$$\det A=\begin{vmatrix}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{vmatrix}\begin{matrix}R_1-5R_2\\ \\R_3-2R_2\\ \\ \end{matrix}\overset{6)}{=}\begin{vmatrix}0&-6&0&-16\\1&2&1&4\\0&-1&3&-4\\0&2&1&1\end{vmatrix}=\sum_{k=1}^{4}a_{k1}A_{k1}=a_{21}A_{21}=$$

$$=1\cdot(-1)^{2+1}\begin{vmatrix}-6&0&-16\\-1&3&-4\\2&1&1\end{vmatrix}R_2-3R_3\overset{6)}{=}-\begin{vmatrix}-6&0&-16\\-7&0&-7\\2&1&1\end{vmatrix}=-\sum_{k=1}^{3}a_{k2}A_{k2}=-a_{32}A_{32}=$$

$$=-1\cdot(-1)^{3+2}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}\overset{8)}{=}-7\begin{vmatrix}-6&-16\\1&1\end{vmatrix}\overset{8)}{=}14\begin{vmatrix}3&8\\1&1\end{vmatrix}=14(3-8)=-70.$$

Method 2 (Laplace expansion by the 4th row):

$$\det A=\begin{vmatrix}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{vmatrix}\overset{6)}{=}\begin{vmatrix}5&-6&5&-1\\1&0&1&3\\2&-7&5&-1\\0&0&1&0\end{vmatrix}=\sum_{k=1}^{4}a_{4k}A_{4k}=a_{43}A_{43}=1\cdot(-1)^{4+3}\begin{vmatrix}5&-6&-1\\1&0&3\\2&-7&-1\end{vmatrix}\overset{6)}{=}$$

$$\overset{6)}{=}-\begin{vmatrix}5&-6&-16\\1&0&0\\2&-7&-7\end{vmatrix}=-\sum_{k=1}^{3}a_{2k}A_{2k}=-a_{21}A_{21}=-1\cdot(-1)^{2+1}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=-70.$$

2.7. INVERSE MATRIX

Let A be a real square matrix of order n.

The matrix A is a *regular* if $\det A\neq 0$. The matrix A is *singular* if $\det A=0$.

Only regular matrix has an inverse matrix.

For a regular matrix A of order n there is a unique matrix B such that

$$A\cdot B=B\cdot A=I_n,$$

where I_n is a unit matrix of order n. The matrix B is also a regular matrix of order n, marked by A^{-1}, and is called *the inverse matrix* of the matrix A.

2.8. DETERMINING THE INVERSE MATRIX BY CALCULATING DETERMINANTS

The inverse matrix $\boldsymbol{A^{-1}}$ of a regular matrix $\boldsymbol{A}$ of order $\boldsymbol{n}$ can be determined by the formula:

$$A^{-1}=\frac{1}{\det A}\begin{bmatrix}A_{11}&A_{21}&\cdots&A_{n1}\\A_{12}&A_{22}&\cdots&A_{n2}\\\vdots&\vdots&\ddots&\vdots\\A_{1n}&A_{2n}&\cdots&A_{nn}\end{bmatrix},$$

where $\boldsymbol{A_{ij}}$ is the cofactor of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$.

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This formula is usually applied only for small values of n ($n = 2$ and $n = 3$) as for determining A^{-1} it is necessary to calculate even n^2 determinants of order $n - 1$.

Example 2.23

Determine the inverse matrix of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by calculating the determinants.

Solution: $n = 2$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{1 \cdot 4 - 3 \cdot 2} \begin{bmatrix} (-1)^{1+1} \cdot 4 & (-1)^{2+1} \cdot 2 \\ (-1)^{1+2} \cdot 3 & (-1)^{2+2} \cdot 1 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Example 2.24

Using the same method, find the inverse matrix of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution: $n = 3$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\det A = 2 \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 2$$

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot 1 = 1$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot (-1) = 1$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \cdot (1 + 1) = 2$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot 0 = 0$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot (-2) = -2$$

$$A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -1 \cdot 2 = -2$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 1 \cdot 0 = 0$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = -1 \cdot 0 = 0$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot (-2) = -2$$

$$\Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

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Determination of the inverse matrix using the Gauss-Jordan method

Elementary transformations (on rows or columns) of the real matrix are:

a) replacing the position of any two rows (columns),

b) multiplying any row (column) by a number other than zero,

c) adding any row (column) to another row (column).

If $\boldsymbol{A}$ is a regular matrix of order $\boldsymbol{n} > \boldsymbol{3}$, Gauss-Jordan method is often used to find $\boldsymbol{A}^{-\boldsymbol{1}}$.

Proceed as follows:

1) form $\boldsymbol{n} \times \boldsymbol{2n}$ matrix $\overline{\boldsymbol{A}} = [\boldsymbol{A}|\boldsymbol{I}_{\boldsymbol{n}}]$,

2) transform the matrix $\overline{\boldsymbol{A}}$ into the matrix $[\boldsymbol{I}_{\boldsymbol{n}}|\boldsymbol{B}]$, using elementary transformations exclusively on rows,

Then $\boldsymbol{A}^{-\boldsymbol{1}} = \boldsymbol{B}$.

Example 2.25

Calculate the inverse matrix of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: $n = 4$

$$\bar{A} = [A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + R_3 \\ R_2 - R_3 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_4 \\ R_2 + R_4 \\ R_3 - R_4 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I_4|B] \Rightarrow A^{-1} = B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.26

Calculate the inverse matrix of the matrix $A = \begin{bmatrix} 5 & 4 & 5 & 4 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix}$ using

Gauss-Jordan method.

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Solution: $n = 4$

$$\bar{A} = [A|I_4] = \left[\begin{array}{cccc|cccc} 5 & 4 & 5 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & 3 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} R_2 \\ R_1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 5 & 4 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & 3 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} \\ R_2 - 5R_1 \\ R_3 - 2R_1 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & -16 & 1 & -5 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} \\ R_3 \\ R_2 \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 & -2 & 1 & 0 \\ 0 & -6 & 0 & -16 & 1 & -5 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} \\ -R_2 \\ -R_3 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 6 & 0 & 16 & -1 & 5 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} R_1 - 2R_2 \\ \\ R_3 - 6R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 18 & -8 & -1 & -7 & 6 & 0 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right]\begin{matrix} \\ \\ 2R_3 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 36 & -16 & -2 & -14 & 12 & 0 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right]\begin{matrix} \\ \\ R_3 - 5R_4 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right]\begin{matrix} R_1 - 7R_3 \\ R_2 + 3R_3 \\ \\ R_4 - 7R_3 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & -140 & 14 & -46 & -12 & 36 \end{array}\right]\begin{matrix} \\ \\ \\ -R_4/140 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & -140 & 14 & -46 & -12 & 36 \end{array}\right]\begin{matrix} \\ \\ \\ -R_4/140 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & 1 & -7/70 & 23/70 & 6/70 & -18/70 \end{array}\right]\begin{matrix} R_1 + 137R_4 \\ R_2 - 61R_4 \\ R_3 - 19R_4 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 21/70 & 1/70 & -18/70 & -16/70 \\ 0 & 1 & 0 & 0 & 7/70 & -3/70 & -16/70 & 48/70 \\ 0 & 0 & 1 & 0 & -7/70 & -17/70 & 26/70 & -8/70 \\ 0 & 0 & 0 & 1 & -7/70 & 23/70 & 6/70 & -18/70 \end{array}\right] = [I_4|B]$$

Therefore,

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$$A^{-1} = B = \begin{bmatrix} 21/70 & 1/70 & -18/70 & -16/70 \\ 7/70 & -3/70 & -16/70 & 48/70 \\ -7/70 & -17/70 & 26/70 & -8/70 \\ -7/70 & 23/70 & 6/70 & -18/70 \end{bmatrix} = \frac{1}{-70}\begin{bmatrix} -21 & -1 & 18 & 16 \\ -7 & 3 & 16 & -48 \\ 7 & 17 & -26 & 8 \\ 7 & -23 & -6 & 18 \end{bmatrix}.$$

2.9. MATRIX EQUATIONS

Matrix equations are equations in which matrices are used and at least one of the matrices is unknown.

To solve such equation, we need to find all the matrices for which that equation is valid.

How to solve the equations

$$AX = B \text{ and } YA = B$$

in which A and B are known, and X and Y are unknown real matrices?

Assume that A is a regular matrix of order n.
Then A^{-1} exists and, by multiplying the first equation by A^{-1} on the left, we get

$$A^{-1} \cdot / AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$I_n X = A^{-1}B$$
$$X = A^{-1}B.$$

Multiplying the second equation by A^{-1} on the left results in

$$YA = B / \cdot A^{-1}$$
$$(YA)A^{-1} = BA^{-1}$$
$$Y(AA^{-1}) = BA^{-1}$$
$$YI = BA^{-1}$$
$$Y = BA^{-1}.$$

Note that in general does not have to be $X = Y$ because the matrix multiplication is generally not commutative.

Example 2.27

Solve the equation $AX - B = A^2X - I_2$ if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -85 & -100 \\ -186 & -215 \end{bmatrix}$.

Solution:

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