It may even happen that one of these products exists and the other does not. In *Example 2.5* there is no product of $B \cdot A$. In *Example 2.6* both products exist and are obviously different because the matrices of different sizes have been obtained.

2.3. MATRIX POLYNOMIAL

Let A be any real square matrix of order n and $P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ a polynomial of degree m, where $x, a_0, a_1, \ldots, a_m \in \mathbb{R}$. Then $P_m(A)$ is defined as follows:

$$P_m(A) = a_0 \cdot A^0 + a_1 \cdot A^1 + a_2 \cdot A^2 + \cdots + a_m \cdot A^m$$

where

$$\begin{aligned} A^0 &= I_n, \\ A^1 &= A, \\ A^2 &= A \cdot A, \\ A^3 &= A^2 \cdot A \overset{\text{a)}}{=} A \cdot A^2, \\ &\vdots \\ A^m &= A^{m-1} \cdot A \overset{\text{a)}}{=} A \cdot A^{m-1}. \end{aligned}$$

It can be noticed that $P_m(A)$ is also a real square matrix, and of the same order as the matrix A.

Example 2.16

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $P_3(x) = 3x^3 + 2x^2 + 2x + 3$, determine $P_3(A)$.

Solution:

$$P_3(A) = 3A^3 + 2A^2 + 2A + 3I_2$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P_3(A) &= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4I_2 \end{aligned}$$

Example 2.17

A polynomial $P_3(x) = x^3 - x^2 - 2x$ and a matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ are given.

Prove the following: $P_3(A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

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Solution:

$$P_3(A) = A^3 - A^2 - 2A$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$P_3(A) = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

2.4. DETERMINANT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a real matrix of order n.

The determinant of a matrix A is a number which can be joined to that matrix and is marked by

$$\det A \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

If $\boldsymbol{A = [a_{11}]}$, then $\boldsymbol{\det A = a_{11}}$.

If $\boldsymbol{A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$, then $\boldsymbol{\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}}$.

2.5. DETERMINANT OF THE MATRIX OF ORDER $n \geq 3$

The minor of the element a_{ij} of the matrix A is determinant of the matrix that is formed from the matrix A by deleting its ith row and jth column. We denote that number by M_{ij}.

Example 2.18

For $A = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$ is

$$M_{11} = -4\,, M_{12} = 2\,, M_{21} = 1\,, M_{22} = 3.$$

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