Solution:

$$P_3(A) = A^3 - A^2 - 2A$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$P_3(A) = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

2.4. DETERMINANT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a real matrix of order n.

The determinant of a matrix A is a number which can be joined to that matrix and is marked by

$$\det A \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

If $\boldsymbol{A} = [\boldsymbol{a_{11}}]$, then $\mathbf{det}\,\boldsymbol{A} = \boldsymbol{a_{11}}$.

If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}$, then $\mathbf{det}\,\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{vmatrix} = \boldsymbol{a_{11}a_{22} - a_{21}a_{12}}$.

2.5. DETERMINANT OF THE MATRIX OF ORDER $\boldsymbol{n \geq 3}$

The minor of the element a_{ij} of the matrix A is determinant of the matrix that is formed from the matrix A by deleting its ith row and jth column. We denote that number by M_{ij}.

Example 2.18

For $A = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$ is

$$M_{11} = -4\,, M_{12} = 2\,, M_{21} = 1\,, M_{22} = 3.$$

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Example 2.19

For $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$ is

$$M_{11}=\begin{vmatrix}7 & 0\\ 6 & 8\end{vmatrix}=56\,,M_{12}=\begin{vmatrix}5 & 0\\ 4 & 8\end{vmatrix}=40\,,M_{13}=\begin{vmatrix}5 & 7\\ 4 & 6\end{vmatrix}=2\,,M_{21}=\begin{vmatrix}1 & 2\\ 6 & 8\end{vmatrix}=-4,\dots$$

The algebraic complement or *cofactor* of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$ is marked $\boldsymbol{A_{ij}}$. That is a number defined by the formula:

$$\boldsymbol{A_{ij}=(-1)^{i+j}M_{ij}}.$$

Example 2.20

For $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$ is

$$A_{11}=M_{11}=56\,,A_{12}=-M_{12}=-40\,,A_{13}=M_{13}=2\,,A_{21}=-M_{21}=4,\dots$$

2.6. LAPLACE EXPANSION FOR THE DETERMINANT

The following formulas are applied for a real matrix $\boldsymbol{A}=\left[\boldsymbol{a_{ij}}\right]$ of order $\boldsymbol{n}\geq\mathbf{2}$:

$\det A=\sum_{k=1}^{n}a_{ik}A_{ik}$ (Laplace expansion by the $\boldsymbol{i}$th row),

$\det A=\sum_{k=1}^{n}a_{kj}A_{kj}$ (Laplace expansion by the $\boldsymbol{j}$th column).

Example 2.21

Determine $\det A$ using Laplace expansion:

a) by the 1st row
b) by the 2nd column

if $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$.

Solution: $n=3$

a)

$$\det A=\sum_{k=1}^{3}a_{1k}A_{1k}=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=3\cdot 56+1\cdot(-40)+2\cdot 2=132$$

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