2.11. SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

A system of m linear algebraic equations with unknowns $x_1, x_2, \dots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (2)$$

or

$$A \cdot X = B,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is *a coefficient matrix*, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is *a vector of unknowns*, and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is *a vector of free terms*.

The system (1) is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$. If at least one of the scalars $b_i\,, i = 1,2, \dots, m$, is different than zero, it is said that the system is *inhomogeneous*.

Every homogeneous system has at least *trivial solution*, i.e., solution in which

$$x_1 = x_2 = \cdots = x_n = 0.$$

It is said that X is a nontrivial solution of a system if at least one component of the vector X is different than zero.

An augmented matrix $\tilde{A}$ for the system (1) is also defined.

$$\tilde{A} = [A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Kronecker-Capelli theorem:

The system (1) has at least one solution if and only if the rank $r(A)$ of the coefficient matrix A is equal to the rank $r(\tilde{A})$ of the augmented matrix $\tilde{A}$.

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a) System in which $m = n = r(A) = r(\tilde{A})$ $(\det A \neq 0)$ has unique solution and can be solved by:

I. Gauss method (Note 3),

II. an inverse matrix

$$A \cdot X = B \Rightarrow X = A^{-1} \cdot B\,, \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

III. Cramer's rule

$$x_i = \frac{D(x_i)}{\det A}\,, \quad i = 1,2,\dots,n,$$

where $D(x_i)$ is the determinant, obtained from $\det A$, by replacing its ith column with the column of free terms, i.e.

$$D(x_i) = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{vmatrix}.$$

b) System in which $r(A) = r(\tilde{A}) = r < n$ has infinitely many solutions and can be solved by the Gauss method (Note 3). r linearly independent equations with r unknowns are selected. Those r unknowns are calculated depending on $n - r$ of the remaining unknowns (so-called *parameters*).

It is said that this type of the system is *an indefinite system*.

Note 1:

A homogeneous system in which the number of equations coincides with the number of unknowns has also a non-trivial solution if and only if $\det A = 0$.

Note 2:

If we apply elementary transformations, but only on the rows of matrix $\tilde{A}$, an augmented matrix of the system, that is equivalent to the system (1), is obtained.

It can easily be seen that equivalent systems have the same solutions (if solutions exist).

Note 3:

Any system of linear equations can be solved by the Gauss method of eliminating variables by reducing the system (1) to an equivalent system with an upper triangular matrix. An extension of the Gauss method is the Gauss-Jordan method where the system matrix is reduced to a unit matrix from which directly provides the solution.

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Note 4: For a system that has no solution, we say that it is *incompatible*, *impossible*, or *inconsistent*. The system is impossible if and only if $r(A) \neq r(\tilde{A})$.

Then there is no vector X such that $A \cdot X = B$.

Example 2.37

Solve the system of linear equations

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 10 \\ 3x_1 + 7x_2 + 4x_3 = 3 \\ x_1 + 2x_2 + 2x_3 = 3 \end{cases}$$

a) by Cramer's rule,
b) by inverse matrix,
c) by Gauss method,
d) by Gauss-Jordan method.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 1 & 2 & 2 \end{bmatrix}, \tilde{A} = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 10 \\ 3 & 7 & 4 & 3 \\ 1 & 2 & 2 & 3 \end{array}\right], X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix};$$

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 1 & 2 & 2 \end{vmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \end{matrix} = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} R_2 + R_1 = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 2 \end{vmatrix}$$
$$= 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \neq 0;$$

a)

$$D(x_1) = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 10 & 3 & 5 \\ 3 & 7 & 4 \\ 3 & 2 & 2 \end{vmatrix} \begin{matrix} R_1 - 3R_3 \\ \\ \end{matrix} = \begin{vmatrix} 1 & -3 & -1 \\ 3 & 7 & 4 \\ 3 & 2 & 2 \end{vmatrix} \begin{matrix} \\ R_2 - R_3 \\ R_3 - 3R_1 \end{matrix}$$
$$= \begin{vmatrix} 1 & -3 & -1 \\ 0 & 5 & 2 \\ 0 & 11 & 5 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 5 & 2 \\ 0 & 1 & 1 \end{vmatrix} R_2 - 2R_3 = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = 3 \Rightarrow x_1 = \frac{D(x_1)}{\det A} = \frac{3}{1} = 3,$$

$$D(x_2) = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = \begin{vmatrix} 2 & 10 & 5 \\ 3 & 3 & 4 \\ 1 & 3 & 2 \end{vmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \end{matrix} = \begin{vmatrix} 0 & 4 & 1 \\ 0 & -6 & -2 \\ 1 & 3 & 2 \end{vmatrix} R_2 + 2R_1$$
$$= \begin{vmatrix} 0 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2 \Rightarrow x_2 = \frac{D(x_2)}{\det A} = \frac{-2}{1} = -2$$

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$$D(x_3) = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 10 \\ 3 & 7 & 3 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \end{matrix} = \begin{vmatrix} 0 & -1 & 4 \\ 0 & 1 & -6 \\ 1 & 2 & 3 \end{vmatrix} R_2 + R_1$$

$$= \begin{vmatrix} 0 & -1 & 4 \\ 0 & 0 & -2 \\ 1 & 2 & 3 \end{vmatrix} = 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} -1 & 4 \\ 0 & -2 \end{vmatrix} = 2 \Rightarrow x_3 = \frac{D(x_3)}{\det A} = \frac{2}{1} = 2$$

b)

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 7 & 4 \\ 2 & 2 \end{vmatrix} = 6 \,, A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} = 4 \,, A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 3 & 5 \\ 7 & 4 \end{vmatrix} = -23$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2 \,, A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = -1 \,, A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 7$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix} = -1 \,, A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1 \,, A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 5$$

$$X = A^{-1} \cdot B = \begin{bmatrix} 6 & 4 & -23 \\ -2 & -1 & 7 \\ -1 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \Rightarrow x_1 = 3 \,, x_2 = -2 \,, x_3 = 2$$

c)

$$\tilde{A} = \left[\begin{matrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 1 & 2 & 2 \end{matrix}\middle|\begin{matrix} 10 \\ 3 \\ 3 \end{matrix}\right] \begin{matrix} R_3 \\ \\ R_1 \end{matrix} \sim \left[\begin{matrix} 1 & 2 & 2 \\ 3 & 7 & 4 \\ 2 & 3 & 5 \end{matrix}\middle|\begin{matrix} 3 \\ 3 \\ 10 \end{matrix}\right] \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \left[\begin{matrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} 3 \\ -6 \\ 4 \end{matrix}\right] R_3 + R_2$$

$$\sim \left[\begin{matrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{matrix}\middle|\begin{matrix} 3 \\ -6 \\ -2 \end{matrix}\right]$$

equation 3: $-x_3 = -2 \Rightarrow x_3 = 2$

equation 2: $x_2 - 2x_3 = -6 \Rightarrow x_2 = 2x_3 - 6 = 2 \cdot 2 - 6 = -2$

equation 1: $x_1 + 2x_2 + 2x_3 = 3 \Rightarrow x_1 = 3 - 2x_2 - 2x_3 = 3 - 2 \cdot (-2) - 2 \cdot 2 = 3$

d)

$$\tilde{A} \sim \left[\begin{matrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{matrix}\middle|\begin{matrix} 3 \\ -6 \\ -2 \end{matrix}\right] \begin{matrix} R_1 - 2R_2 \\ \\ -R_3 \end{matrix} \sim \left[\begin{matrix} 1 & 0 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 15 \\ -6 \\ 2 \end{matrix}\right] \begin{matrix} R_1 - 6R_3 \\ R_2 + 2R_3 \\ \end{matrix} \sim \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 3 \\ -2 \\ 2 \end{matrix}\right]$$

$$\Rightarrow x_1 = 3 \,, x_2 = -2 \,, x_3 = 2$$

Example 2.38

Solve the system of linear equations

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$$\begin{cases} 3x_1 & + & 4x_2 & = & 11 \\ 4x_1 & + & 3x_2 & = & 10 \end{cases}$$

by Cramer's rule and by the inverse matrix.

Solution:

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

By Cramer's rule:

$$\det A = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9 - 16 = -7$$

$$D(x_1) = \begin{vmatrix} 11 & 4 \\ 10 & 3 \end{vmatrix} = 33 - 40 = -7 \Rightarrow x_1 = \frac{D(x_1)}{\det A} = 1$$

$$D(x_2) = \begin{vmatrix} 3 & 11 \\ 4 & 10 \end{vmatrix} = 30 - 44 = -14 \Rightarrow x_2 = \frac{D(x_2)}{\det A} = 2$$

By the inverse matrix:

$$A_{11} = (-1)^{1+1} \cdot 3 = 3 \qquad A_{21} = (-1)^{2+1} \cdot 4 = -4$$

$$A_{12} = (-1)^{1+2} \cdot 4 = -4 \qquad A_{22} = (-1)^{2+2} \cdot 3 = 3$$

$$X = A^{-1}B = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 10 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 33 - 40 \\ -44 + 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, x_2 = 2$$

Example 2.39

Find all solutions of the following system.

$$\begin{cases} 4x_1 & + & 3x_2 & - & 3x_3 & - & 3x_4 & + & 3x_5 & = & 2 \\ 2x_1 & + & x_2 & - & x_3 & - & x_4 & - & x_5 & = & 0 \\ 7x_1 & + & 5x_2 & - & 5x_3 & - & 5x_4 & + & 4x_5 & = & 3 \\ x_1 & + & x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & 1 \end{cases}$$

Solution:

The system cannot be solved by Cramer's rule or by the invers matrix because $n > m \geq r(A)$.

We will solve it by applying Gauss method.

$$\tilde{A} = \left[\begin{array}{ccccc|c} 4 & 3 & -3 & -3 & 3 & 2 \\ 2 & 1 & -1 & -1 & -1 & 0 \\ 7 & 5 & -5 & -5 & 4 & 3 \\ 1 & 1 & -1 & -1 & 2 & 1 \end{array}\right] \begin{matrix} R_4 \\ \\ R_1 \\ R_3 \end{matrix} \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -1 & -1 & 0 \\ 4 & 3 & -3 & -3 & 3 & 2 \\ 7 & 5 & -5 & -5 & 4 & 3 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 7R_1 \end{matrix}$$

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$$\sim\left[\begin{array}{ccccc|c}1&1&-1&-1&2&1\\0&-1&1&1&-5&-2\\0&-1&1&1&-5&-2\\0&-2&2&2&-10&-4\end{array}\right]\begin{array}{c}\\ \\R_3-R_2\\R_4-2R_2\end{array}\sim\left[\begin{array}{ccccc|c}1&1&-1&-1&2&1\\0&-1&1&1&-5&-2\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]$$

$$\Rightarrow r(A)=r(\tilde{A})=2$$

Therefore, the system has (at least) one solution.

The last matrix, i.e., the matrix $\left[\begin{array}{ccccc|c}1&1&-1&-1&2&1\\0&-1&1&1&-5&-2\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]$, is an augmented matrix of the system that has the same solutions as the defined system. Note that in this system we actually have only two equations. Namely, the rows with all zeros, i.e., the last two rows represent equations that are valid for each selection of unknowns $x_1, x_2, \ldots, x_5$. So, only the first 2 rows, i.e., two equations, are observed. Since there are 5 unknowns and two equations, 3 unknowns can be chosen in any way (because $5-2=3$), and the remaining two are determined using the selected 3. Unknowns that can be chosen in any way are called *system parameters* and are labelled in a special way.

Accordingly,

$$\text{the number of parameters} = n - r(A) = 5 - 2 = 3.$$

Let the parameters be unknowns x_3, x_4 and x_5. The following labels are introduced:

$$x_3=\alpha\,, x_4=\beta\,, x_5=\gamma.$$

The system

$$\begin{cases}x_1 + x_2 - x_3 - x_4 + 2x_5 = 1\\ \quad\; - x_2 + x_3 + x_4 - 5x_5 = -2\end{cases}$$

with the augmented matrix $\left[\begin{array}{ccccc|c}1&1&-1&-1&2&1\\0&-1&1&1&-5&-2\end{array}\right]$ is solved starting from the end, i.e., from the second equation.

equation 2: $-x_2+\alpha+\beta-5\gamma=-2 \Leftrightarrow x_2=2+\alpha+\beta-5\gamma$

equation 1: $x_1+2+\cancel{\alpha}+\cancel{\beta}-5\gamma\cancel{-\alpha}\cancel{-\beta}+2\gamma=1 \Leftrightarrow x_1=3\gamma-1.$

All system solutions are all arranged fives $(3\gamma-1, 2+\alpha+\beta-5\gamma, \alpha, \beta, \gamma)$, where $\alpha,\beta,\gamma\in\mathbb{R}$ are arbitrary parameters. Therefore, it is written

$$X=\begin{bmatrix}3\gamma-1\\2+\alpha+\beta-5\gamma\\\alpha\\\beta\\\gamma\end{bmatrix}.$$

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Example 2.40

$$\begin{cases} 6x_1 & + & 2x_2 & - & 2x_3 & + & 5x_4 & + & 7x_5 & = & 0 \\ 9x_1 & + & 4x_2 & - & 3x_3 & + & 8x_4 & + & 9x_5 & = & 0 \\ 6x_1 & + & 6x_2 & - & 2x_3 & + & 7x_4 & + & x_5 & = & 0 \\ 3x_1 & + & 4x_2 & - & x_3 & + & 4x_4 & - & x_5 & = & 0 \end{cases}$$

Solution:

A system in which **each** equation on the right side of the equality has zero is called a **homogeneous system**. Such a system obviously always has a solution (in which all unknowns are equal to zero). Such a solution is called a trivial solution. However, this solution does not have to be the only one.

When determining the rank of a matrix of a homogeneous system, it is not necessary to write zeros on the right-hand sides because they do not change by applying elementary transformations.

$$A = \begin{bmatrix} 6 & 2 & -2 & 5 & 7 \\ 9 & 4 & -3 & 8 & 9 \\ 6 & 6 & -2 & 7 & 1 \\ 3 & 4 & -1 & 4 & -1 \end{bmatrix} \begin{matrix} R_4 \\ R_3 \\ R_1 \\ R_2 \end{matrix} \sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 6 & 6 & -2 & 7 & 1 \\ 6 & 2 & -2 & 5 & 7 \\ 9 & 4 & -3 & 8 & 9 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 0 & -2 & 0 & -1 & 3 \\ 0 & -6 & 0 & -3 & 9 \\ 0 & -8 & 0 & -4 & 12 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \\ R_4 - 4R_2 \end{matrix} \sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 0 & -2 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow r(A) = r(\tilde{A}) = 2$$

the number of parameters$= n - r(A) = 5 - 2 = 3$

$$x_3 = \alpha \,, x_4 = \beta \,, x_5 = \gamma$$

equation 2:

$$-2x_2 - x_4 + 3x_5 = 0 \Rightarrow x_2 = \frac{3x_5 - x_4}{2} = \frac{3\gamma - \beta}{2}$$

equation 1:

$$3x_1 + 4x_2 - x_3 + 4x_4 - x_5 = 0 \Rightarrow x_1 = \frac{x_3 + x_5 - 4x_2 - 4x_4}{3} = \frac{\alpha - 2\beta - 5\gamma}{3}$$

All system solutions are all arranged fives $\left(\frac{\alpha-2\beta-5\gamma}{3}, \frac{3\gamma-\beta}{2}, \alpha, \beta, \gamma\right)$, where $\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary parameters.

Therefore,

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$$X = \begin{bmatrix} \frac{\alpha - 2\beta - 5\gamma}{3} \\ \frac{3\gamma - \beta}{2} \\ \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

We do not always have to take for the parameters the unknowns x_3, x_4 and x_5.

For example, if x_1, x_2 and x_5 are chosen parameters:

$$x_1 = u_1 \,, x_2 = u_2 \text{ and } x_5 = u_3 \,,$$

then, the result of the 2nd equation is

$$-2u_2 - x_4 + 3u_3 = 0 \Leftrightarrow x_4 = 3u_3 - 2u_2 \,,$$

and by including this result into the 1st equation, results in

$$3u_1 + 4u_2 - x_3 + 4(3u_3 - 2u_2) - u_3 = 0 \Leftrightarrow x_3 = 3u_1 - 4u_2 + 11u_3.$$

Therefore, in this case we have a simpler notation of the solution:

$$X = \begin{bmatrix} u_1 \\ u_2 \\ 3u_1 - 4u_2 + 11u_3 \\ 3u_3 - 2u_2 \\ u_3 \end{bmatrix}.$$

Of course, if $r(A) = r(\tilde{A}) = n$, then $n - r(A) = 0$, so no unknown can be a parameter, i.e., the system has a unique solution.

Example 2.41

$$\begin{cases} 4x - 4y + z = 8 \\ 6x - 3y - 2z = 21 \\ -x + 3y + 7z = 4 \end{cases}$$

Solution:

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & -4 & 1 & 8 \\ 6 & -3 & -2 & 21 \\ -1 & 3 & 7 & 4 \end{array}\right] \begin{matrix} R_3 \\ R_1 \\ R_2 \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 4 & -4 & 1 & 8 \\ 6 & -3 & -2 & 21 \end{array}\right] \begin{matrix} \\ R_2 + 4R_1 \\ R_3 + 6R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 8 & 29 & 24 \\ 0 & 15 & 40 & 45 \end{array}\right] \begin{matrix} \\ 2R_2 - R_3 \\ R_3/5 \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 1 & 18 & 3 \\ 0 & 3 & 8 & 9 \end{array}\right] \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 1 & 18 & 3 \\ 0 & 0 & -46 & 0 \end{array}\right]$$

$$\Rightarrow r(A) = r(\tilde{A}) = n = 3$$

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$$\Rightarrow \begin{cases} \text{equation 3:} & z = 0, \\ \text{equation 2:} & y = 3, \\ \text{equation 1:} & -x + 3y = 4 \Rightarrow x = 5. \end{cases}$$

Example 2.42

A system of linear equations is given

$$\begin{cases} 4x_1 & - & 2x_2 & + & 5x_3 & + & 3x_4 & = & 0 \\ 3x_1 & + & 6x_2 & + & 5x_3 & - & 4x_4 & = & 0 \\ 3x_1 & + & 3x_2 & + & px_3 & - & 1.5x_4 & = & 0 \\ x_1 & + & 4x_2 & + & 2x_3 & - & 3x_4 & = & 0 \end{cases},$$

where $p \in \mathbb{R}$. It is necessary to determine a p for which an arranged four $(-6,1,4,2)$ is the solution of a given system. Is it possible to choose a p so that the system has a unique solution?

Solution:

$$A = \begin{bmatrix} 4 & -2 & 5 & 3 \\ 3 & 6 & 5 & -4 \\ 3 & 3 & p & -1.5 \\ 1 & 4 & 2 & -3 \end{bmatrix} \begin{matrix} R_4 \\ \\ \\ R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 3 & 6 & 5 & -4 \\ 3 & 3 & p & -1.5 \\ 4 & -2 & 5 & 3 \end{bmatrix} \begin{matrix} \\ \\ 2R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 3 & 6 & 5 & -4 \\ 6 & 6 & 2p & -3 \\ 4 & -2 & 5 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 6R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & -6 & -1 & 5 \\ 0 & -18 & 2p - 12 & 15 \\ 0 & -18 & -3 & 15 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \\ R_4 - 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & -6 & -1 & 5 \\ 0 & 0 & 2p - 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 cases.

1) $2p - 9 = 0 \Leftrightarrow p = \frac{9}{2}$; $r(A) = 2$

$$\text{the number of parameters} = n - r(A) = 4 - 2 = 2$$

$$x_2 = \alpha \,, x_4 = \beta$$

equation 2: $-6x_2 - x_3 + 5x_4 = 0 \Rightarrow x_3 = 5x_4 - 6x_2 = 5\beta - 6\alpha$

equation 1: $x_1 + 4x_2 + 2x_3 - 3x_4 = 0 \Rightarrow x_1 = 3x_4 - 4x_2 - 2x_3 = 8\alpha - 7\beta$

All system solutions are all arranged fours $(8\alpha - 7\beta, \alpha, 5\beta - 6\alpha, \beta)$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary parameters.

If $x_2 = \alpha = 1$ and $x_4 = \beta = 2$, then

$$x_1 = 8\alpha - 7\beta = -6,$$
$$x_3 = 5\beta - 6\alpha = 5.$$

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Accordingly, the point $(-6,1,4,2)$ is the solution of the system for $p=\frac{9}{2}$.

2) $2p-9\neq 0 \Leftrightarrow p\neq\frac{9}{2}$; $r(A)=3$

$$\text{the number of parameters} = n-r(A)=4-3=1$$

$$x_2=5\alpha$$

equation 3: $\underbrace{(2p-9)}_{\neq 0}x_3=0 \Rightarrow x_3=0$

equation 2: $-6x_2-x_3+5x_4=0 \Rightarrow x_4=\frac{6x_2+x_3}{5}=6\alpha$

equation 1: $x_1+4x_2+2x_3-3x_4=0 \Rightarrow x_1=3x_4-4x_2-2x_3=-2\alpha$

Note that the unknown x_3 (in case 2) cannot be a parameter (it must be exactly equal to zero for equation 3 to be valid).

All system solutions are all arranged fours $(-2\alpha,5\alpha,0,6\alpha)$, where $\alpha\in\mathbb{R}$ is an arbitrary parameter.

It can be concluded that there is no $p\in\mathbb{R}$ for which the system has a unique solution.

Example 2.43

Prove that the system

$$\begin{cases} 2x_1 + 2x_2 + 2x_4 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 0 \\ -x_1 - x_2 + 2x_3 + 2x_4 = 2 \\ -5x_1 - 4x_2 + 5x_3 + 7x_4 = 5 \end{cases}$$

is impossible.

Solution:

$$\tilde{A}=\left[\begin{array}{cccc|c} 2 & 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & -1 & 0 \\ -1 & -1 & 2 & 2 & 2 \\ -5 & -4 & 5 & 7 & 5 \end{array}\right] \begin{matrix} R_1/2 \\ \\ \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & -1 & 0 \\ -1 & -1 & 2 & 2 & 2 \\ -5 & -4 & 5 & 7 & 5 \end{array}\right] \begin{matrix} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4+5R_1 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 & -2 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 1 & 5 & 12 & 10 \end{array}\right] \begin{matrix} \\ \\ \\ R_4+R_2 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 & -2 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 6 & 9 & 8 \end{array}\right] \begin{matrix} \\ \\ \\ R_4-3R_3 \end{matrix}$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 & -2 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \Rightarrow r(A) = 3 \neq 4 = r(\tilde{A}).$$

The last line in the last matrix is an equation for which no choice of unknowns can be satisfactory. Namely, for any $x_1, x_2, x_3, x_4 \in \mathbb{R}$ is

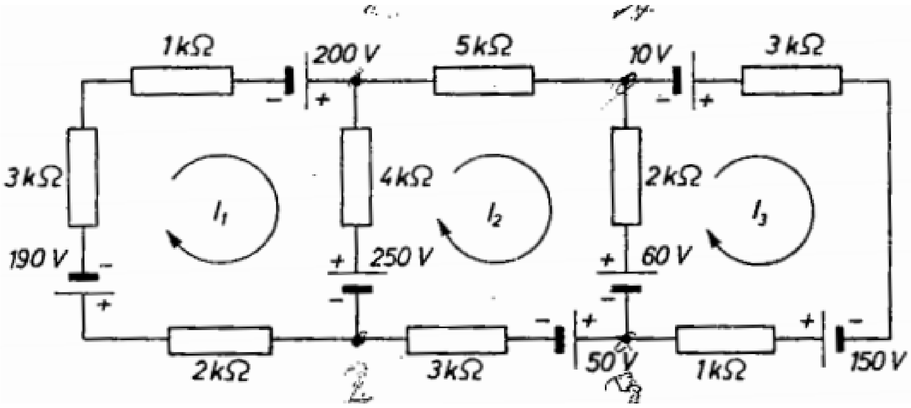
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0,$$

and $0 \neq -1$.

2.12. SOME EXAMPLES OF MATRIX APPLICATION

Example 2.44

The figure below shows one electrical circuit, such as an electrical circuit on a ship. We want to determine the current strength in all branches. It will be shown how this problem is reduced to the problem of solving a system of three linear equations with 3 unknowns, using the method of contour currents.



The circuit has 6 branches and 4 nodes, so it is necessary to select 3 independent contours.

For the selected contours I_1, I_2 and I_3 the contour currents have been drawn in the clockwise direction.

The equations of contour currents are:

$$I_1(3000 + 1000 + 4000 + 2000) - I_2 \cdot 4000 = -190 + 200 - 250,$$

$$I_2(4000 + 5000 + 2000 + 3000) - I_1 \cdot 4000 - I_3 \cdot 2000 = 250 - 60 - 50,$$

$$I_3(2000 + 3000 + 1000) - I_2 \cdot 2000 = 60 + 10 + 150.$$

After addition and shortening, the following system is obtained.

$$\begin{cases} 10I_1 - 4I_2 + 0I_3 = -0.24 \\ -4I_1 + 14I_2 - 2I_3 = 0.14 \\ 0I_1 - 2I_2 + 6I_3 = 0.22 \end{cases}$$

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