$$A^{-1} = B = \begin{bmatrix} 21/70 & 1/70 & -18/70 & -16/70 \\ 7/70 & -3/70 & -16/70 & 48/70 \\ -7/70 & -17/70 & 26/70 & -8/70 \\ -7/70 & 23/70 & 6/70 & -18/70 \end{bmatrix} = \frac{1}{-70}\begin{bmatrix} -21 & -1 & 18 & 16 \\ -7 & 3 & 16 & -48 \\ 7 & 17 & -26 & 8 \\ 7 & -23 & -6 & 18 \end{bmatrix}.$$

2.9. MATRIX EQUATIONS

Matrix equations are equations in which matrices are used and at least one of the matrices is unknown.

To solve such equation, we need to find all the matrices for which that equation is valid.

How to solve the equations

$$AX = B \text{ and } YA = B$$

in which A and B are known, and X and Y are unknown real matrices?

Assume that A is a regular matrix of order n.
Then A^{-1} exists and, by multiplying the first equation by A^{-1} on the left, we get

$$A^{-1} \cdot / AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$I_n X = A^{-1}B$$
$$X = A^{-1}B.$$

Multiplying the second equation by A^{-1} on the left results in

$$YA = B / \cdot A^{-1}$$
$$(YA)A^{-1} = BA^{-1}$$
$$Y(AA^{-1}) = BA^{-1}$$
$$YI = BA^{-1}$$
$$Y = BA^{-1}.$$

Note that in general does not have to be $X = Y$ because the matrix multiplication is generally not commutative.

Example 2.27

Solve the equation $AX - B = A^2X - I_2$ if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -85 & -100 \\ -186 & -215 \end{bmatrix}$.

Solution:

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$$AX - B = A^2X - I_2$$
$$AX - A^2X = B - I_2$$
$$\left(A - A^2\right)^{-1} \cdot / \left(A - A^2\right)X = B - I_2$$
$$I_2X = \left(A - A^2\right)^{-1}\left(B - I_2\right)$$
$$X = \left(A - A^2\right)^{-1}\left(B - I_2\right)$$

if $A - A^2$ is a regular matrix.

$$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \qquad A - A^2 = \begin{bmatrix} -6 & -8 \\ -12 & -18 \end{bmatrix} \Rightarrow \det(A - A^2) = 12 \neq 0$$
$$B - I_2 = \begin{bmatrix} -86 & -100 \\ -186 & -216 \end{bmatrix} \qquad (A - A^2)^{-1} = \frac{1}{12}\begin{bmatrix} -18 & 8 \\ 12 & -6 \end{bmatrix}$$
$$X = (A - A^2)^{-1}(B - I_2) = \frac{1}{12}\begin{bmatrix} -18 & 8 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} -86 & -100 \\ -186 & -216 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 60 & 72 \\ 84 & 96 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Example 2.28

Solve the equation $AX^{-1}B + C = AX^{-1}$ if

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

It can be proven that for regular matrices A and B is valid:

$$(AB)^{-1} = B^{-1}A^{-1}$$

if both products, AB and $B^{-1}A^{-1}$, exist.

$$A^{-1} \cdot / AX^{-1}B + C = AX^{-1}$$
$$\underbrace{\left(A^{-1}A\right)}_{=I_3}X^{-1}B + A^{-1}C = \underbrace{\left(A^{-1}A\right)}_{=I_3}X^{-1}$$
$$X \cdot / X^{-1}B + A^{-1}C = X^{-1}$$
$$\underbrace{\left(XX^{-1}\right)}_{=I_3}B + X\left(A^{-1}C\right) = \underbrace{XX^{-1}}_{=I_3}$$
$$B + X\left(A^{-1}C\right) = I_3$$
$$X\left(A^{-1}C\right) = I_3 - B / \cdot \left(A^{-1}C\right)^{-1}$$
$$X = \left(I_3 - B\right)\left(A^{-1}C\right)^{-1} = \left(I_3 - B\right)C^{-1}A$$

if the matrix X is regular. (Namely, it is easy to notice that the matrices A and C are regular.)

$$I_3 - B = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & 0 & 0 \end{bmatrix} \qquad C^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -1 & -2 \\ -2 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

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$$X=(I_3-B)C^{-1}A=\begin{bmatrix}0&-1&-2\\-1&1&-2\\-2&0&0\end{bmatrix}\frac{1}{-2}\begin{bmatrix}1&-1&-2\\-2&0&4\\0&0&-2\end{bmatrix}\begin{bmatrix}2&1&1\\0&0&-1\\0&1&-1\end{bmatrix}$$
$$=-\frac{1}{2}\begin{bmatrix}4&2&2\\-6&7&-14\\-4&2&-8\end{bmatrix}=\begin{bmatrix}-2&-1&-1\\3&-3.5&7\\2&-1&4\end{bmatrix}$$

The equation $AX+XB=C$ cannot be solved using an inverse matrix. The unknown matrix X is located to the right of matrix A, and to the left of matrix B. As the multiplication of matrices is generally not commutative, on the left side of the equation it is not possible to extract, as a common factor, the matrix X. Namely,

$$AX+XB\neq AX+BX=(A+B)X,$$
$$AX+XB\neq XA+XB=X(A+B).$$

It is not difficult to notice that an equation $AX+XB=C$ only makes sense if all the matrices that appear in it are square. So, we know that the order of the matrix X is equal to the order of the matrix A.

How to solve such an equation is shown in the following example.

Example 2.29

Solve the equation $\begin{bmatrix}4&1\\-3&2\end{bmatrix}X+X\begin{bmatrix}1&3\\5&7\end{bmatrix}=\begin{bmatrix}-4&-1\\16&21\end{bmatrix}$.

Solution:

X is the matrix of the 2nd order, i.e., $X=\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}$. The matrix X with unknown elements is included in the equation, and the result is:

$$\begin{bmatrix}4&1\\-3&2\end{bmatrix}\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}+\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}\begin{bmatrix}1&3\\5&7\end{bmatrix}=\begin{bmatrix}-4&-1\\16&21\end{bmatrix}$$

$$\begin{bmatrix}4x_{11}+x_{21}&4x_{12}+x_{22}\\-3x_{11}+2x_{21}&-3x_{12}+2x_{22}\end{bmatrix}+\begin{bmatrix}x_{11}+5x_{12}&3x_{11}+7x_{12}\\x_{21}+5x_{22}&3x_{21}+7x_{22}\end{bmatrix}=\begin{bmatrix}-4&-1\\16&21\end{bmatrix}$$

$$\begin{bmatrix}5x_{11}+5x_{12}+x_{21}&3x_{11}+11x_{12}+x_{22}\\-3x_{11}+3x_{21}+5x_{22}&-3x_{12}+3x_{21}+9x_{22}\end{bmatrix}=\begin{bmatrix}-4&-1\\16&21\end{bmatrix}.$$

Two matrices of the same dimensions can be equal if and only if their elements, located in the same positions in the matrices, are equal. Therefore, it has to be

$$\begin{cases}5x_{11}+5x_{12}+x_{21}=-4\\3x_{11}+11x_{12}+x_{22}=-1\\-3x_{11}+3x_{21}+5x_{22}=16\\-3x_{12}+3x_{21}+9x_{22}=21\end{cases}.$$

We have obtained a system of linear equations that is easy to solve.

From equation 1: $x_{21}=-4-5x_{11}-5x_{12}$.

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From equation 2: $x_{22} = -1 - 3x_{11} - 11x_{12}$.

By including in the third and fourth equations, the result is:

$$-3x_{11} + 3(-4 - 5x_{11} - 5x_{12}) + 5(-1 - 3x_{11} - 11x_{12}) = 16$$
$$-3x_{12} + 3(-4 - 5x_{11} - 5x_{12}) + 9(-1 - 3x_{11} - 11x_{12}) = 21$$

$$\left.\begin{array}{rcrcr} -33x_{11} & - & 70x_{12} & = & 33 \\ -42x_{11} & - & 117x_{12} & = & 42 \end{array}\right\} \Rightarrow x_{11} = -1\,, x_{12} = 0$$

$$x_{21} = -4 - 5x_{11} - 5x_{12} = -4 - 5 \cdot (-1) - 5 \cdot 0 = 1$$
$$x_{22} = -1 - 3x_{11} - 11x_{12} = -1 - 3 \cdot (-1) - 11 \cdot 0 = 2.$$

Therefore, the matrix $X = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ is the (only) solution of this equation.

Similar to the equation in *Example 3*, equations of the form $AX = B$ and $YA = B$ with unknowns X and Y, in which the matrix A is neither regular nor square, are solved.

2.10. MATRIX RANK

A single-column real matrix is also called *a column vector* (or shorter, a *vector*).

A single-row real matrix is also called *a row vector*.

Example 2.30

$$C = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

is a vector of dimension 3 because it has 3 components: 1,4 and 7.

Vector C is *a zero vector* if all its components are equal to zero.

Analogously, a zero-row vector is defined.

The zero row (column) vector is marked by O.

A non-zero row (column) vector is a row (column) vector for which at least one component is different than zero.

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