2.1. MATRIX

A rectangular table of objects arranged in m rows and n columns is called *a $m \times n$ matrix* or *a matrix of dimensions (size) $m \times n$*. Each matrix is marked with a capital letter, and its objects are set within square or round brackets. Every element of the matrix, i.e., each object in the matrix, is marked by a corresponding small letter with 2 indexes, where the first index indicates the ordinal number of the row in which the element is located, and the second index indicates the ordinal number of the column in which the element is located. If all the elements of a matrix are real numbers, then the matrix is real.

Example 2.1

$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & -1 & -5 \end{bmatrix}$ is 2×3 real matrix with the elements

$$a_{11} = 1, \qquad a_{12} = 4, \qquad a_{13} = 7,$$
$$a_{21} = 3, \qquad a_{22} = -1, \qquad a_{23} = -5.$$

Some special types of matrices

Zero matrix is a matrix whose all elements are equal to zero. Such matrix is marked with O.

Example 2.2

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a 4×2 zero-matrix.

A square matrix is any matrix that has the same number of rows and columns.
Every square matrix, that has n rows and n columns, is a (square) matrix of the nth order or matrix of order n.

Example 2.3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{bmatrix}$$

has 3 rows and 3 columns. So, it is a matrix of order 3.

The elements $a_{11}, a_{22}, \dots, a_{nn}$ form *the main diagonal* of the square matrix of order n.
Therefore, in the previous example, the elements $1, 6, -3$ form the main diagonal of the matrix A.

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Unit matrix is a square matrix in which all elements on the main diagonal are equal to one, while all other elements are equal to zero. This type of matrix is marked I_n, where n is the order of that matrix.

Example 2.4

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is unit matrix of the 4th order.

The upper triangular matrix (*matrix in row echelon form*) is a square matrix in which all the elements below the main diagonal are equal to zero.

The lower triangular matrix is a square matrix in which all the elements above the main diagonal are equal to zero.

Example 2.5

Each unit matrix is both, an upper triangular and a lower triangular matrix.

The transposed matrix A^T of matrix A is obtained by replacing the rows of the matrix A with its columns (and vice versa).
The first column in the matrix A will become the first row in the matrix A^T, the second column in the matrix A will become the second row in the matrix A^T, etc.

Example 2.6

Matrix

$$A^T = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 6 & -2 \\ 3 & 9 & -3 \end{bmatrix}$$

is a transposed matrix of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{bmatrix}.$$

It can be said that matrices A and B are *equal*, and it is written $A = B$, if they have the same dimensions and if they have the same elements in the same positions.

Example 2.7

Determine the dimensions of the given matrix and the required element:

a) $A = \begin{bmatrix} 1 & 5 & \frac{1}{4} & 0 \end{bmatrix}$; $\qquad a_{14}$;

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b) $C = \begin{bmatrix} \frac{5}{2} \\ 8 \\ -2 \\ 1 \end{bmatrix}$; c_{31};

c) $B = \begin{bmatrix} x & y & w & e \\ z & 0 & 3 & t+1 \end{bmatrix}$; b_{23};

d) $D = [u_1 \quad u_2 \quad \cdots \quad u_n]$; d_{1r} (for every $r \in \{1,2,\dots,n\}$).

Solution:

a) A has 1 row and 4 columns so A is 1×4 matrix (i.e., row matrix); $a_{14} = 0$;

b) C has 4 rows and 1 column so C is 4×1 matrix (i.e., column matrix); $c_{31} = -2$;

c) B is 2×4 matrix; $b_{23} = 3$;

d) d) D is $1 \times n$ matrix (i.e., row matrix); $d_{1r} = u_r$.

Example 2.8

Find the values of x, y, z and w from the following equation

$$\begin{bmatrix} x+y & x+z \\ y+z & w \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}.$$

Solution:

Matrices $A = \begin{bmatrix} x+y & x+z \\ y+z & w \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$ are equal only if they have the same elements in the same positions.

This implies that

$$\begin{aligned} x+y &= 3 \\ x+z &= 4 \\ y+z &= 5 \\ w &= 4. \end{aligned}$$

The following is calculated by adding the first three equations:

$$\begin{aligned} 2x+2y+2z &= 3+4+5 \\ 2(x+y+z) &= 12 \\ x+y+z &= 6, \end{aligned}$$

whence it follows

$$\begin{aligned} x &= 6-(y+z) = 6-5 = 1, \\ y &= 6-(x+z) = 6-4 = 2, \\ z &= 6-(x+y) = 6-3 = 3. \end{aligned}$$

Therefore,

$$x = 1, y = 2, z = 3, w = 4.$$

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