Example 2.19

For $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$ is

$$M_{11}=\begin{vmatrix}7 & 0\\ 6 & 8\end{vmatrix}=56\,,M_{12}=\begin{vmatrix}5 & 0\\ 4 & 8\end{vmatrix}=40\,,M_{13}=\begin{vmatrix}5 & 7\\ 4 & 6\end{vmatrix}=2\,,M_{21}=\begin{vmatrix}1 & 2\\ 6 & 8\end{vmatrix}=-4,\dots$$

The algebraic complement or *cofactor* of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$ is marked $\boldsymbol{A_{ij}}$. That is a number defined by the formula:

$$\boldsymbol{A_{ij}=(-1)^{i+j}M_{ij}}.$$

Example 2.20

For $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$ is

$$A_{11}=M_{11}=56\,,A_{12}=-M_{12}=-40\,,A_{13}=M_{13}=2\,,A_{21}=-M_{21}=4,\dots$$

2.6. LAPLACE EXPANSION FOR THE DETERMINANT

The following formulas are applied for a real matrix $\boldsymbol{A}=\left[\boldsymbol{a_{ij}}\right]$ of order $\boldsymbol{n}\geq\boldsymbol{2}$:

$$\det A=\sum_{k=1}^{n}a_{ik}A_{ik}$$ (Laplace expansion by the $\boldsymbol{i}$th row),

$$\det A=\sum_{k=1}^{n}a_{kj}A_{kj}$$ (Laplace expansion by the $\boldsymbol{j}$th column).

Example 2.21

Determine $\det A$ using Laplace expansion:

a) by the 1st row
b) by the 2nd column

if $A=\begin{bmatrix}3 & 1 & 2\\ 5 & 7 & 0\\ 4 & 6 & 8\end{bmatrix}$.

Solution: $n=3$

a)

$$\det A=\sum_{k=1}^{3}a_{1k}A_{1k}=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=3\cdot 56+1\cdot(-40)+2\cdot 2=132$$

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b)

$$\det A = \sum_{k=1}^{3} a_{k2}A_{k2} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$
$$= 1 \cdot (-1)^{1+2} \begin{vmatrix} 5 & 0 \\ 4 & 8 \end{vmatrix} + 7 \cdot (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 4 & 8 \end{vmatrix} + 6 \cdot (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 5 & 0 \end{vmatrix}$$
$$= -40 + 112 + 60 = 132$$

Determinant properties:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be real square matrices.

1) If a row (column) of the matrix $\boldsymbol{A}$ contains only zeros, then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

2) If the matrix $\boldsymbol{A}$ has $\mathbf{2}$ proportional rows (columns), then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

3) If a row (column) of the matrix $\boldsymbol{A}$ is a linear combination of the remaining rows (columns) of the matrix $\boldsymbol{A}$, then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

4) The determinant of the unit matrix is $\mathbf{1}$.

5) If $\boldsymbol{A}$ is a triangular or diagonal matrix, then its determinant is equal to the product of the elements of its main diagonal.

6) If the elements of another row (column) of the matrix $\boldsymbol{A}$ are added to a row (column) of the matrix $\boldsymbol{A}$ multiplied by a number, $\mathbf{det}\,\boldsymbol{A}$ does not change.

7) By changing the place of two rows (columns) of the matrix $\boldsymbol{A}$, $\mathbf{det}\,\boldsymbol{A}$ changes the sign.

8) If the elements of a row (column) of a matrix $\boldsymbol{A}$ are multiplied by the number $\boldsymbol{\lambda}$, the result is a square matrix $\boldsymbol{C}$ with a determinant

$$\mathbf{det}\,\boldsymbol{C} = \boldsymbol{\lambda} \cdot \mathbf{det}\,\boldsymbol{A}.$$

9) $\mathbf{det}\,\boldsymbol{A} = \mathbf{det}\,\boldsymbol{A}^{\boldsymbol{T}}$.

10) $\mathbf{det}(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathbf{det}\,\boldsymbol{A} \cdot \mathbf{det}\,\boldsymbol{B}$.

Example 2.22

Calculate $\det A$, using the properties of the determinant, if $A = \begin{bmatrix} 5 & 4 & 5 & 4 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix}$.

Solution:

Method 1 (Laplace expansion by the 1st column):

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$$\det A=\begin{vmatrix}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{vmatrix}\begin{matrix}R_1-5R_2\\ \\R_3-2R_2\\ \end{matrix}\overset{6)}{=}\begin{vmatrix}0&-6&0&-16\\1&2&1&4\\0&-1&3&-4\\0&2&1&1\end{vmatrix}=\sum_{k=1}^{4}a_{k1}A_{k1}=a_{21}A_{21}=$$

$$=1\cdot(-1)^{2+1}\begin{vmatrix}-6&0&-16\\-1&3&-4\\2&1&1\end{vmatrix}R_2-3R_3\overset{6)}{=}-\begin{vmatrix}-6&0&-16\\-7&0&-7\\2&1&1\end{vmatrix}=-\sum_{k=1}^{3}a_{k2}A_{k2}=-a_{32}A_{32}=$$

$$=-1\cdot(-1)^{3+2}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}\overset{8)}{=}-7\begin{vmatrix}-6&-16\\1&1\end{vmatrix}\overset{8)}{=}14\begin{vmatrix}3&8\\1&1\end{vmatrix}=14(3-8)=-70.$$

Method 2 (Laplace expansion by the 4th row):

$$\det A=\begin{vmatrix}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{vmatrix}\overset{6)}{=}\begin{vmatrix}5&-6&5&-1\\1&0&1&3\\2&-7&5&-1\\0&0&1&0\end{vmatrix}=\sum_{k=1}^{4}a_{4k}A_{4k}=a_{43}A_{43}=1\cdot(-1)^{4+3}\begin{vmatrix}5&-6&-1\\1&0&3\\2&-7&-1\end{vmatrix}\overset{6)}{=}$$

$$\overset{6)}{=}-\begin{vmatrix}5&-6&-16\\1&0&0\\2&-7&-7\end{vmatrix}=-\sum_{k=1}^{3}a_{2k}A_{2k}=-a_{21}A_{21}=-1\cdot(-1)^{2+1}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=-70.$$

2.7. INVERSE MATRIX

Let A be a real square matrix of order n.

The matrix A is a *regular* if $\det A \neq 0$. The matrix A is *singular* if $\det A = 0$.

Only regular matrix has an inverse matrix.

For a regular matrix A of order n there is a unique matrix B such that

$$A \cdot B = B \cdot A = I_n,$$

where I_n is a unit matrix of order n. The matrix B is also a regular matrix of order n, marked by A^{-1}, and is called *the inverse matrix* of the matrix A.

2.8. DETERMINING THE INVERSE MATRIX BY CALCULATING DETERMINANTS

The inverse matrix $\boldsymbol{A^{-1}}$ of a regular matrix $\boldsymbol{A}$ of order $\boldsymbol{n}$ can be determined by the formula:

$$\boldsymbol{A^{-1}=\frac{1}{\det A}\begin{bmatrix}A_{11}&A_{21}&\cdots&A_{n1}\\A_{12}&A_{22}&\cdots&A_{n2}\\\vdots&\vdots&\ddots&\vdots\\A_{1n}&A_{2n}&\cdots&A_{nn}\end{bmatrix}},$$

where $\boldsymbol{A_{ij}}$ is the cofactor of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$.

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