From equation 2: $x_{22} = -1 - 3x_{11} - 11x_{12}$.

By including in the third and fourth equations, the result is:

$$-3x_{11} + 3(-4 - 5x_{11} - 5x_{12}) + 5(-1 - 3x_{11} - 11x_{12}) = 16$$
$$-3x_{12} + 3(-4 - 5x_{11} - 5x_{12}) + 9(-1 - 3x_{11} - 11x_{12}) = 21$$

$$\left.\begin{array}{rcrcr} -33x_{11} & - & 70x_{12} & = & 33 \\ -42x_{11} & - & 117x_{12} & = & 42 \end{array}\right\} \Rightarrow x_{11} = -1\,, x_{12} = 0$$

$$x_{21} = -4 - 5x_{11} - 5x_{12} = -4 - 5 \cdot (-1) - 5 \cdot 0 = 1$$
$$x_{22} = -1 - 3x_{11} - 11x_{12} = -1 - 3 \cdot (-1) - 11 \cdot 0 = 2.$$

Therefore, the matrix $X = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ is the (only) solution of this equation.

Similar to the equation in *Example 3*, equations of the form $AX = B$ and $YA = B$ with unknowns X and Y, in which the matrix A is neither regular nor square, are solved.

2.10. MATRIX RANK

A single-column real matrix is also called *a column vector* (or shorter, a *vector*).

A single-row real matrix is also called *a row vector*.

Example 2.30

$$C = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

is a vector of dimension 3 because it has 3 components: 1,4 and 7.

Vector C is *a zero vector* if all its components are equal to zero.

Analogously, a zero-row vector is defined.

The zero row (column) vector is marked by O.

A non-zero row (column) vector is a row (column) vector for which at least one component is different than zero.

Co-funded by the Erasmus+ Programme of the European Union

Example 2.31

Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ can be written in the form

$$A = [C_1 \quad C_2],$$

where $C_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ are column vectors,

or in the form

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

where $R_1 = [1 \quad 2]$, $R_2 = [3 \quad 4]$ and $R_3 = [5 \quad 6]$ are row vectors.

A linear combination of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is any vector C defined by the formula

$$C = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n \, ,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers.

Analogously, a linear combination of row vectors is defined.

Notice that from $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ follows $C = 0$.

The opposite does not need to be valid, i.e., if $C = O$, then it does not necessarily have to be

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

It is said that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is *linearly independent*, i.e., that the vectors $C_1, C_2, \dots, C_n$ are *linearly independent*, if from $C = O$ necessarily follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

It is said that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is *linearly dependent* if it is not linearly independent, i.e., if from $C = O$ does not necessarily follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

i.e., that at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ can be different than zero.

If $C = O$ and for example $\alpha_1 \neq 0$, the result is

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = O \Leftrightarrow \alpha_1 C_1 = -\alpha_2 C_2 - \cdots - \alpha_n C_n \Leftrightarrow C_1 = \beta_2 C_2 + \cdots + \beta_n C_n \, ,$$

Co-funded by the Erasmus+ Programme of the European Union

where $\beta_2 = -\frac{\alpha_2}{\alpha_1}, \dots, \beta_n = -\frac{\alpha_n}{\alpha_1}$ are well defined numbers because $\alpha_1 \neq 0$.

Vector C_1 is written in the form of a linear combination of vectors $C_2, \dots, C_n$.

It can be concluded that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is linearly dependent if and only if at least one of these vectors can be represented as a linear combination of the remaining vectors of that set.

Theorem:

In each real matrix, the maximum number of linearly independent column vectors is equal to the maximum number of linearly independent row vectors. This number is called *the rank of the matrix* $\boldsymbol{A}$ and is marked by $\boldsymbol{r(A)}$.

The rank of the real matrix does not change if elementary transformations are performed on the matrix.

We say that two real matrices of the same dimensions, A and B, are *equivalent* if one can be transformed from the other by applying finally many elementary transformations.

In such case we write $A \sim B$.

It means that equivalent matrices have the same rank.

The rank of the real matrix A is determined using *the Gauss method* – by elementary transformations the matrix A is transformed into an equivalent matrix B in which all elements below the diagonal determined by the elements a $b_{11}, b_{22}, \dots$ are equal to zero.

Note that, by applying this procedure, real square matrices are transformed into upper triangular matrices.

In applying this procedure, the following should be taken into account:

1) If one component of a non-zero row vector is equal to zero, and the same component of another non-zero row vector is different from zero, then those two row vectors are linearly independent. The same is also true for vector columns.

2) Each null row vector reduces the rank of the matrix by 1 which is obvious because every set of row-vectors containing a null row vector is linearly dependent. The same is true for column vectors.

Co-funded by the Erasmus+ Programme of the European Union

Example 2.32

Determine the rank of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 7 \\ -1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 17 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 0 & 1 & 2 & 7 \\ -1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 17 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \end{matrix} \sim \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 2 & 3 & 0 & 17 \end{bmatrix} R_3 + 2R_1 \sim \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 3 & 6 & 21 \end{bmatrix} R_3 - 3R_2$$

$$\sim \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A) = 2$$

Some remarks about the solution:

We can write matrix A in the form $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, where $R_1 = [0 \quad 1 \quad 2 \quad 7]$, $R_2 = [-1 \quad 0 \quad 3 \quad 2]$, $R_3 = [2 \quad 3 \quad 0 \quad 17]$ are row vectors of dimension 4.

It is not difficult to notice that each of the sets $\{R_1, R_2\}$, $\{R_1, R_3\}$, $\{R_2, R_3\}$ is linearly independent. For example, R_1 and R_2 are linearly independent because the first component of R_1 is the number 0, while the first component of R_2 is the number -1.

Let us check this using the definition of linear independence. It should be proven that from

$$\alpha_1 R_1 + \alpha_2 R_2 = O$$

necessarily follows that $\alpha_1 = \alpha_2 = 0$.

$$\alpha_1 R_1 + \alpha_2 R_2 = O$$

$$\alpha_1 [0 \quad 1 \quad 2 \quad 7] + \alpha_2 [-1 \quad 0 \quad 3 \quad 2] = [0 \quad 0 \quad 0 \quad 0]$$

$$[0 \quad \alpha_1 \quad 2\alpha_1 \quad 7\alpha_1] + [-\alpha_2 \quad 0 \quad 3\alpha_2 \quad 2\alpha_2] = [0 \quad 0 \quad 0 \quad 0]$$

$$[-\alpha_2 \quad \alpha_1 \quad 2\alpha_1 + 3\alpha_2 \quad 7\alpha_1 + 2\alpha_2] = [0 \quad 0 \quad 0 \quad 0] \Leftrightarrow \begin{cases} \quad\; - \alpha_2 = 0 \\ \alpha_1 \qquad\; = 0 \\ 2\alpha_1 + 3\alpha_2 = 0 \\ 7\alpha_1 + 2\alpha_2 = 0 \end{cases}$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = 0$$

The result is that $r(A) = 2$, so it can be concluded that the set $\{R_1, R_2, R_3\}$ is linearly dependent (because otherwise the rank of the matrix would be 3, not 2). Let us check that too!

According to the definition, it should be proven that from

$$\beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 = O$$

Co-funded by the Erasmus+ Programme of the European Union

does not necessarily follow that $\beta_1 = \beta_2 = \beta_3 = 0$.

$$\beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 = O$$

$$\beta_1[0 \quad 1 \quad 2 \quad 7] + \beta_2[-1 \quad 0 \quad 3 \quad 2] + \beta_3[2 \quad 3 \quad 0 \quad 17] = [0 \quad 0 \quad 0 \quad 0]$$

$$[-\beta_2 + 2\beta_3 \quad \beta_1 + 3\beta_3 \quad 2\beta_1 + 3\beta_2 \quad 7\beta_1 + 2\beta_2 + 17\beta_3] = [0 \quad 0 \quad 0 \quad 0]$$

$$\Leftrightarrow \begin{cases} \quad -\beta_2 + 2\beta_3 = 0 \\ \beta_1 \quad + 3\beta_3 = 0 \\ 2\beta_1 + 3\beta_2 \quad = 0 \\ 7\beta_1 + 2\beta_2 + 17\beta_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = -3\beta_3 \\ \beta_2 = 2\beta_3 \\ \beta_3 \text{ is any real number} \end{cases}$$

It can be seen that for $\beta_3 = 0$ we have $\beta_1 = \beta_2 = \beta_3 = 0$. However, this is not the only possibility.

Thus, e.g., $\beta_3 = 1 \Rightarrow \beta_1 = -3, \beta_2 = 2$.

Therefore:

$$-3R_1 + 2R_2 + R_3 = O \Leftrightarrow R_3 = 3R_1 - 2R_2.$$

It is proven that a row vector R_3 can be written in a form of a linear combination of row vectors R_1 and R_2.

Exercise (for better understanding of the topic)

Show that for any three real numbers k_1, k_2 and k_3 different from zero and any $a, b, c \in \mathbb{R}$ the upper triangular matrix $A = \begin{bmatrix} k_1 & a & b \\ 0 & k_2 & c \\ 0 & 0 & k_3 \end{bmatrix}$ has a rank 3.

Solution:

$$R_1 = [k_1 \quad a \quad b], R_2 = [0 \quad k_2 \quad c], R_3 = [0 \quad 0 \quad k_3]$$

$$\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3 = O$$

$$\alpha_1[k_1 \quad a \quad b] + \alpha_2[0 \quad k_2 \quad c] + \alpha_3[0 \quad 0 \quad k_3] = [0 \quad 0 \quad 0]$$

$$[k_1\alpha_1 \quad a\alpha_1 + k_2\alpha_2 \quad b\alpha_1 + c\alpha_2 + k_3\alpha_3] = [0 \quad 0 \quad 0] \Leftrightarrow \begin{cases} k_1\alpha_1 = 0 \\ a\alpha_1 + k_2\alpha_2 = 0 \\ b\alpha_1 + c\alpha_2 + k_3\alpha_3 = 0 \end{cases}$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Co-funded by the
Erasmus+ Programme
of the European Union

This is the so-called lower-triangular system which is simply solved by determining, from the first equation, α_1, then from the second α_2, and finally from the third equation α_3.

A more detailed description of the Gauss method

The rank of the $m \times n$ real matrix A needs to be determined.

If $a_{11} = 0$, then using elementary transformations, the matrix A is transformed into the matrix A_1 in which $a_{11} \neq 0$. If such matrix A_1 does not exist, then the matrix A is a null-matrix with the rank $r(A) = 0$. Otherwise, i.e., if such a matrix A_1 exists, using elementary transformations, the matrix A_1 is transformed into the matrix A_1' in which $a_{21} = a_{31} = \cdots = a_{m1} = 0$. Then, using elementary transformations (but not on the 1st row nor on the 1st column of the matrix A_1'), the matrix A_1' is transformed into the matrix A_2 in which $a_{22} \neq 0$. If such a matrix A_2 does not exist, then $r(A) = r(A_1') = 1$. Otherwise, using elementary transformations (but not on the 1st row nor on the 1st column of the matrix A_2) the matrix A_2 is transformed into the matrix A_2' in which $a_{32} = a_{42} = \cdots = a_{m2} = 0$. Then, using elementary transformations (but not on the first two rows nor on the first two columns of the matrix A_2'), the matrix A_2' is transformed into the matrix A_3 in which $a_{33} \neq 0$. If such a matrix A_3 does not exist, then $r(A) = r(A_2') = 2$. Otherwise, using elementary transformations (but not on the first two rows nor on the first two columns of the matrix A_3), the matrix A_3 is transformed into the matrix A_3' in which $a_{43} = a_{53} = \cdots = a_{m3} = 0$. Then, using elementary transformations (but not on the first three rows nor on the first three columns of the matrix A_3'), the matrix A_3' is transformed into the matrix A_4 in which $a_{44} \neq 0$. If such a matrix A_4 does not exist, then $r(A) = r(A_3') = 3$. Etc.

Sometimes it is convenient to put 1s on the diagonal, which can always be easily achieved, although it is sometimes difficult to avoid calculating with fractions. So, with the usage of elementary transformations, 1 is put on the position a_{11}. The new elements of the new matrix are marked in the same way as the elements of the matrix A. Then the 1st row is multiplied by $-a_{21}, -a_{31}, \dots, -a_{m1}$ and is added to the 2nd row, the 3rd row, …, the mth row. In this way, a new matrix is created (with the same element labels as in the matrix A) in which all the elements in the 1st column below the element a_{11} are equal to zero. After that, neither the 1st row nor the 1st column are going to be changed in elementary transformations. Then, 1 is put on the position a_{22}. The new elements of the new matrix are marked in the same way as the elements of the matrix A. Then the 2nd row is multiplied by $-a_{32}, -a_{42}, \dots, -a_{m2}$ and is added to the 3rd row, the 4th row, …, the mth row. In this way, a new matrix is created (with the same element labels as in the matrix A) in which all the elements in the 1st column below the element a_{11} and all the elements in the 2nd column below the element a_{22} are equal to zero. After that, neither the first two rows nor the first two columns are going to be changed in elementary transformations. Etc.

At the end of the procedure, if A is not a null-matrix (each null-matrix has a rank 0), we get:

a) the matrix $B = \left[\frac{B_1 | B_3}{B_2}\right]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, and B_2 is a null-matrix, so $r(A) = r(B) = r(B_1) =$ order of B_1,

Co-funded by the Erasmus+ Programme of the European Union

or

b) the matrix $B=\left[\frac{B_1}{B_2}\right]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, and B_2 is a null-matrix, so $r(A)=r(B)=r(B_1)=$ order of $B_1=n$ (which is possible only when $m>n$),

or

c) the matrix $B=[B_1|B_3]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, then $r(A)=r(B)=r(B_1)=$ order of $B_1=m$ (which is possible only when $m<n$),

or

d) the upper triangular matrix B that has no zeros on the main diagonal, so

$r(A)=r(B)=r(B_1)=$ order of $B_1=m=n$ (which is possible only when $m=n$).

Notice that in any case $0\le r(A)\le \min\{m,n\}$.

Thus, the rank of the matrix A, which is equivalent to the matrix B, in one of the previous 4 cases is easily determined.

In the *Example 3.3*, the following result is obtained

$$A\sim\begin{bmatrix}-1&0&3&2\\0&1&2&7\\0&0&0&0\end{bmatrix}=\left[\frac{B_1|B_3}{B_2}\right], \text{ where } B_1=\begin{bmatrix}-1&0\\0&1\end{bmatrix}, B_2=\begin{bmatrix}0&0&0&0\end{bmatrix}, B_3=\begin{bmatrix}3&2\\2&7\end{bmatrix}.$$

Therefore, $r(A)=r(B)=r(B_1)=$ order of $B_1=2$. (case a)).

$$\text{Thus, } A_1=\begin{bmatrix}-1&0&3&2\\0&1&2&7\\2&3&0&17\end{bmatrix}, A_1'=A_2=\begin{bmatrix}-1&0&3&2\\0&1&2&7\\0&3&6&21\end{bmatrix}.$$

Example 2.33

Determine the rank of the matrix $A=\begin{bmatrix}9&20&6\\10&9&-5\\8&31&17\end{bmatrix}$.

Solution:

$$A=\begin{bmatrix}9&20&6\\10&9&-5\\8&31&17\end{bmatrix}\begin{matrix}R_1-R_3\\ \\ \end{matrix}\sim\begin{bmatrix}1&-11&-11\\10&9&-5\\8&31&17\end{bmatrix}\begin{matrix}\\R_2-10R_1\\R_3-8R_1\end{matrix}\sim\begin{bmatrix}1&-11&-11\\0&119&105\\0&119&105\end{bmatrix}\begin{matrix}\\ \\R_3-R_2\end{matrix}$$

$$\sim\underbrace{\begin{bmatrix}1&-11&-11\\0&119&105\\0&0&0\end{bmatrix}}_{=B}=\left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=\begin{bmatrix}1&-11\\0&119\end{bmatrix}, B_2=\begin{bmatrix}0&0&0\end{bmatrix}, B_3=\begin{bmatrix}-11\\105\end{bmatrix}$.

Then $r(A)=r(B)=r(B_1)=$ order of $B_1=2$.

Co-funded by the
Erasmus+ Programme
of the European Union

Example 2.34

Determine the rank of the matrix $A=\begin{bmatrix}1&1&1\\1&m&1\\1&m^2&m^2\end{bmatrix}$ depending on the real parameter m.

Solution:

$$A=\begin{bmatrix}1&1&1\\1&m&1\\1&m^2&m^2\end{bmatrix}\begin{matrix}\\R_2-R_1\\R_3-R_1\end{matrix}\sim\begin{bmatrix}1&1&1\\0&m-1&0\\0&m^2-1&m^2-1\end{bmatrix};$$

$$m=1\Rightarrow A\sim\underbrace{\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}}_{=B}=\begin{bmatrix}B_1|B_3\\\hline B_2\end{bmatrix},$$

where $B_1=[1]\,,B_2=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\,,B_3=[1\quad 1]$.
Thus, $r(A)=r(B)=r(B_1)=$ order of $B_1=1$.

$$m\neq 1\Rightarrow A\sim\begin{bmatrix}1&1&1\\0&m-1&0\\0&(m-1)(m+1)&(m-1)(m+1)\end{bmatrix}\begin{matrix}\\R_2/(m-1)\\R_3/(m-1)\end{matrix}$$

$$\sim\begin{bmatrix}1&1&1\\0&1&0\\0&m+1&m+1\end{bmatrix}\begin{matrix}\\\\R_3-(m+1)R_2\end{matrix}\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&m+1\end{bmatrix};$$

$$m=-1\Rightarrow A\sim\underbrace{\begin{bmatrix}1&1&1\\0&1&0\\0&0&0\end{bmatrix}}_{=B}=\begin{bmatrix}B_1|B_3\\\hline B_2\end{bmatrix},$$

where $B_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}\,,B_2=[0\quad 0\quad 0]\,,B_3=\begin{bmatrix}1\\0\end{bmatrix}$;
then $r(A)=r(B)=r(B_1)=\text{order of } B_1=2$.

$$m\notin\{-1,1\}\Rightarrow A\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&m+1\end{bmatrix}\begin{matrix}\\\\R_3/(m+1)\end{matrix}\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow r(A)=r(B)=3 \text{ (case d))}.$$

Therefore,

$$m=1\Rightarrow r(A)=1,$$
$$m=-1\Rightarrow r(A)=2,$$
$$m\notin\{-1,1\}\Rightarrow r(A)=3.$$

Example 2.35

For what value of the real parameter t is the rank of the matrix $A=\begin{bmatrix}t&1&1&1\\1&t&1&1\\1&1&t&1\\1&1&1&t\end{bmatrix}$ equal to 3?

Co-funded by the Erasmus+ Programme of the European Union

Solution:

$$A=\begin{bmatrix} t & 1 & 1 & 1 \\ 1 & t & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & t & 1 & 1 \\ t & 1 & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{bmatrix} \begin{matrix} \\ R_2-tR_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix} \sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1-t^2 & 1-t & 1-t \\ 0 & 1-t & t-1 & 0 \\ 0 & 1-t & 0 & t-1 \end{bmatrix};$$

case 1: $t=1$

$$A\sim \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{=B} = \left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=[1]\,,B_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,,B_3=[1\quad 1\quad 1];$
then $r(A)=r(B)=r(B_1)=\text{order of } B_1=1$.

case 2: $t\neq 1$

$$A\sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & (1-t)(1+t) & 1-t & 1-t \\ 0 & 1-t & -(1-t) & 0 \\ 0 & 1-t & 0 & -(1-t) \end{bmatrix} \begin{matrix} \\ R_2/(1-t) \\ R_3/(1-t) \\ R_4/(1-t) \end{matrix} \sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1+t & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ R_3 \\ R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+t & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3-(1+t)R_2 \\ R_4-R_2 \end{matrix} \sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2+t & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_4 \\ R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2+t & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4-(2+t)R_3 \end{matrix} \sim \begin{bmatrix} 1 & t & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3+t \end{bmatrix};$$

$$t=-3 \Rightarrow A\sim \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A)=3.$$

Thus, for $t=-3$

$$r(A)=3.$$

Example 2.36

Determine the rank of the matrix $A=\begin{bmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & 8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{bmatrix}$.

Co-funded by the
Erasmus+ Programme
of the European Union

Solution:

$$A=\begin{bmatrix}1&-2&3&-1&-1&-2\\2&-1&1&0&-2&-2\\-2&-5&8&-4&3&-1\\6&0&-1&2&-7&-5\\-1&-1&1&-1&2&1\end{bmatrix}\begin{matrix}\\R_2-2R_1\\R_3+2R_1\\R_4-6R_1\\R_5+R_1\end{matrix}\sim\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&-9&14&-6&1&-5\\0&12&-19&8&-1&7\\0&-3&4&-2&1&-1\end{bmatrix}\begin{matrix}\\ \\R_3+3R_2\\R_4-4R_2\\R_5+R_2\end{matrix}$$

$$\sim\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&0&-1&0&1&1\\0&0&1&0&-1&-1\\0&0&-1&0&1&1\end{bmatrix}\begin{matrix}\\ \\ \\R_4+R_3\\R_5-R_3\end{matrix}\sim\underbrace{\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&0&-1&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}}_{=B}=\left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=\begin{bmatrix}1&-2&3\\0&3&-5\\0&0&-1\end{bmatrix}, B_2=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}, B_3=\begin{bmatrix}-1&-1&-2\\2&0&2\\0&1&1\end{bmatrix};$

then $r(A)=r(B)=r(B_1)=\text{order of } B_1=3$.

Co-funded by the
Erasmus+ Programme
of the European Union