2.2. MATRIX ARITHMETIC

Matrix addition

Only matrices of the same dimensions can be added.

If A and B are $m \times n$ real matrices, then $C = A + B$ is also $m \times n$ real matrix. Element c_{ij} of the matrix C is calculated using the following formula

$$c_{ij} = a_{ij} + b_{ij}.$$

Therefore, the elements in the same positions are added.

Example 2.9

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \end{bmatrix};$$

$$C = A + B = \begin{bmatrix} 2+1 & 3+3 & 7+4 \\ 1+2 & 0+(-1) & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 11 \\ 3 & -1 & 2 \end{bmatrix}.$$

Multiplying the matrix by a number

The real matrix is multiplied by a number so that each element of the matrix is multiplied by that number.

Example 2.10

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix}, \lambda = \frac{1}{2};$$

$$D = \lambda A = \frac{1}{2}\begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 3 & \frac{1}{2}\cdot 7 \\ \frac{1}{2}\cdot 1 & \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

The following is also defined

$$-A = (-1) \cdot A.$$

Matrix subtraction

Only matrices of the same dimensions can be subtracted.

If A and B are real matrices of the same dimensions, then their difference, matrix E, is defined as follows:

$$E = A - B = A + (-B) = A + (-1) \cdot B.$$

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Example 2.11

$$A=\begin{bmatrix}2 & 3 & 7\\ 1 & 0 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 3 & 4\\ 2 & -1 & 1\end{bmatrix};$$

$$E=A-B=A+(-1)\cdot B=\begin{bmatrix}2 & 3 & 7\\ 1 & 0 & 1\end{bmatrix}+\begin{bmatrix}-1 & -3 & -4\\ -2 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2+(-1) & 3+(-3) & 7+(-4)\\ 1+(-2) & 0+1 & 1+(-1)\end{bmatrix}=\begin{bmatrix}2-1 & 3-3 & 7-4\\ 1-2 & 0-(-1) & 1-1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 3\\ -1 & 1 & 0\end{bmatrix}.$$

Example 2.12

The matrices A, B, C are defined as follows

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\\ 1 & -2\end{bmatrix},\ B=\begin{bmatrix}0.25 & 1\\ 0 & -0.5\\ 1 & 3\end{bmatrix}\text{ and } C=\begin{bmatrix}-1 & 1\\ -1 & -1\\ 1 & 1\end{bmatrix}.$$

Calculate:

a) $A+B$
b) $A-C$
c) $A+B-C$
d) $12B$
e) $2A-C$
f) $2A+0.5C$
g) $2A^T$
h) A^T+3C^T.

Solution:

a) $$A+B=\begin{bmatrix}0 & 1\\ -1 & 0\\ 1 & -2\end{bmatrix}+\begin{bmatrix}0.25 & 1\\ 0 & -0.5\\ 1 & 3\end{bmatrix}=\begin{bmatrix}0+0.25 & 1+1\\ -1+0 & 0+(-0.5)\\ 1+1 & -2+3\end{bmatrix}=\begin{bmatrix}0.25 & 2\\ -1 & -0.5\\ 2 & 1\end{bmatrix}$$

b) $$A-C=\begin{bmatrix}0 & 1\\ -1 & 0\\ 1 & -2\end{bmatrix}-\begin{bmatrix}-1 & 1\\ -1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}0-(-1) & 1-1\\ -1-(-1) & 0-(-1)\\ 1-1 & -2-1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & -3\end{bmatrix}$$

c) $$A+B-C=(A+B)-C=\begin{bmatrix}0.25 & 2\\ -1 & -0.5\\ 2 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ -1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1.25 & 1\\ 0 & 0.5\\ 1 & 0\end{bmatrix}$$

d) $$12B=12\begin{bmatrix}0.25 & 1\\ 0 & -0.5\\ 1 & 3\end{bmatrix}=\begin{bmatrix}12\cdot 0.25 & 12\cdot 1\\ 12\cdot 0 & 12\cdot(-0.5)\\ 12\cdot 1 & 12\cdot 3\end{bmatrix}=\begin{bmatrix}3 & 12\\ 0 & -6\\ 12 & 36\end{bmatrix}$$

e) $$2A-C=\begin{bmatrix}0 & 2\\ -2 & 0\\ 2 & -4\end{bmatrix}-\begin{bmatrix}-1 & 1\\ -1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & 1\\ 1 & -5\end{bmatrix}$$

f) $$2A+0.5C=\begin{bmatrix}0 & 2\\ -2 & 0\\ 2 & -4\end{bmatrix}+\begin{bmatrix}-0.5 & 0.5\\ -0.5 & -0.5\\ 0.5 & 0.5\end{bmatrix}=\begin{bmatrix}-0.5 & 2.5\\ -2.5 & -0.5\\ 2.5 & -3.5\end{bmatrix}$$

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g) $2A^T = 2\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -4 \end{bmatrix}$

h) $A^T + 3C^T = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} + 3\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}.$

Matrix multiplication

Product $F = A \cdot B$ of the real matrices A and B exists only if the number of rows of the matrix B is equal to the number of columns of the matrix A.

If A is $m \times n$ real matrix and B is $n \times k$ real matrix, then $F = A \cdot B$ is $m \times k$ real matrix and the following is valid

$$f_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}.$$

Example 2.13

$$A = \begin{bmatrix} 5 & 2 \\ 7 & -1 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 2 & -2 & 4 & 0 \end{bmatrix};$$

$$F = A \cdot B = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} = \begin{bmatrix} 14 & 11 & 13 & 20 \\ 12 & 23 & 3 & 28 \\ -8 & 13 & -19 & 4 \end{bmatrix}.$$

$$f_{12} = \sum_{l=1}^{2} a_{1l} b_{l2} = a_{11} b_{12} + a_{12} b_{22} = 5 \cdot 3 + 2 \cdot (-2) = 11,$$

$$f_{23} = \sum_{l=1}^{2} a_{2l} b_{l3} = a_{21} b_{13} + a_{22} b_{23} = 7 \cdot 1 + (-1) \cdot 4 = 3,$$

$$f_{31} = \sum_{l=1}^{2} a_{3l} b_{l1} = a_{31} b_{11} + a_{32} b_{21} = 1 \cdot 2 + (-5) \cdot 2 = -8,$$

$$\ldots$$

Example 2.14

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = [3 \cdot 1 + 4 \cdot 2 + 5 \cdot (-3)] = [-4];$$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ -9 & -12 & -15 \end{bmatrix}.$$

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Example 2.15

Determine the matrix $F = A \cdot B$ if

a) $A = \begin{bmatrix} 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -1 & 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 5 & 2 \\ -1 & 8 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & 8 & 0 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$.

Solution:

a)

Matrix F is not defined (i.e., does not exist) because the number of rows of matrix B is not equal to the number of columns of matrix A. Namely, matrix B has 1 row, and matrix A has 2 columns.

b) $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \end{bmatrix}$, i.e., F is 1×3 matrix because A is 1×4 matrix and B is 4×3 matrix;

$$\begin{aligned} f_{11} = \sum_{l=1}^{4} a_{1l} b_{l1} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} \\ &= -1 \cdot (-1) + 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) = 0, \end{aligned}$$

$$\begin{aligned} f_{12} = \sum_{l=1}^{4} a_{1l} b_{l2} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} \\ &= -1 \cdot 2 + 1 \cdot (-1) + 2 \cdot 5 + 3 \cdot 8 = 31, \end{aligned}$$

$$\begin{aligned} f_{13} = \sum_{l=1}^{4} a_{1l} b_{l3} &= a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} + a_{14}b_{43} \\ &= -1 \cdot 0 + 1 \cdot 0 + 2 \cdot 2 + 3 \cdot 1 = 7. \end{aligned}$$

Therefore, $F = \begin{bmatrix} 0 & 31 & 7 \end{bmatrix}$.

c) $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix}$, i.e., F is 2×3 matrix because A is 2×3 matrix and B is 3×3 matrix.

$$f_{11} = \sum_{l=1}^{3} a_{1l} b_{l1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 4 = 4,$$

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$$f_{12} = \sum_{l=1}^{3} a_{1l}b_{l2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 8 = 9,$$

$$f_{13} = \sum_{l=1}^{3} a_{1l}b_{l3} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 0 = -1,$$

$$f_{21} = \sum_{l=1}^{3} a_{2l}b_{l1} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = -1 \cdot 0 + 1 \cdot 1 + 2 \cdot 4 = 9,$$

$$f_{22} = \sum_{l=1}^{3} a_{2l}b_{l2} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = -1 \cdot 1 + 1 \cdot 0 + 2 \cdot 8 = 15,$$

$$f_{23} = \sum_{l=1}^{3} a_{2l}b_{l3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-1) \cdot (-1) + 1 \cdot 1 + 2 \cdot 0 = 2.$$

Therefore, $F = \begin{bmatrix} 4 & 9 & -1 \\ 9 & 15 & 2 \end{bmatrix}$.

d) The required matrix is $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$. Namely, the matrices A and B are 3×3 matrices.

$$f_{12} = \sum_{l=1}^{3} a_{1l}b_{l2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 4 = 3,$$

$$f_{23} = \sum_{l=1}^{3} a_{2l}b_{l3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = 2 \cdot 4 + (-2) \cdot 0 + 1 \cdot 1 = 9,$$

$$f_{31} = \sum_{l=1}^{3} a_{3l}b_{l1} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} = 0 \cdot 1 + 0 \cdot 1 + (-1) \cdot 0 = 0.$$

Similarly, the remaining elements of the matrix F are determined. The following is obtained

$$F = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 9 \\ 0 & -4 & -1 \end{bmatrix}.$$

The two most important properties of matrix multiplication

a) Matrix multiplication is associative, i.e., $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ when all products are defined.

b) Multiplication of matrices is generally not commutative, i.e., if the products $A \cdot B$ and $B \cdot A$ of the matrices A and B exist, then $A \cdot B$ does not have to be (and most often is not) equal to $B \cdot A$.

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It may even happen that one of these products exists and the other does not. In *Example 2.5* there is no product of $B \cdot A$. In *Example 2.6* both products exist and are obviously different because the matrices of different sizes have been obtained.

2.3. MATRIX POLYNOMIAL

Let A be any real square matrix of order n and $P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ a polynomial of degree m, where $x, a_0, a_1, \dots, a_m \in \mathbb{R}$. Then $P_m(A)$ is defined as follows:

$$P_m(A) = a_0 \cdot A^0 + a_1 \cdot A^1 + a_2 \cdot A^2 + \cdots + a_m \cdot A^m$$

where

$$\begin{aligned} &A^0 = I_n, \\ &A^1 = A, \\ &A^2 = A \cdot A, \\ &A^3 = A^2 \cdot A \overset{\text{a)}}{=} A \cdot A^2, \\ &\vdots \\ &A^m = A^{m-1} \cdot A \overset{\text{a)}}{=} A \cdot A^{m-1}. \end{aligned}$$

It can be noticed that $P_m(A)$ is also a real square matrix, and of the same order as the matrix A.

Example 2.16

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $P_3(x) = 3x^3 + 2x^2 + 2x + 3$, determine $P_3(A)$.

Solution:

$$P_3(A) = 3A^3 + 2A^2 + 2A + 3I_2$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P_3(A) &= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4I_2 \end{aligned}$$

Example 2.17

A polynomial $P_3(x) = x^3 - x^2 - 2x$ and a matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ are given.

Prove the following: $P_3(A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

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