$$\sim\left[\begin{array}{cccc|c}1 & 1 & 0 & 1 & 1\\ 0 & -1 & 1 & -3 & -2\\ 0 & 0 & 2 & 3 & 3\\ 0 & 0 & 0 & 0 & -1\end{array}\right] \Rightarrow r(A) = 3 \neq 4 = r(\tilde{A}).$$

The last line in the last matrix is an equation for which no choice of unknowns can be satisfactory. Namely, for any $x_1, x_2, x_3, x_4 \in \mathbb{R}$ is

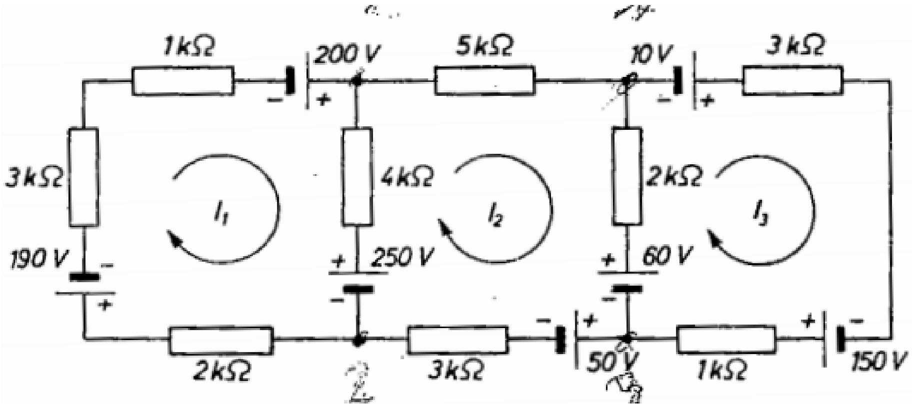
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0,$$

and $0 \neq -1$.

2.12. SOME EXAMPLES OF MATRIX APPLICATION

Example 2.44

The figure below shows one electrical circuit, such as an electrical circuit on a ship. We want to determine the current strength in all branches. It will be shown how this problem is reduced to the problem of solving a system of three linear equations with 3 unknowns, using the method of contour currents.



The circuit has 6 branches and 4 nodes, so it is necessary to select 3 independent contours.

For the selected contours I_1, I_2 and I_3 the contour currents have been drawn in the clockwise direction.

The equations of contour currents are:

$$I_1(3000 + 1000 + 4000 + 2000) - I_2 \cdot 4000 = -190 + 200 - 250,$$

$$I_2(4000 + 5000 + 2000 + 3000) - I_1 \cdot 4000 - I_3 \cdot 2000 = 250 - 60 - 50,$$

$$I_3(2000 + 3000 + 1000) - I_2 \cdot 2000 = 60 + 10 + 150.$$

After addition and shortening, the following system is obtained.

$$\begin{cases} 10I_1 - 4I_2 + 0I_3 = -0.24 \\ -4I_1 + 14I_2 - 2I_3 = 0.14 \\ 0I_1 - 2I_2 + 6I_3 = 0.22 \end{cases}$$

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Its coefficient matrix $A = \begin{bmatrix} 10 & -4 & 0 \\ -4 & 14 & -2 \\ 0 & -2 & 6 \end{bmatrix}$ is regular. Using Cramer's rule, the following is found:

$$I_1 = \frac{D_1}{\det A}, I_2 = \frac{D_2}{\det A} \text{ and } I_3 = \frac{D_3}{\det A},$$

where

$$\det A = \begin{vmatrix} 10 & -4 & 0 \\ -4 & 14 & -2 \\ 0 & -2 & 6 \end{vmatrix} = 704,$$

$$D_1 = \begin{vmatrix} -0.24 & -4 & 0 \\ 0.14 & 14 & -2 \\ 0.22 & -2 & 6 \end{vmatrix} = -14.08,$$

$$D_2 = \begin{vmatrix} 10 & -0.24 & 0 \\ -4 & 0.14 & -2 \\ 0 & 0.22 & 6 \end{vmatrix} = 7.04,$$

$$D_3 = \begin{vmatrix} 10 & -4 & -0.24 \\ -4 & 14 & 0.14 \\ 0 & -2 & 0.22 \end{vmatrix} = 28.16.$$

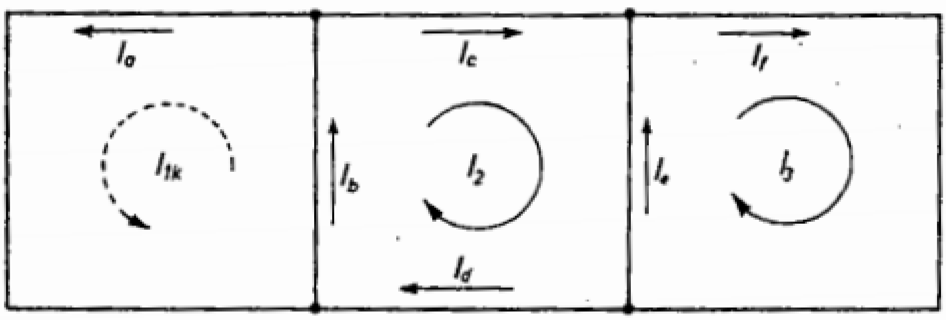
Therefore,

$$I_1 = -20\, mA, I_2 = 10\, mA \text{ and } I_3 = 40\, mA.$$

Currents in some branches can now be calculated.

As $I_1 < 0$, the presumed direction of that current was incorrect.

After correcting the direction of I_1



the currents I_a, I_b, I_c, I_d, I_e and I_f are as follows:

$$I_a = I_{1k} = 20\, mA,$$

$$I_b = I_{1k} + I_2 = 30\, mA,$$

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$$I_c = I_2 = 10\ mA,$$

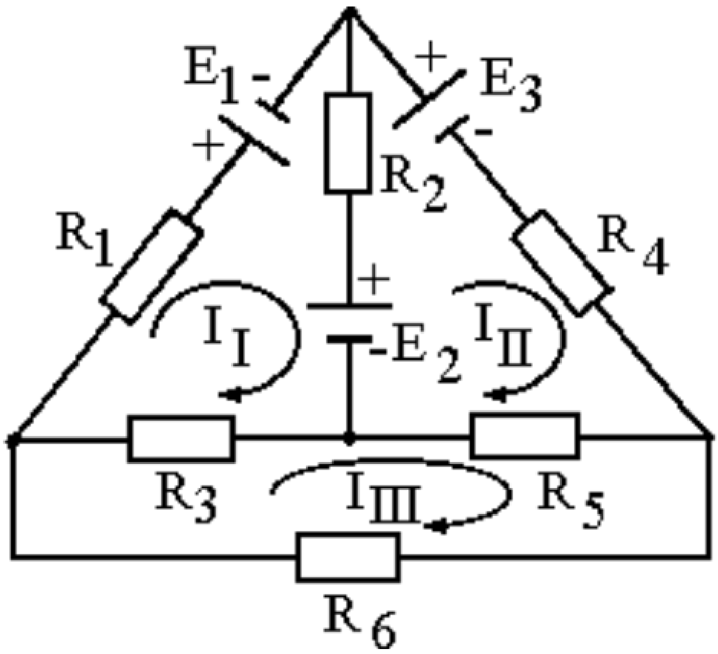
$$I_d = I_2 = 10\ mA,$$

$$I_e = I_3 - I_2 = 30\ mA,$$

$$I_f = I_3 = 40\ mA.$$

Example 2.45

The contour current method should be used to determine the strength of the electric current through all the resistors in the figure.



It is known that:

$E_1 = 10\ V\ , E_2 = 20\ V\ , E_3 = 5\ V\ , R_1 = 20\ \Omega\ , R_2 = 5\ \Omega\ , R_3 = 25\ \Omega\ , R_4 = 15\ \Omega\ ,$
$R_5 = 30\ \Omega\ , R_6 = 10\ \Omega.$

The equations of contour currents are:

$$\begin{aligned} -E_1 - E_2 &= (R_1 + R_2 + R_3) \cdot I_I - R_2 \cdot I_{II} - R_3 \cdot I_{III}\ , \\ E_2 - E_3 &= -R_2 \cdot I_I + (R_2 + R_4 + R_5) \cdot I_{II} - R_5 \cdot I_{III}\ , \\ 0 &= -R_3 \cdot I_I - R_5 \cdot I_{II} + (R_3 + R_5 + R_6) \cdot I_{III}\ . \end{aligned}$$

By entering and arranging data, the following system is obtained.

$$\begin{cases} 10I_I - I_{II} - 5I_{III} = -6 \\ -I_I + 10I_{II} - 6I_{III} = 3 \\ -5I_I - 6I_{II} + 13I_{III} = 0 \end{cases}$$

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The coefficient matrix $A=\begin{bmatrix}10 & -1 & -5\\ -1 & 10 & -6\\ -5 & -6 & 13\end{bmatrix}$ is regular. Using Cramer's rule, the following is found:

$$I_I=\frac{D_I}{\det A},\ I_{II}=\frac{D_{II}}{\det A}\ \text{ and }\ I_{III}=\frac{D_{III}}{\det A},$$

where

$$\det A=\begin{vmatrix}10 & -1 & -5\\ -1 & 10 & -6\\ -5 & -6 & 13\end{vmatrix}=617,$$

$$D_I=\begin{vmatrix}-6 & -1 & -5\\ 3 & 10 & -6\\ 0 & -6 & 13\end{vmatrix}=-435,$$

$$D_{II}=\begin{vmatrix}10 & -6 & -5\\ -1 & 3 & -6\\ -5 & 0 & 13\end{vmatrix}=57,$$

$$D_{III}=\begin{vmatrix}10 & -1 & -6\\ -1 & 10 & 3\\ -5 & -6 & 0\end{vmatrix}=-141.$$

Therefore,

$$I_I=-0.71\,A,\ I_{II}=92\,mA\ \text{ and }\ I_{III}=-0.23\,A.$$

The negative sign of the first and third contour currents means that the direction for these two contour currents is incorrectly assumed, and the direction should be drawn correctly in the diagram. The amounts of these currents are correct. The strengths of the currents through the individual resistors are determined from the circuit with the corrected direction of the contour currents.

$$I_{R_1}=I_I=0.71\,A$$
$$I_{R_2}=I_I+I_{II}=0.802\,A$$
$$I_{R_3}=I_I-I_{III}=0.48\,A$$
$$I_{R_4}=I_{II}=0.092\,A$$
$$I_{R_5}=I_{II}+I_{III}=0.32\,A$$
$$I_{R_6}=I_{III}=0.23\,A$$

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Applications of matrix multiplication in geometry and computer graphics.

Example 2.46 Symmetry of an object with respect to the line

The coordinate axes can be chosen so that the equation of that line is $x = 0$.

The point A with coordinates (x_A, y_A) is unambiguously associated with the vector $\begin{bmatrix} x_A \\ y_A \end{bmatrix}$ so it can be written $A = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$. If $S_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then

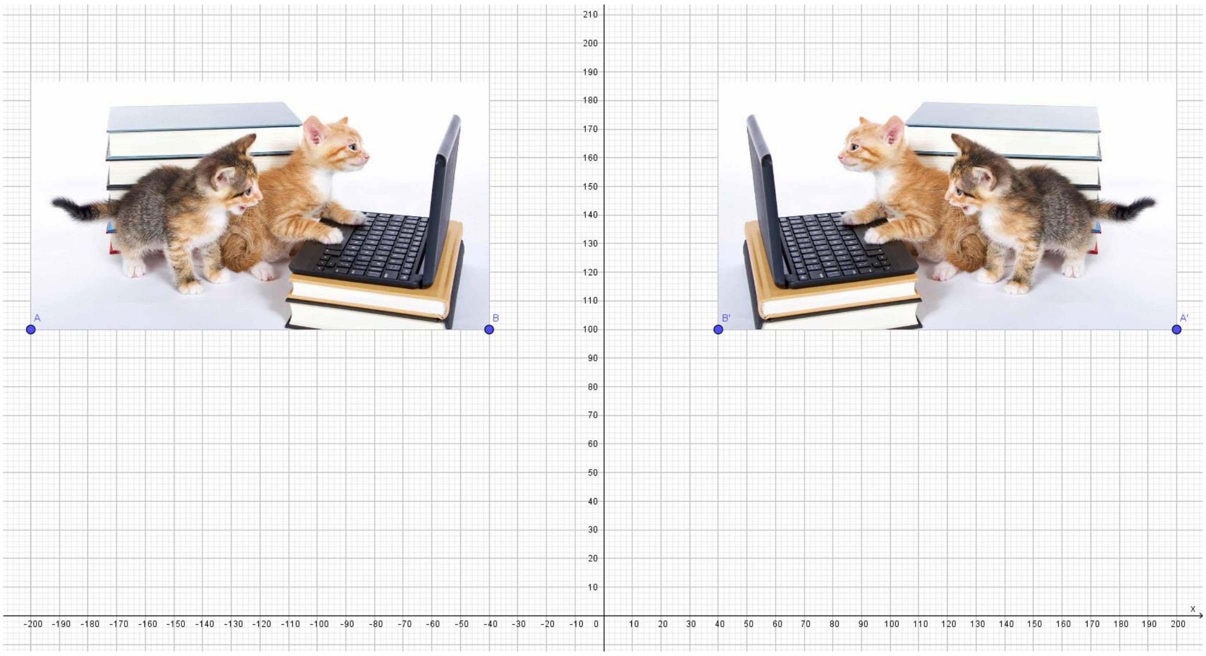
$$S_y \cdot A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} -x_A \\ y_A \end{bmatrix}.$$

So, it is said that by the action of the matrix S_y on the point A with coordinates (x_A, y_A), the point A' with coordinates $(-x_A, y_A)$ is obtained. The point A' is symmetric to the point A with respect to the straight line $x = 0$ and that's why S_y is called the symmetry matrix (with respect to the line $x = 0$).

The action of the matrix S_y on any object O in plane xy creates an object O' that is symmetrical to the object O with respect to the line $x = 0$.

S_y acts on the photograph (i.e., on each point of the photograph) above the line segment $\overline{AB}$.

This creates a photograph above the line segment $\overline{A'B'}$.



Example 2.47 Rotation of an object in the positive direction for an angle $\boldsymbol{\alpha}$ around a point

Coordinate axes can be chosen so that they intersect exactly at the point $(0,0)$ around which the rotation takes place. If $A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$, where r and φ are polar coordinates of the point A.

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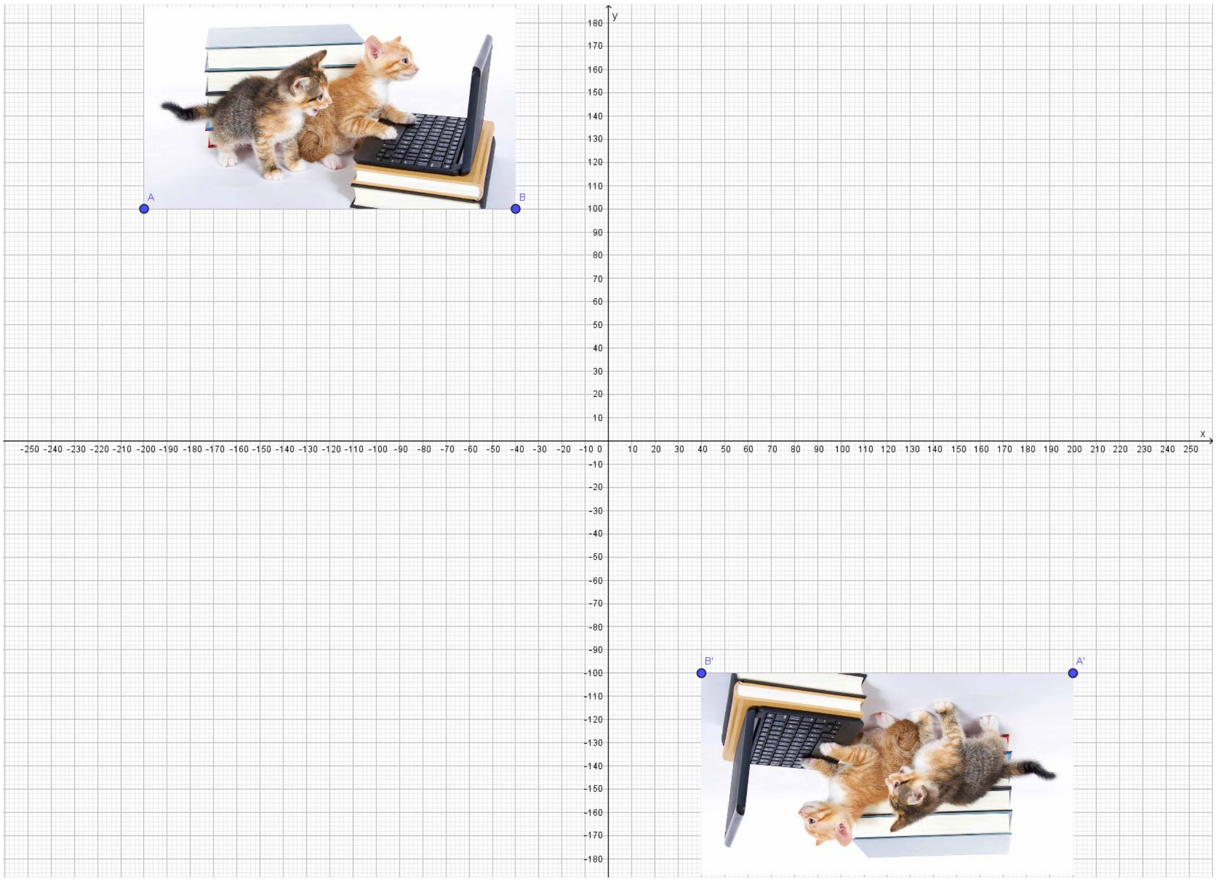
If $R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then

$$R_\alpha \cdot A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix} = \begin{bmatrix} r(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi) \\ r(\sin\alpha\cos\varphi + \cos\alpha\sin\varphi) \end{bmatrix} = \begin{bmatrix} r\cos(\alpha+\varphi) \\ r\sin(\alpha+\varphi) \end{bmatrix}$$

so, by the action of the matrix R_α on the point A with the coordinates $(r\cos\varphi, r\sin\varphi)$, the point A' is obtained with coordinates $(r\cos(\alpha+\varphi), r\sin(\alpha+\varphi))$. Note that the point A' is obtained by rotating the point A in the positive direction by the angle α around the origin. R_α is therefore called the rotation matrix in the positive direction for the angle α.

By acting of the matrix R_α on any object O in the plane xy the object O' is obtained, and each point of the object O' is obtained by rotating the corresponding point of the object O in the positive direction by the angle α around the origin.

$R_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ rotates the photograph (i.e., each point of the photograph) above the line segment $\overline{AB}$ in the positive direction by the angle $\pi = 180°$ around the origin. This creates an image below the line segment $\overline{A'B'}$.



Images on websites or obtained with a mobile phone camera are called digital photos. Such images are matrices. Suppose that in the details of one such image we have discovered that it is an image of the dimensions 1200×640. Then that image is a 640×1200 real matrix whose elements are pixels. A pixel is the smallest graphic element composed of a single colour. If the image contains

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only black and white colour, then the only elements of the matrix are the numbers 0 and 1. These numbers determine the colour of each pixel: zero represents black and one represents white. Digital photos that have only two colours are called binary images.

If the selected image is a black and white image, then it is also a 640×1200 real matrix, but its elements are integers between 0 and 255. Zero is again black (the colour of the minimum intensity), 255 is white (the colour of the maximum intensity), and numbers $1 - 254$ represent shades of grey from the darkest represented by number 1 to the lightest represented by number 254.

Colour images are created by overlapping of (i.e., by adding) three matrices - red, green, and blue.

The elements, i.e., the pixels, of the red matrix are arranged triplets $(r, 0,0)$, where r is the integer between 0 and 255. $(0,0,0)$ is black, $(255,0,0)$ is red, and the triplets $(r, 0,0)$ when r growing from 1 to 254 represent shades of red from a darker colour to a lighter colour.

The elements of the green matrix are arranged triplets $(0, g, 0)$, where g is the integer between 0 and 255. $(0,0,0)$ is black, $(0,255,0)$ is green, and the triplets $(0, g, 0)$ when g growing from 1 to 254 represent shades of green from a darker colour to a lighter colour.

The elements of the blue matrix are arranged triplets $(0,0, b)$, where b is the integer between 0 and 255. $(0,0,0)$ is black, $(0,0,255)$ is blue, and the triplets $(0,0, b)$ when b growing from 1 to 254 represent shades of blue from a darker colour to a lighter colour.

This type of a colour system is called an RGB system. In the RGB system, it is possible to display

$256^3 = 2^{24} = 16777216$ different colours.

The white colour of this system is a triplet

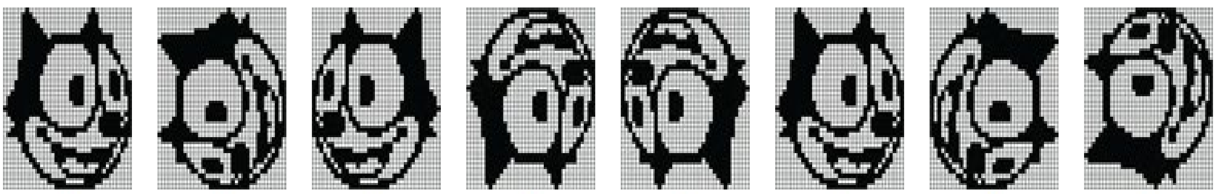
$$(255,255,255) = (255,0,0) + (0,255,0) + (0,0,255).$$

The shades of grey show triples of the shape (a, a, a), where a is the integer between 1 and 254.

Digital image processing and matrix operations

As digital photographs can be displayed using matrices, the question arises how operations on the elements of the matrix affect the corresponding photograph. This is shown in the following example.

Example 2.48



A B C D E F G H

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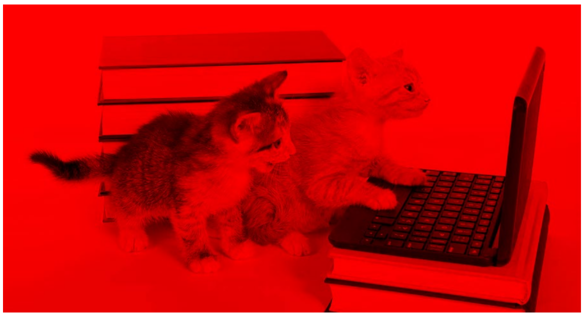
If $A = [a_{i,j}]$ is the given matrix (binary image) of the dimensions 35×35 then:

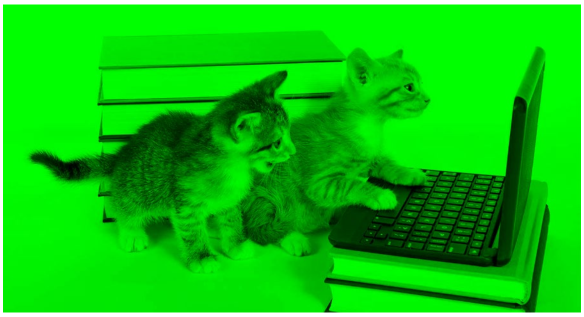
$B = [b_{i,j}] = [a_{j,i}] = A^T$, $C = [c_{i,j}] = [a_{i,35-j+1}]$, $D = [d_{i,j}] = [a_{35-i+1,j}]$,

$E = [e_{i,j}] = [a_{35-i+1,35-j+1}]$, $G = [g_{i,j}] = [a_{j,35-i+1}]$, $H = [h_{i,j}] = [a_{35-j+1,i}]$.

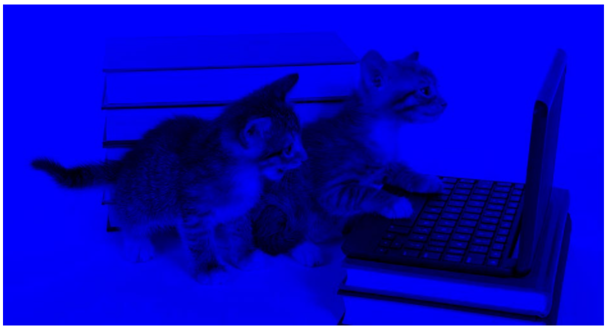
Example 2.49 Digital colour photography created by summing its red, green, and blue matrix



+



+



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Example 5 The effect of switching from one image to another image often used in PowerPoint presentations, slides, and slide shows

Take two digital colour photos of the same dimensions. The first photograph is represented by the matrix A, and the second by the matrix Z in the RGB system. For each number $t \in [0,1]$ the matrix $M(t) = (1-t)A + tZ$ is defined.

Notice that $M(0) = A$ and $M(1) = Z$. The larger $t \in (0,1)$, the less the matrix $M(t)$ resembles the matrix A, and is more similar to the matrix Z.

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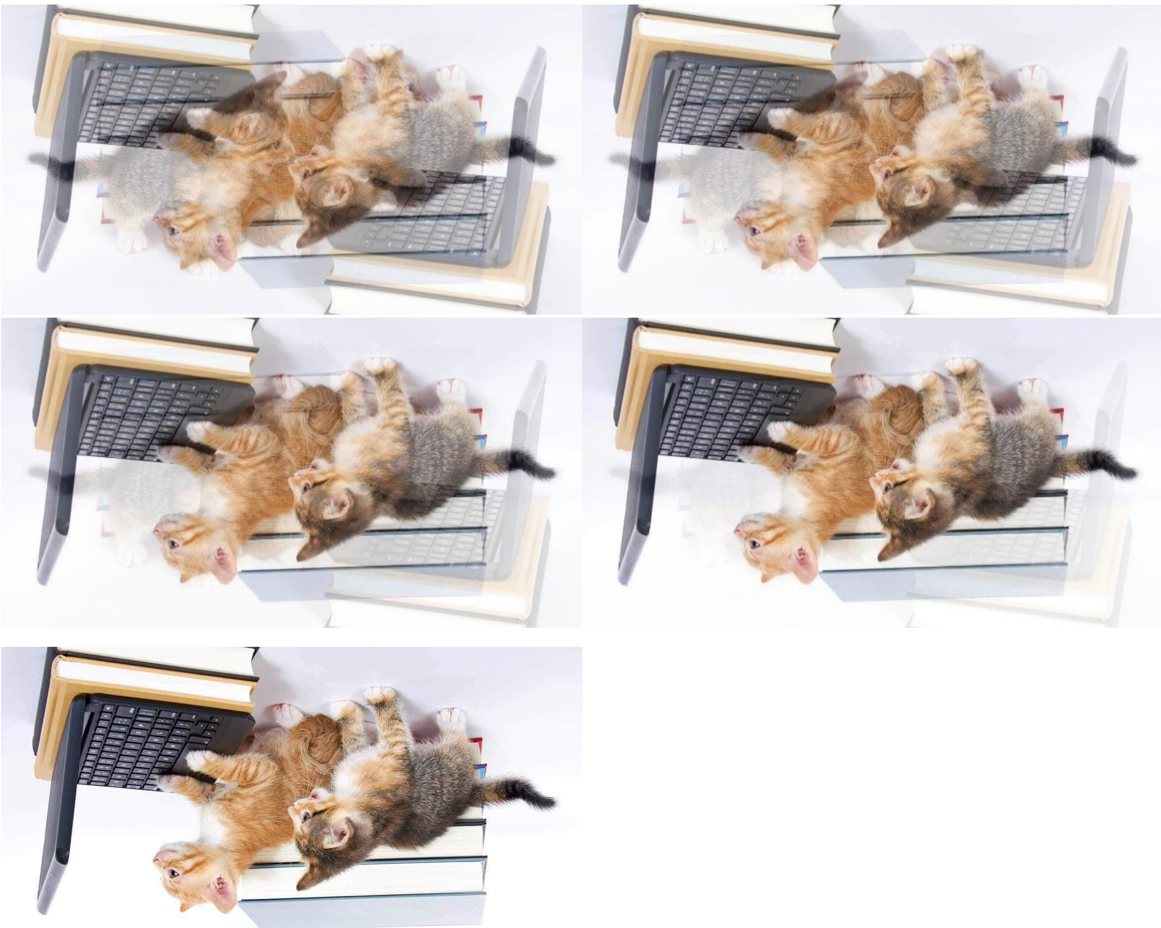

Example 6 Application of the singular value decomposition of the matrix in image approximation

The real matrix U of order n is orthogonal if: $U^T U = I_n$.

If A is any $m \times n$ real matrix, it can be proven that A can be written in the form of a product of three matrices as follows: $A = USV^T$,

where U is the orthogonal matrix of order m, V is the orthogonal matrix of order n,

and $S = [s_{i,j}]$ $m \times n$ real matrix for whose elements is valid

$$s_{i,j} = 0 \text{ when } i \neq j$$
$$s_{1,1} \geq s_{2,2} \geq \cdots \geq s_{k,k} \geq 0 \text{ with a label } k = \min\{m, n\}.$$

Such notation of the matrix A is called *a singular decomposition* of that matrix.

It is briefly said and written:

USV^T is SVD (singular value decomposition) of the matrix A.

If the columns of the matrix U are in the order $u_1, u_2, \dots, u_m$ and the columns of the matrix V in the order $v_1, v_2, \dots, v_n$, it can easily be checked that the following is valid:

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$$USV^T = \sum_{l=1}^{k} s_{l,l} u_l v_l^T .$$

Therefore,

$$A = \sum_{l=1}^{k} A_l ,$$

where $A_l = s_{l,l} u_l v_l^T$ is the $m \times n$ real matrix for each $l \in \{1, \dots, k\}$.

As $s_{1,1} \geq s_{2,2} \geq \cdots \geq s_{k,k}$, it is obvious that

$$\lim_{r \to k} \sum_{l=1}^{r} A_l = \sum_{l=1}^{k} A_l = A.$$

Accordingly, the closer r is to k, the matrix

$$C = \sum_{l=1}^{r} A_l$$

is better approximation of the matrix A.

Suppose that a space probe is programmed to send to a laboratory on Earth large amounts of black-and-white images of dimensions 1000×1000. For each such image, the probe would have to send $1000 \cdot 1000 = 1000000$ pixels (i.e., 1000000 numbers, one number for each pixel). However, before sending, the computer in the probe makes an SVD of each image, i.e., of the real matrix A. Then it takes the first 40 columns of the matrix U ($40 \cdot 1000 = 40000$ numbers), the first 40 columns of the matrix V ($40 \cdot 1000 = 40000$ numbers) and numbers $s_{1,1}, s_{2,2}, \dots, s_{40,40}$ (40 numbers), i.e., the total of $40000 + 40000 + 40 = 80040$ numbers, which are, instead of the elements of the matrix A, sent to Earth. So, instead of 1000000 numbers per image, the probe sends only 80040 numbers per image to Earth. On Earth, then, for each image, the matrix $\sum_{l=1}^{40} A_l$ is determined which is an approximation of the image, i.e., the matrix A. In that way, it is possible to obtain each image faster with a loss of quality.

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Figure 2.1 Black and white image of the dimensions 1000×1000 (k=1000)

Figure 2.2 Approximation of the same image (r=40)

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Figure 2.3 Image in colour of the dimensions 1200×640 (k=640)

Figure 2.4 Approximation of the same image (r=50)

Example 7 Removing interference (noise) from the image

Noise is defined as an unwanted random signal. Such a signal is mixed with the useful signal and affects its quality. Median filter is a technique used in digital processing to reduce the impact of noise in order to increase quality. Let us briefly describe what a median filter does. In all possible ways, 3 adjacent rows and 3 adjacent columns of the image (matrix A) are selected. The intersection of the selected rows and columns is the matrix B of order 3. The elements of the

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matrix B are arranged in an ascending order. The central element of the matrix B, i.e., the element located in the second row and the second column, is replaced by the fifth member (the so-called median) of the obtained sequence.

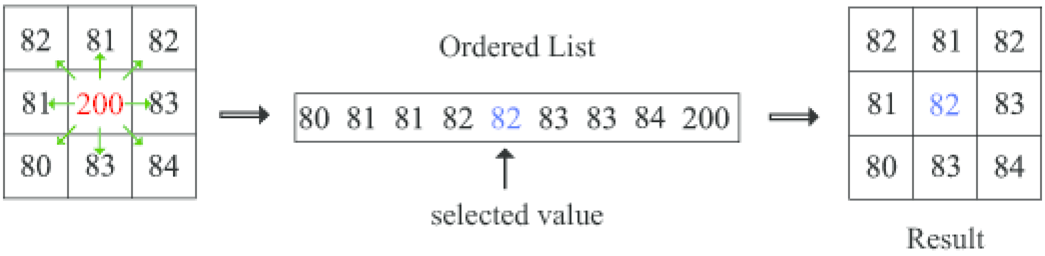

Figure 2.5 Black and white image with noise

Figure 2.6 The same image after the application of a median filter

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