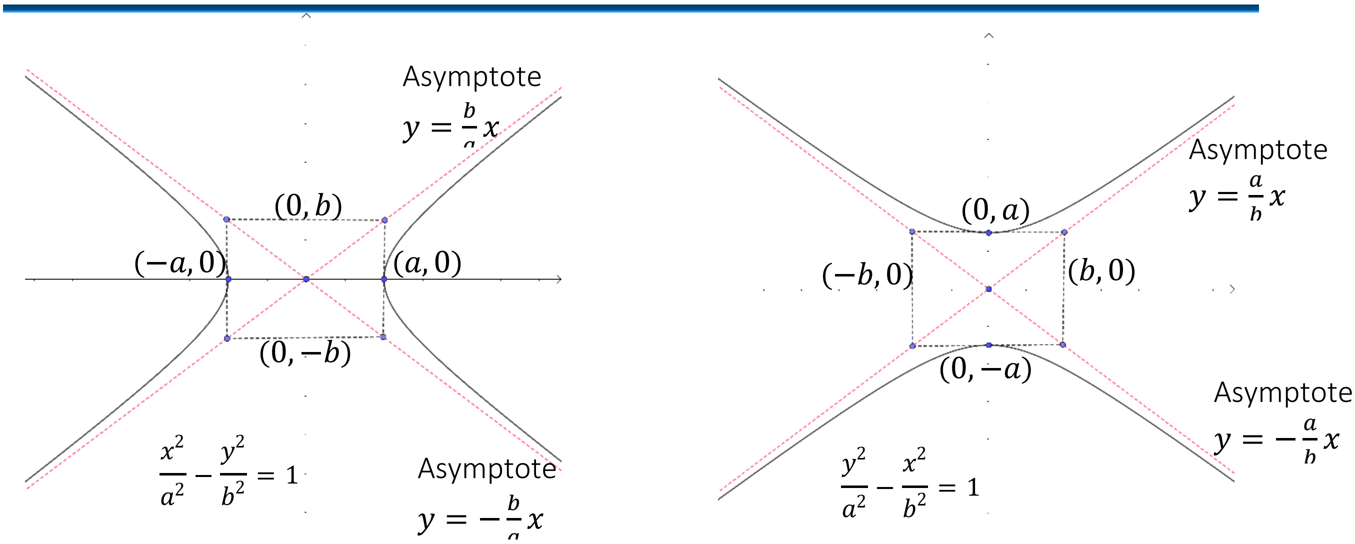

Figure 5.111 Asymptotes of a hyperbola

The asymptotes of a hyperbola centered at the origin

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a horizontal transverse axis and two asymptotes

$$y = -\frac{b}{a}x$$

and

$$y = \frac{b}{a}x.$$

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has a vertical transverse axis and two asymptotes

$$y = -\frac{a}{b}x$$

and

$$y = \frac{a}{b}x.$$

Graphing hyperbolas centered at the origin

Hyperbolas are graphed using vertices and asymptotes.

Graphing Hyperbolas:

1. Locate the vertices.
2. Use dashed lines to draw the rectangle centered at the origin with sides parallel to the axes, crossing one axis at $\pm a$ and the other at $\pm b$.
3. Use dashed lines to draw the diagonals of this rectangle and extend them to obtain the asymptotes.
4. Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

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Example 5.78 : GRAPHING A HYPERBOLA

Graph and locate the foci: $\frac{x^2}{25} - \frac{y^2}{16} = 1.$ What are the equations of the asymptotes?

Solution:

1. ***Locate the vertices.*** The given equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ with $a^2 = 25$ and $b^2 = 16.$ Based on the standard form of the equation with the transverse axis on the x −axis, we know that the vertices are $(-a, 0)$ and $(a, 0).$ Because $a^2 = 25$ $a = 5.$ Thus, the vertices are $(-5,0)$ and $(5, 0)$, shown in **Figure 7a**.
2. ***Draw a rectangle.*** Because $a^2 = 25$ and $b^2 = 16,$ $a = 5$ and $b = 4.$ We construct a rectangle to find the asymptotes, using -5 and 5 on the x −axis (the vertices are located here) and -4 and 4 on the y −axis. The rectangle passes through these four points, shown using dashed lines in **Figure 7a.**
3. ***Draw extended diagonals for the rectangle to obtain the asymptotes.*** We draw dashed lines through the opposite corners of the rectangle, shown in *Figure 5.112* **a**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are

   d. $y = \pm\frac{b}{a}x$

   e. or

   f. $y = \pm\frac{4}{5}x.$

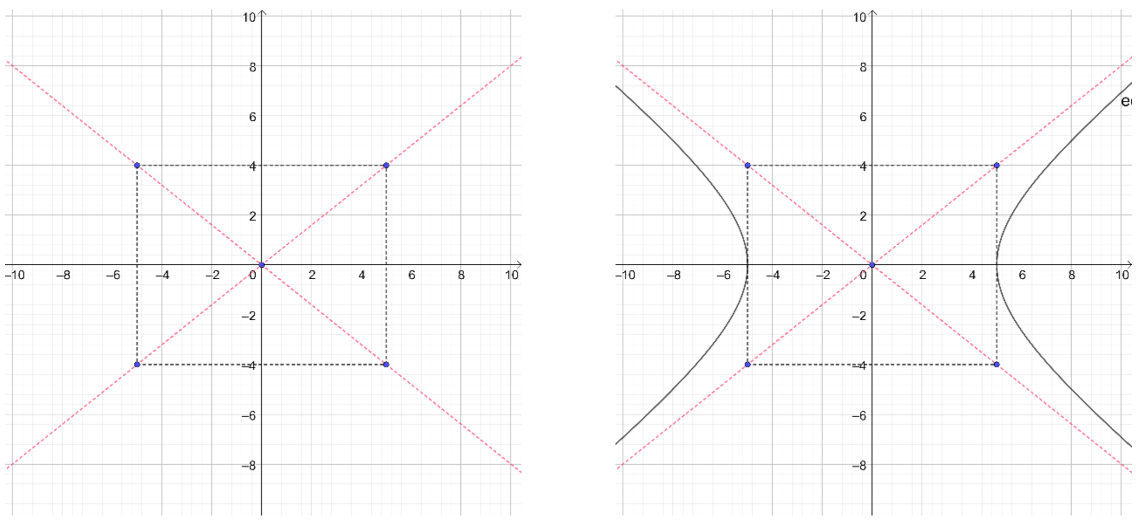

Figure 5.112 a) Preparing to graph $\frac{x^2}{25} - \frac{y^2}{16} = 1;$ b) The graph of $\frac{x^2}{25} - \frac{y^2}{16} = 1.$

4. ***Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.*** The hyperbola is shown in *Figure 5.112***b**.

We now consider the foci, located at $(-c, 0)$ and $(c, 0).$ We find c using $c^2 = a^2 + b^2.$

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$$c^2 = 25 + 16 = 41.$$

Because $c^2 = 41, c = \sqrt{41}.$ The foci are located at $\left(-\sqrt{41}, 0\right)$ and $\left(\sqrt{41}, 0\right)$, approximately $(-6.4, 0)$ and $(6.4, 0)$.

Example 5.79 : GRAPHING A HYPERBOLA

Graph and locate the foci: $9y^2 - 4x^2 = 36.$ What are the equations of the asymptotes?

Solution: We begin by writing the equation in standard form. The right side should be 1, so we divide both sides by 36.

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36},$$

simplifying

$$\frac{y^2}{4} - \frac{x^2}{9} = 1.$$

Now we are ready to use our four-step procedure for graphing hyperbolas.

1. ***Locate the vertices.*** The equation that we obtained is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, with $a^2 = 4$ and $b^2 = 9.$ Based on the standard form of the equation with the transverse axis on the y −axis, we know that the vertices are $(0, -a,)$ and $(0, a).$ Because $a^2 = 4 \;\; a = 2.$ Thus, the vertices are $(0, -2)$ and $(0,2)$, shown in *Figure 5.113***a**.
2. ***Draw a rectangle.*** Because $a^2 = 4$ and $b^2 = 9$, $a = 2$ and $b = 3.$ We construct a rectangle to find the asymptotes, using -2 and 2 on the y −axis (the vertices are located here) and -3 and 3 on the x −axis. The rectangle passes through these four points, shown using dashed lines in **Figure 8a.**
3. ***Draw extended diagonals for the rectangle to obtain the asymptotes.*** We draw dashed lines through the opposite corners of the rectangle, shown in *Figure 5.113***a**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are
$$y = \pm\frac{a}{b}x \text{ or } y = \pm\frac{2}{3}x.$$
4. ***Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.*** The hyperbola is shown in *Figure 5.113 b*.

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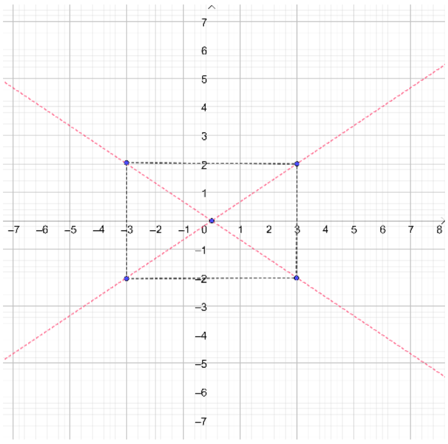

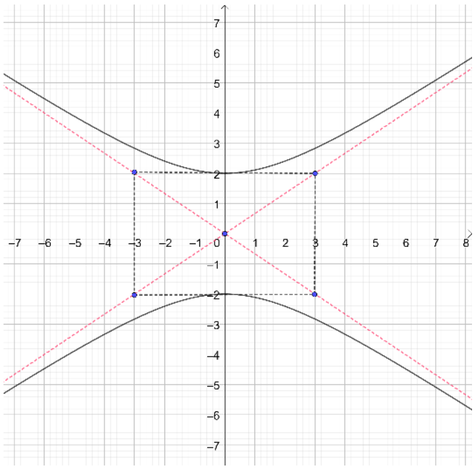

Figure 5.113 *a) Preparing to graph* $\frac{y^2}{4}-\frac{x^2}{9}=1$; *b) The graph of* $\frac{y^2}{4}-\frac{x^2}{9}=1$;

We now consider the foci, located at $(0,-c)$ and $(0,c)$. We find c using $c^2=a^2+b^2$.

$$c^2=4+9=13.$$

Because $c^2=13, c=\sqrt{13}.$ The foci are located at $(0,-\sqrt{13})$ and $(0,\sqrt{13})$, approximately $(0,-3.6)$ and $(0,3.6)$.

Translations of hyperbolas

The graph of a hyperbola can be centered at (h,k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x-h$ and y with $y-k$ in the standard form of the hyperbola's equation.

Table 1 gives the standard forms of equations of hyperbolas centered at (h,k), and shows their graphs.

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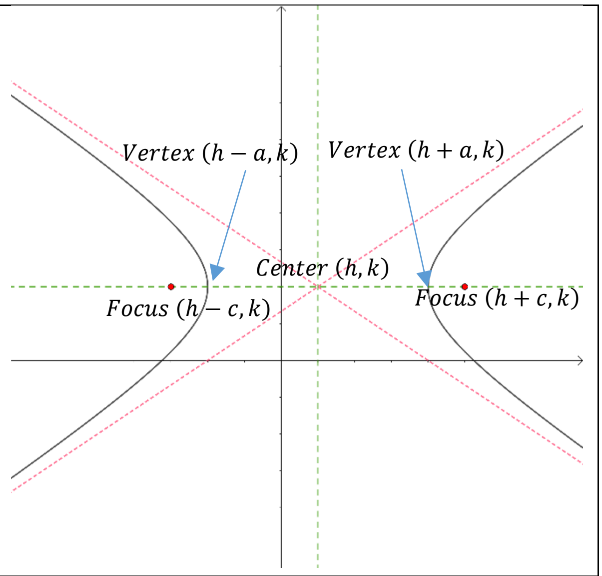
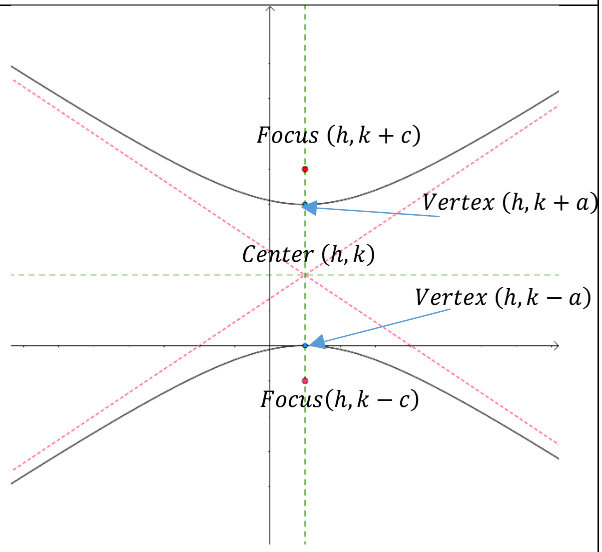
Table 1 Standard Forms of Equations of Hyperbolas Centered at (h,k)

Equation: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$; Vertices are a units right and a units left of center. Foci are c units right and c units left of center, where $c^2=a^2+b^2$.	Center: (h,k) Transverse Axis: Parallel to the x −axis; horizontal Vertices: $(h-a,k)$, $(h+a,k)$	
Equation: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$; Vertices are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2=a^2+b^2$.	Center: (h,k) Transverse Axis: Parallel to the y −axis; vertical Vertices: $(h,k-a)$, $(h,k+a)$	

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Example 5.80 : GRAPHING A HYPERBOLA CENTERED AT (h, k)

Graph: $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1.$ Where are the foci located? What are the equations of the asymptotes?

Solution: In order to graph the hyperbola, we need to know its center (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and h is the number subtracted from y. Where is in

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

we see that $h = 2$ and $k = 3$. Thus, the center of the hyperbola, (h, k), is $(2, 3)$. We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.

1. ***Locate the vertices***. To do this, we must identify a^2. The form of our equation is
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$
we see that $a^2 = 16$ and $b^2 = 9$. Based on the standard form of the equation with a horizontal transverse axis, the vertices are a units to the left and right of the center. Because $a^2 = 16$, $a = 4$. This means that the vertices are 4 units to the left and right of the center, $(2, 3)$. Four units to the left of $(2, 3)$ puts one vertex at $(2-4, 3)$ or $(-2, 3)$. Four units to the right

   of $(2, 3)$ puts the other vertex at $(2+4, 3)$, or $(6, 3)$. The vertices are shown in Figure 5.114.

2. ***Draw a rectangle***. Because $a^2 = 16$ and $b^2 = 9$, $a = 4$ and $b = 3$. The rectangle passes through points that are 4 units to the right and left of the center (the vertices are located here) and 3 units above and below the center. The rectangle is shown using dashed lines in Figure 5.114.

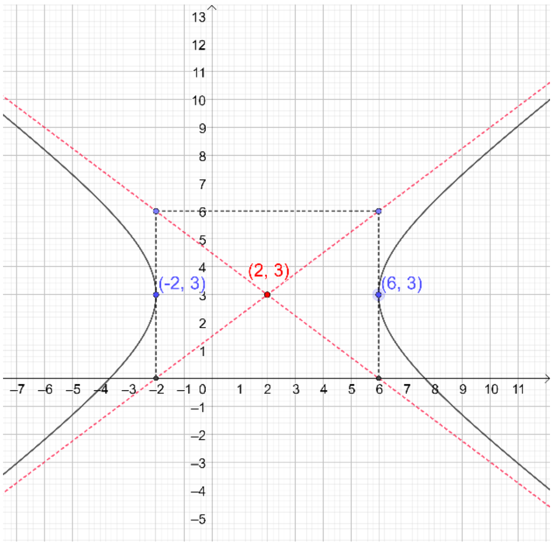

Figure 5.114 The graph of $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$

3. ***Draw the extended diagonals of the rectangle to obtain the asymptotes***. We draw dashed lines through the opposite corners of the rectangle, shown in Figure 5.114., to obtain the graph of the asymptotes. The equations of the asymptotes of the upshifted hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{3}{4}x$.

Thus, the asymptotes for the hyperbola that is shifted two units to the right and three units up, namely

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

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have equations that can be expressed as

$$y-3=\pm\frac{3}{4}(x-2).$$

4. ***Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes***. The hyperbola is shown in **Figure 9**. We now consider the foci, located c units to the right and left of the center. We find c using $c^2=a^2+b^2$.

$$c^2=16+9=25.$$

Because $c^2=25, c=5.$ This means that the foci are 5 units to the left and right of the center, (2, 3). Five units to the left of $(2,3)$ puts one focus at $(2-5,3)$ or $(-3,3)$. Five units to the right of $(2,3)$ puts the other focus at $(2+5,3)$, or $(7,3)$.

In our next example, it is necessary to convert the equation of a hyperbola to standard form by completing the square on x and y.

Example 5.81 : GRAPHING A HYPERBOLA CENTERED AT(h,k)

Graph: $4x^2-24x-25y^2+250y-489=0.$ Where are the foci located? What are the equations of the asymptotes?

Solution: We begin by completing the square on x and y.
This is given equation

$$4x^2-24x-25y^2+250y-489=0.$$

Group terms and add 489 to both sides

$$(4x^2-24x)+(-25y^2+250y)=489;$$
$$4(x^2-6x+?)-25(y^2+10y+?)=489$$

Factor out 4 and -25, respectively, so coefficients of x^2 and y^2 are 1. Complete each square by adding the square of half the coefficient of x and y, respectively.

$$4(x^2-6x+\ 9)-25(y^2+10y+25)=489+36+(-625).$$

Factoring

$$4(x-3)^2-25(y-5)^2=-100$$

and dividing both sides by -100

$$\frac{4(x-3)^2}{-100}-\frac{25(y-5)^2}{-100}=-\frac{100}{-100}.$$

Simplifying

$$\frac{(y-5)^2}{4}-\frac{(x-3)^2}{25}=1.$$

We see that $h=3$ and $k=5$. Thus, the center of the hyperbola, (h,k), is $(3,5)$. Because the x^2 −term is being subtracted, the transverse axis is vertical and the hyperbola opens upward and downward.

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We use our four-step procedure to obtain the graph of

$$\frac{(y-5)^2}{4}-\frac{(x-3)^2}{25}=1.$$

1. ***Locate the vertices.*** Based on the standard form of the equation with a vertical transverse axis, the vertices are a units above and below the center. Because $a^2=4$, $a=2$. This means that the vertices are 2 units above and below the center, $(3,5)$. This puts the vertices at $(3,7)$ and $(3,3)$, shown in Figure 5.115.

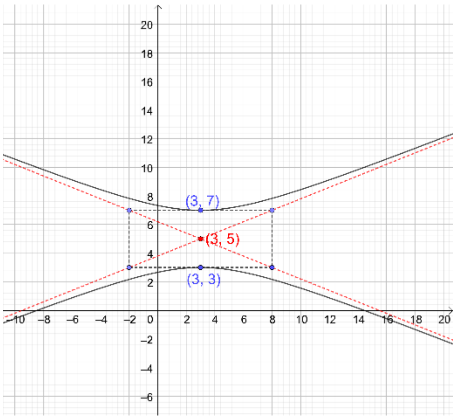

Figure 5.115 The graph of $\frac{(y-5)^2}{4}-\frac{(x-3)^2}{25}=1$

2. ***Draw a rectangle***. Because $b^2=25, a^2=4$ and $a=2$, $b=5$. The rectangle passes through points that are 2 units above and below the center (the vertices are located here) and 5 units to the right and left of the center. The rectangle is shown using dashed lines in Figure 5.115.
3. ***Draw extended diagonals of the rectangle to obtain the asymptotes.*** We draw dashed lines through the opposite corners of the rectangle, shown in Figure 5.115, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{y^2}{4}-\frac{x^2}{25}=1$ are $y=\pm\frac{a}{b}x$ or $y=\pm\frac{2}{5}x$. Thus, the asymptotes for the hyperbola that is shifted three units to the right and five units up, namely

$$\frac{(y-5)^2}{4}-\frac{(x-3)^2}{25}=1$$

have equations that can be expressed as $y-5=\pm\frac{2}{5}(x-3)$.

4. ***Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.*** The hyperbola is shown in Figure 5.115. We now consider the foci, located c units above and below the center, $(3,5)$. We find c using $c^2=a^2+b^2$.

$$c^2=4+25=29.$$

Because $c^2=29, c=\sqrt{29}$. This means that the foci are located at $(3,5+\sqrt{29})$ and $(3,5-\sqrt{29})$.

APPLICATIONS

Hyperbolas have many applications. When a jet flies at a speed greater than the speed of sound, the shock wave that is created is heard as a sonic boom. The wave has the shape of a cone. The shape formed as the cone hits the ground is one branch of a hyperbola. Halley's Comet, a permanent part of our solar system, travels around the sun in an elliptical orbit. Other comets pass through the solar system only once, following a hyperbolic path with the sun as a focus. Hyperbolas are of practical importance in fields ranging from architecture to navigation. Cooling

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towers used in the design for nuclear power plants have cross sections that are both ellipses and hyperbolas. Three-dimensional solids whose cross sections are hyperbolas are used in some rather unique architectural creations. A hyperbolic mirror is used in some telescopes. Such a mirror has the property that a light ray directed at one focus will be reflected to the other focus.

Long-range navigation (LORAN) is a radio navigation system developed during World War II. The system enables a pilot to guide aircraft by maintaining a constant difference between the aircraft's distances from two fixed points: the master station and the slave station.

Example 5.82 : AN APPLICATION INVOLVING HYPERBOLAS

An explosion is recorded by two microphones that are 2 miles apart. Microphone M_1 received the sound 4 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

Solution: We begin by putting the microphones in a coordinate system. Because $1\,mile = 5280\,feet$, we place M_1 $5280\,feet$ on a horizontal axis to the right of the origin and M_2 $5280\,feet$ on a horizontal axis to the left of the origin. Figure 5.116 illustrates that the two microphones are 2 miles apart.

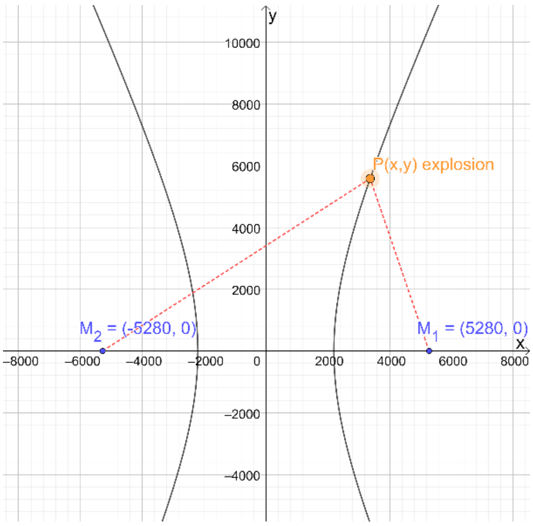

Figure 5.116 Locating an explosion on the branch of a hyperbola

We know that M_2 received the sound 4 seconds after M_1. Because sound travels at $1100\,feet$ per second, the difference between the distance from P to M_1 and the distance from P to M_2 is $4400\,feet$. The set of all points P (or locations of the explosion) satisfying these conditions fits the definition of a hyperbola, with microphones M_1 and M_2 at the foci. Use the standard form of the hyperbola's equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

$P(x,y)$, the explosion point, lies on the hyperbola. We must find a^2 and b^2. The difference between the distances, represented by $2a$ in the derivation of the hyperbola's equation, is $4400\,feet$. Thus, $2a = 4400$ and $a = 2200$. Substitute 2200 for a into hyperbola's equation

$$\frac{x^2}{2200^2}-\frac{y^2}{b^2}=1,$$

then square 2200

$$\frac{x^2}{4\,840\,000}-\frac{y^2}{b^2}=1.$$

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We must still find b^2. We know that $a = 2200$. The distance from the center $(0,0)$, to either focus, $(-5280,0)$ or $(5280,0)$, is 5280. Thus, $c = 5280$. Using $c^2 = a^2 + b^2$, we have

$$5280^2 = 2200^2 + b^2$$

and

$$b^2 = 5280^2 - 2200^2 = 23\,038\,400.$$

The equation of the hyperbola with a microphone at each focus is

$$\frac{x^2}{4\,840\,000} - \frac{y^2}{23\,038\,400} = 1.$$

We can conclude that the explosion occurred somewhere on the right branch (the branch closer to M_1) of the hyperbola given by this equation.

In Example 7, we determined that the explosion occurred somewhere along one branch of a hyperbola, but not exactly where on the hyperbola. If, however, we had received the sound from another pair of microphones, we could locate the sound along a branch of another hyperbola. The exact location of the explosion would be the point where the two hyperbolas intersect.

PRACTICE EXERCISES

Find the vertices and locate the foci of each hyperbola with the given equation.

1. $\frac{x^2}{4} - \frac{y^2}{1} = 1;$
2. $\frac{y^2}{4} - \frac{x^2}{1} = 1;$
3. $\frac{x^2}{1} - \frac{y^2}{4} = 1;$
4. $\frac{y^2}{1} - \frac{x^2}{4} = 1;$

Find the standard form of the equation of each hyperbola satisfying the given conditions.

5. Foci: $(0,-3)$, $(0,3)$; vertices: $(0,-1)$, $(0,1)$;
6. Foci: $(0,-6)$, $(0,6)$; vertices: $(0,-2)$, $(0,2)$;
7. Foci: $(-4,0)$, $(4,0)$; vertices: $(-3,0)$, $(3,0)$;
8. Foci: $(-7,0)$, $(7,0)$; vertices: $(-5,0)$, $(5,0)$;
9. Endpoints of transverse axis: $(0,-6)$, $(0,6)$; asymptote: $y = 2x$;
10. Endpoints of transverse axis: $(-4,0)$, $(4,0)$; asymptote: $y = 2x$;
11. Center: $(4,-2)$; Focus: $(7,-2)$; vertex: $(6,-2)$;
12. Center: $(-2,1)$; Focus: $(-2,6)$; vertex: $(-2,4)$.

Use vertices and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

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13. $\frac{x^2}{9} - \frac{y^2}{25} = 1;$
14. $\frac{x^2}{100} - \frac{y^2}{64} = 1;$
15. $\frac{y^2}{16} - \frac{x^2}{36} = 1;$
16. $\frac{x^2}{16} - \frac{y^2}{25} = 1;$
17. $\frac{x^2}{144} - \frac{y^2}{81} = 1;$
18. $\frac{y^2}{25} - \frac{x^2}{64} = 1;$
19. $4y^2 - x^2 = 1;$
20. $9x^2 - 25y^2 = 36;$
21. $9y^2 - 25x^2 = 225;$
22. $9y^2 - x^2 = 1;$
23. $4x^2 - 25y^2 = 100;$
24. $16y^2 - 9x^2 = 144;$
25. $y = \pm\sqrt{x^2 - 3};$
26. $y = \pm\sqrt{x^2 - 2}.$

Find the standard form of the equation of each hyperbola.

27. $9x^2 - 4y^2 - 18x + 8y - 31 = 0$
28. $16x^2 - 4y^2 + 64x - 24y - 36 = 0$
29. $y^2 - x^2 - 4y + 2x - 6 = 0$
30. $4y^2 - 16x^2 - 24y + 96x - 172 = 0$
31. $9y^2 - x^2 + 18y - 4x - 4 = 0$

Use the center, vertices, and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

32. $\frac{(x+4)^2}{9} - \frac{(y+3)^2}{16} = 1;$
33. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{25} = 1;$
34. $\frac{(x+3)^2}{25} - \frac{y^2}{16} = 1;$
35. $\frac{(x+2)^2}{9} - \frac{y^2}{25} = 1;$
36. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1;$
37. $\frac{(y-2)^2}{36} - \frac{(x+1)^2}{49} = 1;$
38. $(x-3)^2 - 4(y+3)^2 = 4;$
39. $(x-1)^2 - (y-2)^2 = 3;$
40. $(x+3)^2 - 9(y-4)^2 = 9;$
41. $(y-2)^2 - (x+3)^2 = 5.$

Convert each equation to standard form by completing the square on x and y. Then graph the hyperbola. Locate the foci and find the equations of the asymptotes.

42. $x^2 - y^2 - 2x - 4y - 4 = 0;$
43. $4x^2 - y^2 + 32x + 6y + 39 = 0;$
44. $16x^2 - y^2 + 64x - 2y + 67 = 0;$
45. $-4x^2 + 9y^2 + 24x - 18y - 63 = 0;$
46. $4x^2 - 9y^2 - 16x + 54y - 101 = 0;$

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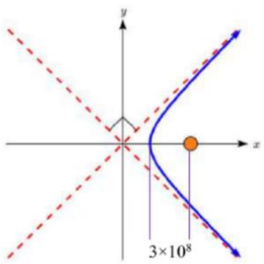
47. $4x^2 - 9y^2 + 8x - 18y - 6 = 0;$
48. $4x^2 - 25y^2 - 32x + 164 = 0;$
49. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

Application exercises

50. Two microphones that are 1 mile apart record an explosion. Microphone M_1 received the sound 2 seconds before Microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.
51. Radio towers A and B, 200 kilometres apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 microseconds before the signal from A.
    a. Assuming that the radio signals travel 300 meters per microsecond, determine the equation of the hyperbola on which the ship is located.
    b. If the ship lies due north of tower B how far out at sea is it?
52. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250\ 000$, where and are in yards. How far apart are the houses at their closest point?
53. Stations A and B are 100 kilometres apart and send a simultaneous radio signal to a ship. The signal from A arrives 0.0002 seconds before the signal from B. If the signal travels 300,000 kilometres per second, find an equation of the hyperbola on which the ship is positioned if the foci are located at A and B.
54. Anna and Julia are standing 3050 feet apart when they see a bolt of light strike the ground. Anna hears the thunder 0.5 seconds before Julia does. Sound travels at 1100 feet per second. Find an equation of the hyperbola on which the lighting strike is positioned if Anna and Julia are located at the foci.
55. A comet passes through the solar system following a hyperbolic trajectory with the sun as a focus. The closest it gets to the sun is 3×108 miles. The figure shows the trajectory of the comet, whose path of entry is at a right angle to its path of departure. Find an equation for the comet's trajectory. Round to two decimal places



56. Write the standard form equation for the ship's location $P(x)$ in the diagram below. Assume that two stations, 300 miles apart, are positioned as pictured

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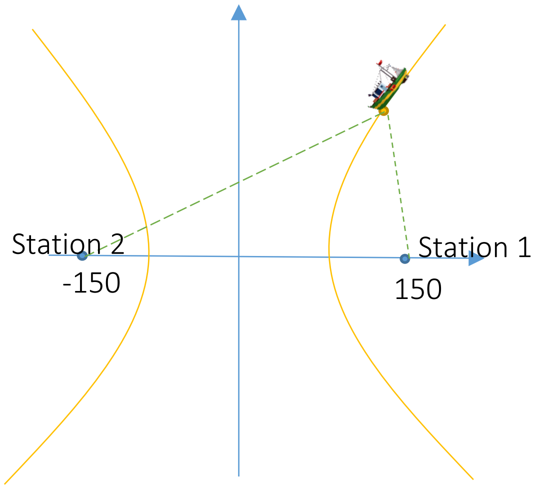

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5.9.3. THE PARABOLA

ABSTRACT;

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. In this section, we study parabolas and their applications, including parabolic shapes.

OBJECTIVES AND OUTCOMES:

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Rewrite equations of parabolas in standard form.
- Solve applied problems involving parabolas.

CONTENTS

Definition of a parabola

Here is a summary of what you should already know about graphing parabolas.

Graphing $y = a(x-h)^2 + k$ and $y = ax^2 + bx + c$.

1. If $a > 0,$ the graph opens upward. If $a < 0,$ the graph opens downward.
2. The vertex of $y = a(x-h)^2 + k$ is (h, k).

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3. The x −coordinate of the vertex of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

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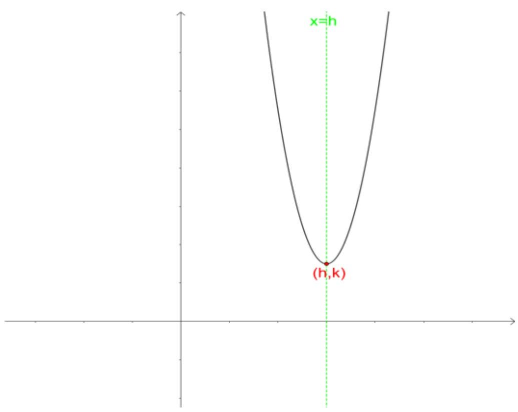
<td>
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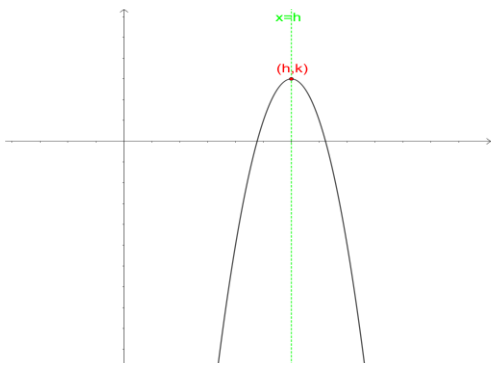
<td>
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<tr>
<td>$y = a(x-h)^2 + k, a > 0.$</td>
<td>$y = a(x-h)^2 + k, a < 0.$</td>
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Parabolas can be given a geometric definition that enables us to include graphs that open to the left or to the right, as well as those that open obliquely. The definitions of ellipses and hyperbolas involved two fixed points, the foci. By contrast, the definition of a parabola is based on one point and a line.

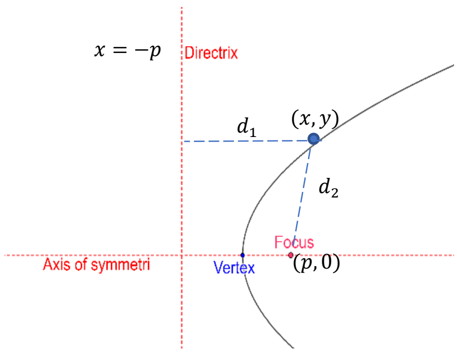

Figure 1

Definition: A parabola is the set of all points in a plane that are equidistant from a fixed line, called the directrix, and a fixed point not on the line, the focus. (see **Figure 1**).

In other words, if given a red line - the directrix, and a point - the focus, then (x, y) is a point on the parabola if the shortest distance from it to the focus and from it to the line is equal.

In **Figure 1**, find the line passing through the focus and perpendicular to the directrix. This is the axis of symmetry of the parabola. The vertex of the parabola is the point where the shortest distance to the directrix is at a minimum. Notice that the vertex is midway between the focus and the directrix. In addition, a parabola is formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone.

Standard form of the equation of a parabola

The rectangular coordinate system enables us to translate a parabola's geometric definition into an algebraic equation. **Figure 1** is our starting point for obtaining an equation. We place the focus on the x −axis at the point $(p, 0)$. The directrix has an equation given by $x = -p$. The vertex, located midway between the focus and the directrix, is at the origin.

For any point (x, y) on the parabola, the distance d_1 to the directrix is equal to the distance d_2 to the focus. Thus, the point (x, y) is on the parabola if and only if

$$d_1 = d_2.$$

Using distance formula

$$\sqrt{(x+p)^2 + (y-y)^2} = \sqrt{(x-p)^2 + (y-0)^2}$$

and squaring both sides of the equation we find

$$(x+p)^2 = (x-p)^2 + y^2.$$

$$x^2 + 2xp + p^2 = x^2 - 2xp + p^2 + y^2$$

$$y^2 = 4px.$$

This last equation is called the standard form of the equation of a parabola with its vertex at the origin. There are two such equations, one for a focus on the x −axis and one for a focus on the y −axis.

Standard forms of the equations of a parabola

The standard form of the equation of a parabola with vertex at the origin is

$$y^2 = 4px$$

Or

$$x^2 = 4py.$$

Figure 2 illustrates that for the equation on the left, the focus is on the x–axis, which is the axis of symmetry. **Figure 3** illustrates that for the equation on the right, the focus is on the y −axis, which is the axis of symmetry.

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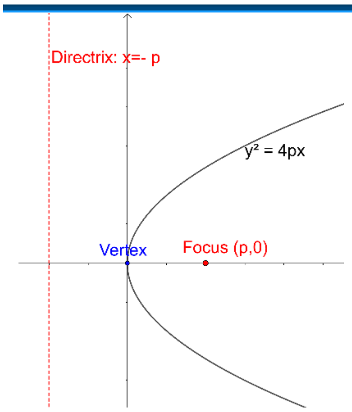

Figure 2 Parabola with the x −axis as the axis of symmetry. If $p > 0$, the graph opens to the right. If $p < 0$, the graph opens to the left

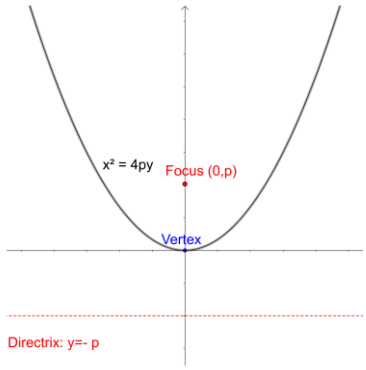

Figure 3 Parabola with the y −axis as the axis of

Using the standard form of the equation of a parabola

We can use the standard form of the equation of a parabola to find its focus and directrix. Observing the graph's symmetry from its equation is helpful in locating the focus.

Although the definition of a parabola is given in terms of its focus and its directrix, the focus and directrix are not part of the graph. The vertex, located at the origin, is a point on the graph of $y^2 = 4px$ and $x^2 = 4py$. Example 1 illustrates how you can find two additional points on the parabola.

EXAMPLE 1 FINDING THE FOCUS AND DIRECTRIX OF A PARABOLA

Find the focus and directrix of the parabola given by $y^2 = 12x$. Then graph the parabola.

Solution: The given equation, $y^2 = 12x$ is in the standard form $y^2 = 4px$, so $4p = 12$.

We can find both the focus and the directrix by finding p.

$$4p = 12.$$

Diving both sides by 4 we get

$$p = 3.$$

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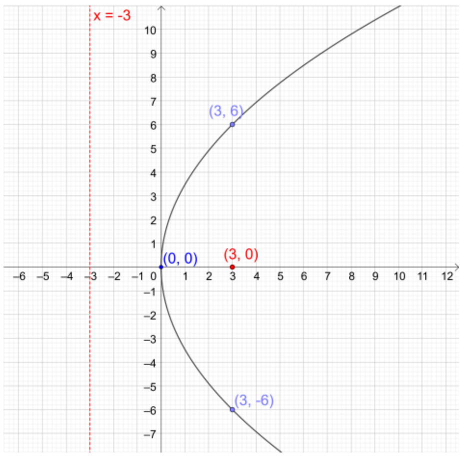

Figure 4 The graph of $y^2 = 12x$

Because p is positive, the parabola, with its x–axis symmetry, opens to the right. The focus is 3 units to the right of the vertex, $(0, 0)$. The focus, $(3, 0)$, and directrix, $x = -3$, are shown in **Figure 4**.

To graph the parabola, we will use two points on the graph that lie directly above and below the focus. Because the focus is at $(3, 0)$, substitute 3 for x in the parabola's equation, $y^2 = 12x$

$$y^2 = 12 \cdot 3$$
$$y^2 = 36.$$

Applying the square root property, we find

$$y = \pm\sqrt{36} = \pm 6.$$

The points on the parabola above and below the focus are $(3, 6)$ and $(3, -6)$. The graph is sketched in **Figure 4**.

In general, the points on a parabola $y^2 = 4px$ that lie above and below the focus, $(p, 0)$, are each at a distance $|2p|$ from the focus. This is because if $x = p$, then $y^2 = 4px = 4p^2$, so $y = \pm 2p$. The line segment joining these two points is called the latus rectum; its length is $|4p|$.

The latus rectum and graphing parabolas

The latus rectum of a parabola is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola. **Figure 5** shows that the length of the latus rectum for the graphs of $y^2 = 4px$ and $x^2 = 4py$ is $|4p|$.

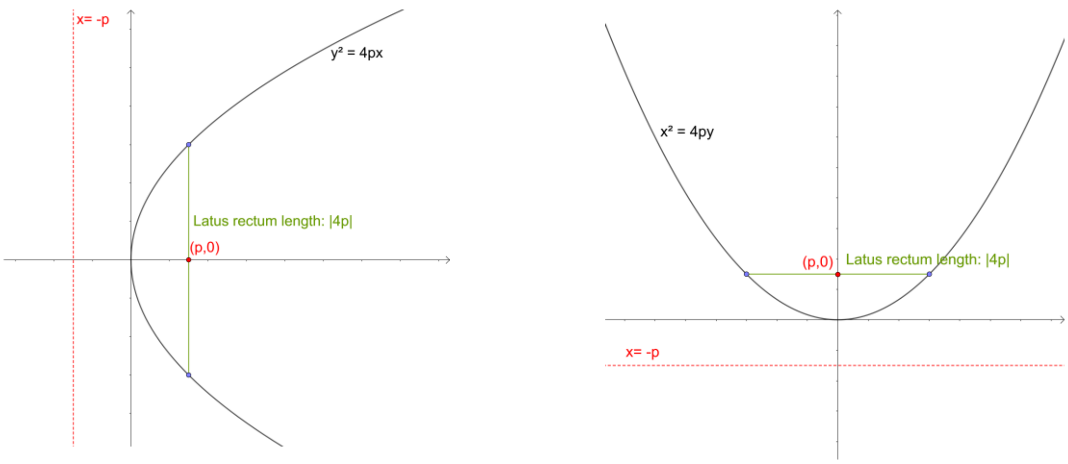

Figure 5 Endpoints of the latus rectum are helpful in determining a parabola's "width," o0r how it opens

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EXAMPLE 2 FINDING THE FOCUS AND DIRECTRIX OF A PARABOLA

Find the focus and directrix of the parabola given by $x^2 = -8y$. Then graph the parabola.
Solution: The given equation, $x^2 = -8y$, is in the standard form $x^2 = 4py$, so $4p = -8$. We can find both the focus and directix by finding p

$$4p = -8,$$

Dividing both sides by 4 we get

$$p = -2.$$

Because is negative, the parabola, with its $y -$ axis symmetry, opens downward. The focus is 2 units below the vertex, $(0, 0)$.

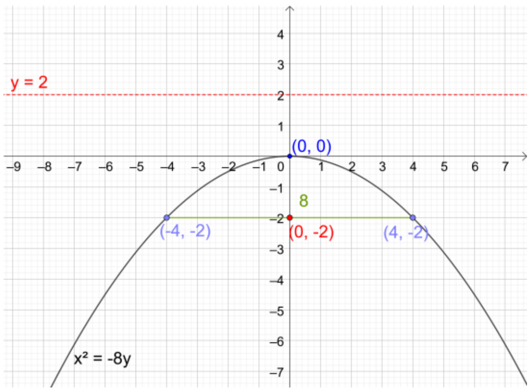

Figure 6 The graph of $x^2 = -8y$

Focus: $(0, p) = (0, -2)$

Directrix: $x = -p;\ y = 2$

The focus and directrix are shown in **Figure 6**.

To graph the parabola, we will use the vertex, $(0, 0)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-2)| = 8.$$

Because the graph has $y -$ axis symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(0, -2)$. The endpoints of the latus rectum are $(-4, -2)$ and $(4, -2)$. Passing a smooth curve through the vertex and these two

points, we sketch the parabola, shown in **Figure 6**.

In examples above, we used the equation of a parabola to find its focus and directrix. In the next example, we reverse this procedure.

EXAMPLE 3 FINDING THE EQUATION OF A PARABOLA FROM ITS FOCUS AND DIRECTRIX

Find the standard form of the equation of a parabola with focus $(5, 0)$ and directrix $x = -5$, shown in **Figure 7**.

Solution: The focus is $(5, 0)$. Thus, the focus is on $x -$axis. We use the standard from of the equation in which there is $x -$axis, namely $y^2 = 4px$.

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We need to determine the value of p. Figure 7 shows that the focus is 5 units to the right of vertex, $(0,0)$. Thus, p is positive and $p = 5$. We substitute 5 for p in $y^2 = 4px$ to obtain the standard form of the equation of the parabola. The equation is

$$y^2 = 4 \cdot 5x \quad or \quad y^2 = 20x.$$

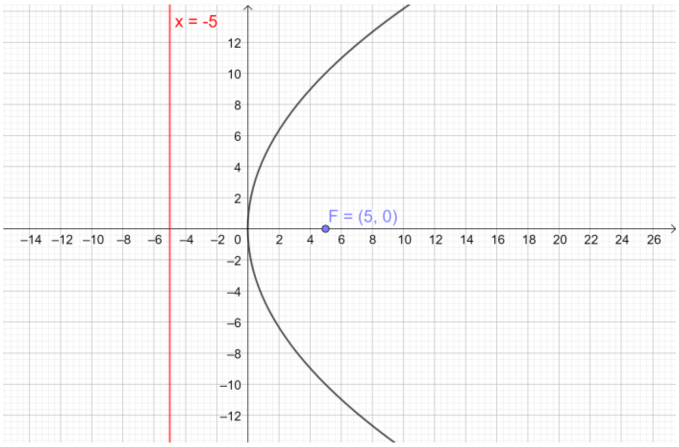

Figure 7

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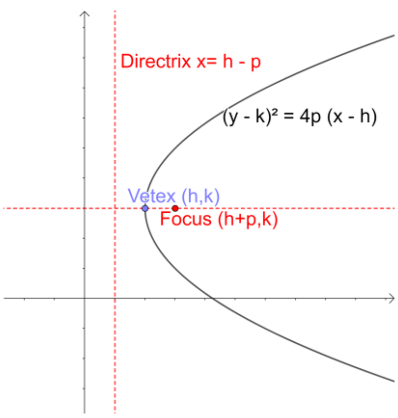
Translations of parabolas

The graph of a parabola can have its vertex at (h, k) rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the parabola's equation. **Table 1** gives the standard forms of equations of parabolas with vertex at (h, k). **Figure 8** shows their graphs.

Table 1 Standard Forms of Equations of Parabolas with Vertex at (h,k)

Equation	Vertex	Axis of Symmetry	Focus	Directrix	Description
$(y-k)^2 = 4p(x-h)$	(h,k)	Horizontal	$(h+p,k)$	$x = h - p$	If $p > 0$, opens to the right. If $p < 0$, opens to the left.
$(x-h)^2 = 4p(y-k)$	(h,k)	Vertical	$(h,k+p)$	$y = k - p$	If $p > 0$, opens to the upward. If $p < 0$, opens to the downward.



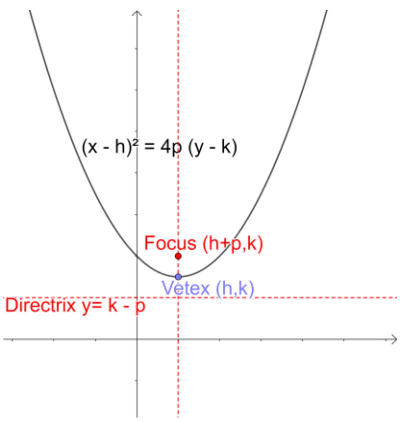


Figure 8 Graphs of parabolas with vertex at (h,k) *and* $p > 0$

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EXAMPLE 4 GRAPHING A PARABOLA WITH VERTEX AT (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$(x-3)^2 = 8(y+1).$$

Then graph the parabola.

Solution: In order to find the focus and directrix, we need to know the vertex. In the standard forms of equations with vertex at (h, k), h is the number subtracted from x and k is the number subtracted from y.

$$(x-3)^2 = 8(y-(-1)).$$

We see that $h = 3$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (3, -1)$. Now that we have the vertex, $(3, -1)$, we can find both the focus and directrix by finding p. The equation

$$(x-3)^2 = 8(y+1)$$

Is in the standard form $(x-h)^2 = 4p(y-k)$. Because x is the square term, there is vertical symmetry and the parabola's equation is a function. Because $4p = 8$, $p = 2$. Based on the standard form of the equation, the axis of symmetry is vertical. With a positive value for and a vertical axis of symmetry, the parabola opens upward. Because $p = 2$, the focus is located 2 units above the vertex, $(3, -1)$. Likewise, the directrix is located 2 units below the vertex.

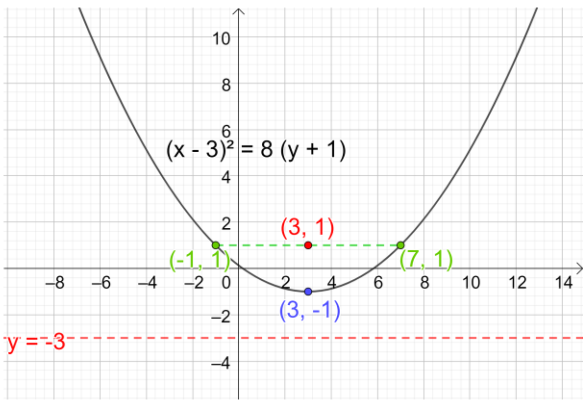

Figure 9 The graph of $(x-3)^2 = 8(y-(-1))$.

Focus: $(h, k+p) = (3, -1+2) = (3,1)$;

Directrix: $y = k - p$, $y = -1 - 2 = -3$.

Thus, the focus is $(3, 1)$ and the directrix is $y = -3$. They are shown in **Figure 9**. To graph the parabola, we will use the vertex, $(3, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4 \cdot 2| = |8| = 8.$$

Because the graph has vertical symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(3, 1)$. The endpoints of the latus rectum are $(3-4, 1)$, or $(-1, 1)$, and $(3+4, 1)$, or $(7, 1)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in **Figure 9**.

In some cases, we need to convert the equation of a parabola to standard form by completing the square on x or y, whichever variable is squared. Let's see how this is done.

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EXAMPLE 5 GRAPHING A PARABOLA WITH VERTEX AT (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$y^2 + 2y + 12x - 23 = 0.$$

Then graph the parabola.

Solution: We convert the given equation to standard form by completing the square on the variable y. We isolate the terms involving y on the left side. This is the given equation

$$y^2 + 2y + 12x - 23 = 0.$$

Isolate the terms involving y

$$y^2 + 2y = -12x + 23.$$

Complete square by adding the square og half the coefficient of y

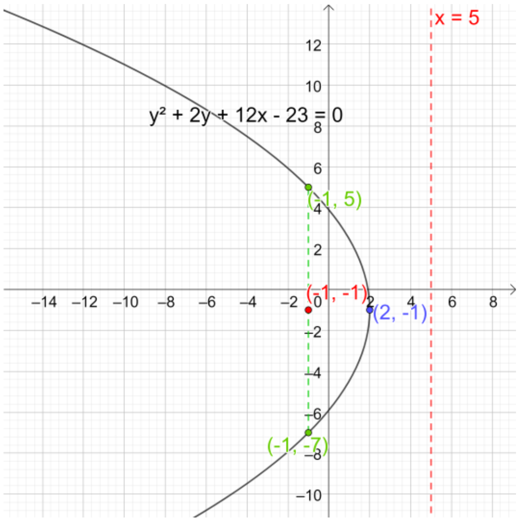

Figure 10 The graph of $y^2 + 2y + 12x - 23 = 0$

$$y^2 + 2y + 1 = -12x + 23 + 1.$$

Factoring

$$(y + 1)^2 = -12x + 24.$$

To express the equation $(y - 1)^2 = -12x + 24$ to identify the vertex, (h, k), and the value p needed to locate the focus and directrix

$$\left(y - (-1)\right)^2 = -12(x - 2).$$

We see that $h = 2$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (2, -1)$. Because $4p = -12$, $p = -3$. Based on the standard form of the equation, the axis of symmetry is horizontal. With a negative value for p and a horizontal axis of symmetry, the parabola opens to the left. Because, $p = -3$, the focus is located 3 units to the left of the vertex, $(2, -1)$. Likewise, the directrix is located 3 units to the right of the vertex.

Focus: $(h + p, k) = (2 + (-3), -1) = (-1, -1)$;

Directrix: $x = h - p, \quad x = 2 - (-3) = 5$.

Thus, the focus is $(-1, -1)$ and the directrix is $x = 5$. They are shown in **Figure 10**. To graph the parabola, we will use the vertex, $(2, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-3)| = |-12| = 12.$$

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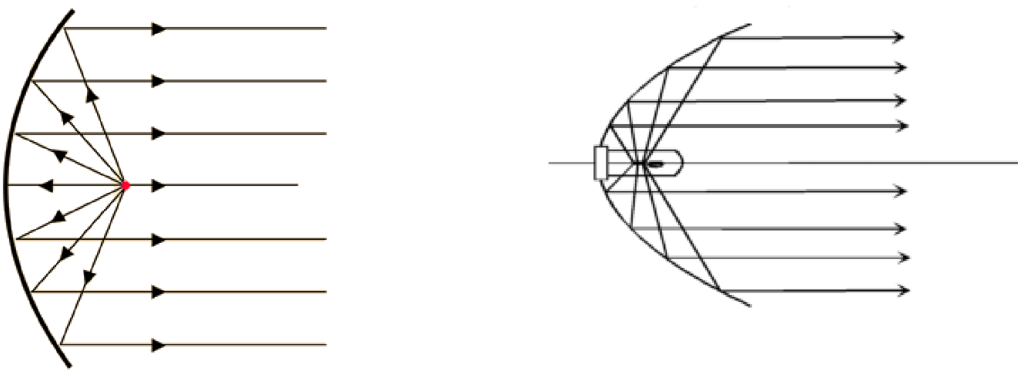
Because the graph has horizontal symmetry, the latus rectum extends 6 units above and 6 units below the focus, $(-1,-1)$. The endpoints of the latus rectum are $(-1,-1+6)$, or $(-1,5)$ and $(-1,-1-6)$ or $(-1,-7)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola shown in Figure 10.

APPLICATIONS

Parabolas have many applications. Cables hung between structures to form suspension bridges form parabolas. Arches constructed of steel and concrete, whose main purpose is strength, are usually parabolic in shape.

Figure 11 Suspension and arch bridges

We have seen that comets in our solar system travel in orbits that are ellipses and hyperbolas. Some comets follow parabolic paths. Only comets with elliptical orbits, such as Halley's Comet,



return to our part of the galaxy.

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Figure 12 a) Parabolic surface reflecting light b)Light from the focus is reflected parallel to the axis of symmetry

If a parabola is rotated about its axis of symmetry, a parabolic surface is formed. Figure 12a shows how a parabolic surface can be used to reflect light. Light originates at the focus. Note how the light is reflected by the parabolic surface, so that the outgoing light is parallel to the axis of symmetry. The reflective properties of parabolic surfaces are used in the design of searchlights [see Figure 12b], automobile headlights, and parabolic microphones.

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EXAMPLE 6 USING THE REFLECTION PROPERTY OF PARABOLAS

An engineer is designing a using a parabolic reflecting mirror and a light source, shown in **Figure 13**. The casting has a diameter of 4 inches and a depth of 2 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

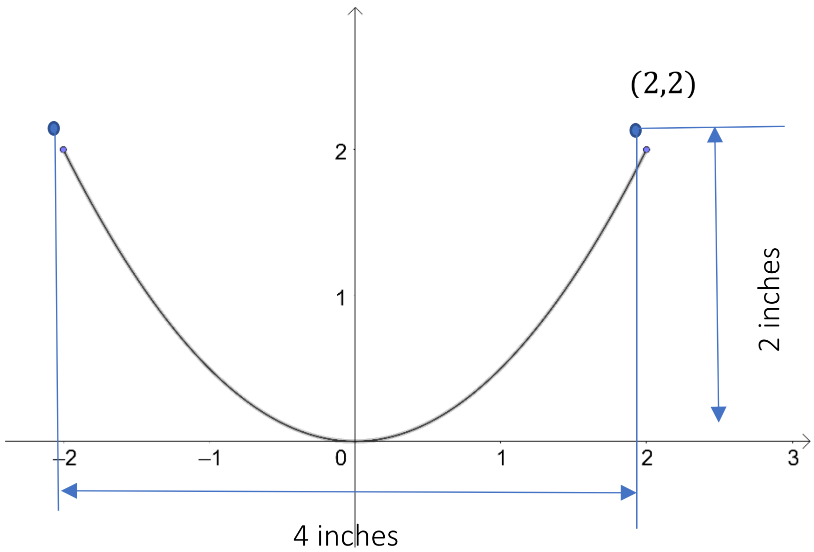

Figure 117 Designing a flashlight

Solution: We position the parabola with its vertex at the origin and opening upward (see **Figure 13**). Thus, the focus is on the y –axis, located at $(0, p)$. We use the standard form of the equation in which there is y –axis symmetry, namely $x^2 = 4py$. We need to find p. Because $(2,2)$ lies on the parabola, we let $x = 2$ and $y = 2$ in $x^2 = 4py$.
Substitute 2 for x and 2 for y

$$2^2 = 4p \cdot 2.$$

Simplifying and dividing both sides of equation by 8 and reducing the resulting fraction we get

$$4 = 8p,$$
$$p = \frac{1}{2}.$$

We substitute $\frac{1}{2}$ for p in $x^2 = 4py$ to obtain the standard form of the equation of the parabola. The equation of the parabola used to shape the mirror is

$$x^2 = 4 \cdot \frac{1}{2} y$$

or

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$$x^2 = 2y.$$

The light source shold be placed at the focus, $(0, p)$. Because $p = \frac{1}{2}$, the light should be placed at $\left(0, \frac{1}{2}\right)$, or $\frac{1}{2}$ inch above the vertex.

Practice exercises

Find the focus and directrix of each parabola with the given equation. $y^2 = 4x$;

1. $x^2 = 4y$;
2. $x^2 = -4y$;
3. $y^2 = -4x$.

Find the focus and directrix of the parabola with the given equation. Then graph the parabola.

4. $y^2 = 16x$;
5. $y^2 = -8x$;
6. $y^2 = 16x$;
7. $x^2 = -16y$;
8. $y^2 - 6x = 0$;
9. $8x^2 + 4y = 0$;
10. $y^2 = 4x$;
11. $y^2 = -16x$;
12. $x^2 = 8y$;
13. $x^2 = -20y$;
14. $x^2 - 6y = 0$;
15. $8y^2 + 4x = 0$.

Find the standard form of the equation of each parabola satisfying the given conditions.

16. Focus: $(7, 0)$; Directrix: $x = -7$;
17. Focus: $(9, 0)$; Directrix: $x = -9$;
18. Focus: $(-5, 0)$; Directrix: $x = 5$;
19. Focus: $(-10, 0)$; Directrix: $x = 10$;
20. Focus: $(0, 15)$; Directrix: $y = -15$;
21. Focus: $(0, 20)$; Directrix: $y = -20$;
22. Focus: $(0, -25)$; Directrix: $y = 25$;
23. Focus: $(0, -15)$; Directrix: $y = 15$;
24. Vertex: $(2, -3)$; Focus: $(2, -5)$;
25. Vertex: $(5, -2)$; Focus: $(7, -2)$;

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26. Focus: $(3,2)$; Directrix: $x = -1$;
27. Focus: $(2,4)$; Directrix: $x = -4$;
28. Focus: $(-3,4)$; Directrix: $y = 2$;
29. Focus: $(7,-1)$; Directrix: $y = -9$.

Find the vertex, focus, and directrix of each parabola with the given equation.

30. $(y-1)^2 = 4(x-1)$;
31. $(y-1)^2 = -4(x-1)$;
32. $(x+1)^2 = 4(y+1)$;
33. $(x+1)^2 = -4(y+1)$;

Find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

34. $(x-2)^2 = 8(y-1)$;
35. $(x+1)^2 = -8(y+1)$;
36. $(y+3)^2 = 12(x+1)$;
37. $(y+1)^2 = 8x$;
38. $(x+2)^2 = 4(y+1)$;
39. $(x+2)^2 = -8(y+2)$;
40. $(y+4)^2 = 12(x+2)$;
41. $(y-1)^2 = -8x$.

Convert each equation to standard form by completing the square on x or y. Then find the vertex, focus, and directrix of the parabola. Finally, graph the parabola.

42. $x^2 - 2x - 4y + 9 = 0$;
43. $y^2 - 2y + 12x - 35 = 0$;
44. $x^2 + 6x - 4y + 1 = 0$;
45. $x^2 + 6x + 8y + 1 = 0$;
46. $y^2 - 2y - 8x + 1 = 0$;
47. $x^2 + 8x - 4y + 8 = 0$.

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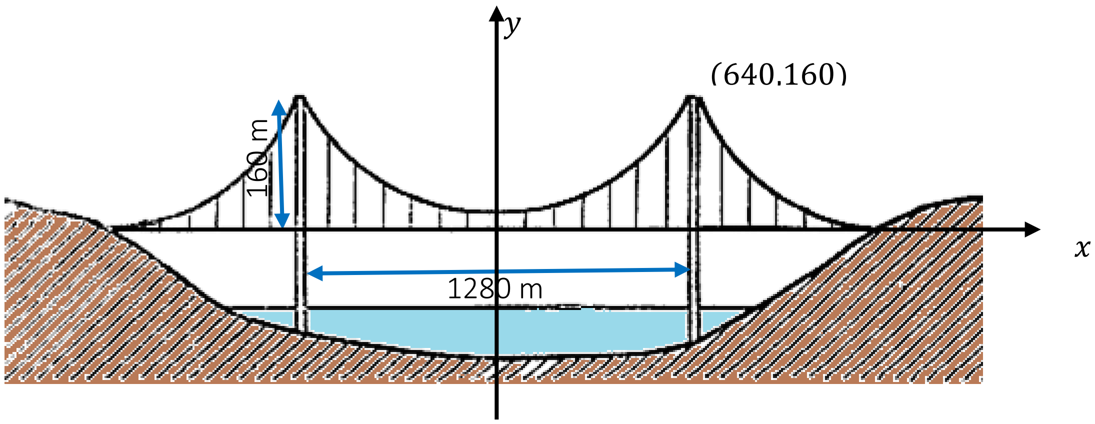
Application exercises

1. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 4 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
2. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 8 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
3. A satellite dish, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish has a diameter of 12 feet and a depth of 2 feet. How far from the base of the dish should the receiver be placed?
4. In Exercise 3, if the diameter of the dish is halved and the depth stays the same, how far from the base of the smaller dish should the receiver be placed?
5. The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 meters apart and rise 160 meters above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower? Round to the nearest meter.



6. The towers of a suspension bridge are 800 feet apart and rise 160 feet above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 feet from a tower?

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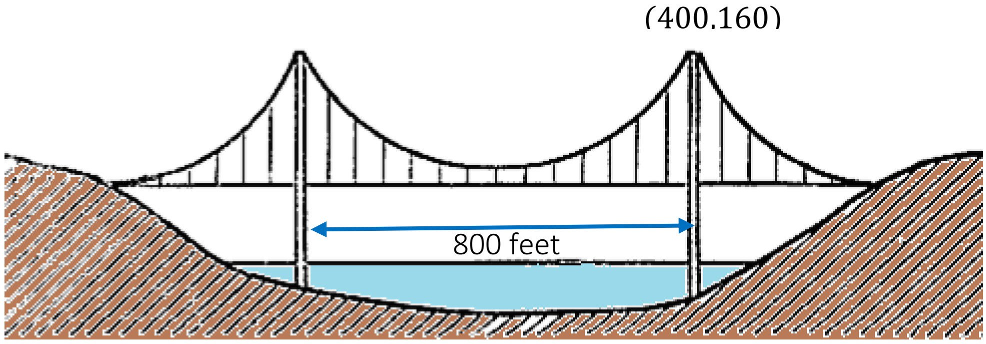
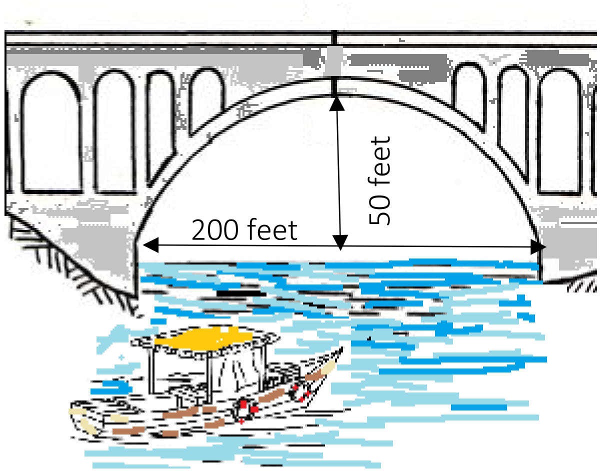


7. The parabolic arch shown in the figure is 50 feet above the water at the center and 200 feet wide at the base. Will a boat that is 30 feet tall clear the arch 30 feet from the center?



8. A satellite dish in the shape of a parabolic surface has a diameter of 20 feet. If the receiver is to be placed 6 feet from the base, how deep should the dish be?
9. A domed ceiling is a parabolic surface. Ten meters down from the top of the dome, the ceiling is 15 meters wide. For the best lighting on the floor, a light source should be placed at the focus of the parabolic surface. How far from the top of the dome, to the nearest tenth of a meter, should the light source be placed?

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