Chapter 2. MATRICES AND DETERMINANTS

ABSTRACT:

This chapter introduces the concept of matrix. Firstly, some operations with matrices are defined, then the determinant of the square matrix and the inverse matrix of the regular matrix. After that, it is demonstrated how to solve matrix equations and how a system of linear equations can be solved using matrices. Each topic is covered in detail, and in addition to some solved examples, it also contains exercises. At the end of the chapter is a knowledge test. Additional applications in MATLAB, Excel and Geogebra can be used to check the obtained solutions or to make faster calculations (if the procedure for solving exercises is not required).

AIM: **To learn how to calculate the determinant and inverse matrix, and learn how to solve a system of linear equations using matrices.**

Previous knowledge of mathematics:

The student should know the basic arithmetic operations (addition, subtraction, multiplication, and division) with real numbers.

Learning Outcomes:

1. Know how and when arithmetic operations with matrices are defined.
2. Calculate the determinant using the basic properties of the determinant.
3. Determine the inverse matrix using the Gauss-Jordan method.
4. Solve a linear system using matrices.

Application:

- in geometry
- in processing of digital photography
- model of consumer preference
- encryption and decoding of messages in cryptography
- analysis of an economic system
- regression line with the least square deviation from the given data set
- problem of transport and distribution
- traveling salesman problem

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2.1. MATRIX

A rectangular table of objects arranged in m rows and n columns is called *a $m \times n$ matrix* or *a matrix of dimensions (size) $m \times n$*. Each matrix is marked with a capital letter, and its objects are set within square or round brackets. Every element of the matrix, i.e., each object in the matrix, is marked by a corresponding small letter with 2 indexes, where the first index indicates the ordinal number of the row in which the element is located, and the second index indicates the ordinal number of the column in which the element is located. If all the elements of a matrix are real numbers, then the matrix is real.

Example 2.1

$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & -1 & -5 \end{bmatrix}$ is 2×3 real matrix with the elements

$$a_{11} = 1, \qquad a_{12} = 4, \qquad a_{13} = 7,$$
$$a_{21} = 3, \qquad a_{22} = -1, \qquad a_{23} = -5.$$

Some special types of matrices

Zero matrix is a matrix whose all elements are equal to zero. Such matrix is marked with O.

Example 2.2

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a 4×2 zero-matrix.

A square matrix is any matrix that has the same number of rows and columns.
Every square matrix, that has n rows and n columns, is a (square) matrix of the nth order or matrix of order n.

Example 2.3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{bmatrix}$$

has 3 rows and 3 columns. So, it is a matrix of order 3.

The elements $a_{11}, a_{22}, \dots, a_{nn}$ form *the main diagonal* of the square matrix of order n.
Therefore, in the previous example, the elements $1,6,-3$ form the main diagonal of the matrix A.

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Unit matrix is a square matrix in which all elements on the main diagonal are equal to one, while all other elements are equal to zero. This type of matrix is marked I_n, where n is the order of that matrix.

Example 2.4

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is unit matrix of the 4th order.

The upper triangular matrix (matrix in row echelon form) is a square matrix in which all the elements below the main diagonal are equal to zero.

The lower triangular matrix is a square matrix in which all the elements above the main diagonal are equal to zero.

Example 2.5

Each unit matrix is both, an upper triangular and a lower triangular matrix.

The transposed matrix A^T of matrix A is obtained by replacing the rows of the matrix A with its columns (and vice versa).
The first column in the matrix A will become the first row in the matrix A^T, the second column in the matrix A will become the second row in the matrix A^T, etc.

Example 2.6

Matrix

$$A^T = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 6 & -2 \\ 3 & 9 & -3 \end{bmatrix}$$

is a transposed matrix of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{bmatrix}.$$

It can be said that matrices A and B are *equal*, and it is written $A = B$, if they have the same dimensions and if they have the same elements in the same positions.

Example 2.7

Determine the dimensions of the given matrix and the required element:

a) $A = \begin{bmatrix} 1 & 5 & \frac{1}{4} & 0 \end{bmatrix}$; a_{14};

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b) $C = \begin{bmatrix} \frac{5}{2} \\ 8 \\ -2 \\ 1 \end{bmatrix}$; c_{31};

c) $B = \begin{bmatrix} x & y & w & e \\ z & 0 & 3 & t+1 \end{bmatrix}$; b_{23};

d) $D = [u_1 \quad u_2 \quad \cdots \quad u_n]$; d_{1r} (for every $r \in \{1,2,\dots,n\}$).

Solution:

a) A has 1 row and 4 columns so A is 1×4 matrix (i.e., row matrix); $a_{14} = 0$;
b) C has 4 rows and 1 column so C is 4×1 matrix (i.e., column matrix); $c_{31} = -2$;
c) B is 2×4 matrix; $b_{23} = 3$;
d) d) D is $1 \times n$ matrix (i.e., row matrix); $d_{1r} = u_r$.

Example 2.8

Find the values of x, y, z and w from the following equation

$$\begin{bmatrix} x+y & x+z \\ y+z & w \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}.$$

Solution:

Matrices $A = \begin{bmatrix} x+y & x+z \\ y+z & w \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$ are equal only if they have the same elements in the same positions.

This implies that

$$\begin{aligned} x + y &= 3 \\ x + z &= 4 \\ y + z &= 5 \\ w &= 4. \end{aligned}$$

The following is calculated by adding the first three equations:

$$\begin{aligned} 2x + 2y + 2z &= 3 + 4 + 5 \\ 2(x + y + z) &= 12 \\ x + y + z &= 6, \end{aligned}$$

whence it follows

$$\begin{aligned} x &= 6 - (y + z) = 6 - 5 = 1, \\ y &= 6 - (x + z) = 6 - 4 = 2, \\ z &= 6 - (x + y) = 6 - 3 = 3. \end{aligned}$$

Therefore,

$$x = 1, y = 2, z = 3, w = 4.$$

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2.2. MATRIX ARITHMETIC

Matrix addition

Only matrices of the same dimensions can be added.

If A and B are $m \times n$ real matrices, then $C = A + B$ is also $m \times n$ real matrix. Element c_{ij} of the matrix C is calculated using the following formula

$$c_{ij} = a_{ij} + b_{ij}.$$

Therefore, the elements in the same positions are added.

Example 2.9

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \end{bmatrix};$$

$$C = A + B = \begin{bmatrix} 2+1 & 3+3 & 7+4 \\ 1+2 & 0+(-1) & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 11 \\ 3 & -1 & 2 \end{bmatrix}.$$

Multiplying the matrix by a number

The real matrix is multiplied by a number so that each element of the matrix is multiplied by that number.

Example 2.10

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix}, \lambda = \frac{1}{2};$$

$$D = \lambda A = \frac{1}{2}\begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot 2 & \frac{1}{2} \cdot 3 & \frac{1}{2} \cdot 7 \\ \frac{1}{2} \cdot 1 & \frac{1}{2} \cdot 0 & \frac{1}{2} \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

The following is also defined

$$-A = (-1) \cdot A.$$

Matrix subtraction

Only matrices of the same dimensions can be subtracted.

If A and B are real matrices of the same dimensions, then their difference, matrix E, is defined as follows:

$$E = A - B = A + (-B) = A + (-1) \cdot B.$$

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Example 2.11

$$A=\begin{bmatrix}2&3&7\\1&0&1\end{bmatrix},B=\begin{bmatrix}1&3&4\\2&-1&1\end{bmatrix};$$

$$E\ =A-B=A+(-1)\cdot B=\begin{bmatrix}2&3&7\\1&0&1\end{bmatrix}+\begin{bmatrix}-1&-3&-4\\-2&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}2+(-1)&3+(-3)&7+(-4)\\1+(-2)&0+1&1+(-1)\end{bmatrix}=\begin{bmatrix}2-1&3-3&7-4\\1-2&0-(-1)&1-1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&3\\-1&1&0\end{bmatrix}.$$

Example 2.12

The matrices A,B,C are defined as follows

$$A=\begin{bmatrix}0&1\\-1&0\\1&-2\end{bmatrix},B=\begin{bmatrix}0.25&1\\0&-0.5\\1&3\end{bmatrix}\text{ and }C=\begin{bmatrix}-1&1\\-1&-1\\1&1\end{bmatrix}.$$

Calculate:

a) $A+B$
b) $A-C$
c) $A+B-C$
d) $12B$
e) $2A-C$
f) $2A+0.5C$
g) $2A^T$
h) A^T+3C^T.

Solution:

a) $$A+B=\begin{bmatrix}0&1\\-1&0\\1&-2\end{bmatrix}+\begin{bmatrix}0.25&1\\0&-0.5\\1&3\end{bmatrix}=\begin{bmatrix}0+0.25&1+1\\-1+0&0+(-0.5)\\1+1&-2+3\end{bmatrix}=\begin{bmatrix}0.25&2\\-1&-0.5\\2&1\end{bmatrix}$$

b) $$A-C=\begin{bmatrix}0&1\\-1&0\\1&-2\end{bmatrix}-\begin{bmatrix}-1&1\\-1&-1\\1&1\end{bmatrix}=\begin{bmatrix}0-(-1)&1-1\\-1-(-1)&0-(-1)\\1-1&-2-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\0&-3\end{bmatrix}$$

c) $$A+B-C=(A+B)-C=\begin{bmatrix}0.25&2\\-1&-0.5\\2&1\end{bmatrix}-\begin{bmatrix}-1&1\\-1&-1\\1&1\end{bmatrix}=\begin{bmatrix}1.25&1\\0&0.5\\1&0\end{bmatrix}$$

d) $$12B=12\begin{bmatrix}0.25&1\\0&-0.5\\1&3\end{bmatrix}=\begin{bmatrix}12\cdot 0.25&12\cdot 1\\12\cdot 0&12\cdot(-0.5)\\12\cdot 1&12\cdot 3\end{bmatrix}=\begin{bmatrix}3&12\\0&-6\\12&36\end{bmatrix}$$

e) $$2A-C=\begin{bmatrix}0&2\\-2&0\\2&-4\end{bmatrix}-\begin{bmatrix}-1&1\\-1&-1\\1&1\end{bmatrix}=\begin{bmatrix}1&1\\-1&1\\1&-5\end{bmatrix}$$

f) $$2A+0.5C=\begin{bmatrix}0&2\\-2&0\\2&-4\end{bmatrix}+\begin{bmatrix}-0.5&0.5\\-0.5&-0.5\\0.5&0.5\end{bmatrix}=\begin{bmatrix}-0.5&2.5\\-2.5&-0.5\\2.5&-3.5\end{bmatrix}$$

g) $2A^T = 2\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -4 \end{bmatrix}$

h) $A^T + 3C^T = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} + 3\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}.$

Matrix multiplication

Product $F = A \cdot B$ of the real matrices A and B exists only if the number of rows of the matrix B is equal to the number of columns of the matrix A.

If A is $m \times n$ real matrix and B is $n \times k$ real matrix, then $F = A \cdot B$ is $m \times k$ real matrix and the following is valid

$$f_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}.$$

Example 2.13

$$A = \begin{bmatrix} 5 & 2 \\ 7 & -1 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 2 & -2 & 4 & 0 \end{bmatrix};$$

$$F = A \cdot B = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} = \begin{bmatrix} 14 & 11 & 13 & 20 \\ 12 & 23 & 3 & 28 \\ -8 & 13 & -19 & 4 \end{bmatrix}.$$

$$f_{12} = \sum_{l=1}^{2} a_{1l} b_{l2} = a_{11}b_{12} + a_{12}b_{22} = 5 \cdot 3 + 2 \cdot (-2) = 11,$$

$$f_{23} = \sum_{l=1}^{2} a_{2l} b_{l3} = a_{21}b_{13} + a_{22}b_{23} = 7 \cdot 1 + (-1) \cdot 4 = 3,$$

$$f_{31} = \sum_{l=1}^{2} a_{3l} b_{l1} = a_{31}b_{11} + a_{32}b_{21} = 1 \cdot 2 + (-5) \cdot 2 = -8,$$

$$\dots$$

Example 2.14

$$[3 \quad 4 \quad 5] \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = [3 \cdot 1 + 4 \cdot 2 + 5 \cdot (-3)] = [-4];$$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot [3 \quad 4 \quad 5] = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ -9 & -12 & -15 \end{bmatrix}.$$

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Example 2.15

Determine the matrix $F = A \cdot B$ if

a) $A = \begin{bmatrix} 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -1 & 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 5 & 2 \\ -1 & 8 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & 8 & 0 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$.

Solution:

a)

Matrix F is not defined (i.e., does not exist) because the number of rows of matrix B is not equal to the number of columns of matrix A. Namely, matrix B has 1 row, and matrix A has 2 columns.

b) $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \end{bmatrix}$, i.e., F is 1×3 matrix because A is 1×4 matrix and B is 4×3 matrix;

$$f_{11} = \sum_{l=1}^{4} a_{1l} b_{l1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}$$

$$= -1 \cdot (-1) + 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) = 0,$$

$$f_{12} = \sum_{l=1}^{4} a_{1l} b_{l2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}$$

$$= -1 \cdot 2 + 1 \cdot (-1) + 2 \cdot 5 + 3 \cdot 8 = 31,$$

$$f_{13} = \sum_{l=1}^{4} a_{1l} b_{l3} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} + a_{14}b_{43}$$

$$= -1 \cdot 0 + 1 \cdot 0 + 2 \cdot 2 + 3 \cdot 1 = 7.$$

Therefore, $F = \begin{bmatrix} 0 & 31 & 7 \end{bmatrix}$.

c) $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix}$, i.e., F is 2×3 matrix because A is 2×3 matrix and B is 3×3 matrix.

$$f_{11} = \sum_{l=1}^{3} a_{1l} b_{l1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 4 = 4,$$

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$$f_{12} = \sum_{l=1}^{3} a_{1l}b_{l2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 8 = 9,$$

$$f_{13} = \sum_{l=1}^{3} a_{1l}b_{l3} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 0 = -1,$$

$$f_{21} = \sum_{l=1}^{3} a_{2l}b_{l1} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = -1 \cdot 0 + 1 \cdot 1 + 2 \cdot 4 = 9,$$

$$f_{22} = \sum_{l=1}^{3} a_{2l}b_{l2} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = -1 \cdot 1 + 1 \cdot 0 + 2 \cdot 8 = 15,$$

$$f_{23} = \sum_{l=1}^{3} a_{2l}b_{l3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-1) \cdot (-1) + 1 \cdot 1 + 2 \cdot 0 = 2.$$

Therefore, $F = \begin{bmatrix} 4 & 9 & -1 \\ 9 & 15 & 2 \end{bmatrix}$.

d) The required matrix is $F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$. Namely, the matrices A and B are 3×3 matrices.

$$f_{12} = \sum_{l=1}^{3} a_{1l}b_{l2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 4 = 3,$$

$$f_{23} = \sum_{l=1}^{3} a_{2l}b_{l3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = 2 \cdot 4 + (-2) \cdot 0 + 1 \cdot 1 = 9,$$

$$f_{31} = \sum_{l=1}^{3} a_{3l}b_{l1} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} = 0 \cdot 1 + 0 \cdot 1 + (-1) \cdot 0 = 0.$$

Similarly, the remaining elements of the matrix F are determined. The following is obtained

$$F = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 9 \\ 0 & -4 & -1 \end{bmatrix}.$$

The two most important properties of matrix multiplication

a) Matrix multiplication is associative, i.e., $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ when all products are defined.

b) Multiplication of matrices is generally not commutative, i.e., if the products $A \cdot B$ and $B \cdot A$ of the matrices A and B exist, then $A \cdot B$ does not have to be (and most often is not) equal to $B \cdot A$.

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It may even happen that one of these products exists and the other does not. In *Example 2.5* there is no product of $B \cdot A$. In *Example 2.6* both products exist and are obviously different because the matrices of different sizes have been obtained.

2.3. MATRIX POLYNOMIAL

Let A be any real square matrix of order n and $P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ a polynomial of degree m, where $x, a_0, a_1, \ldots, a_m \in \mathbb{R}$. Then $P_m(A)$ is defined as follows:

$$P_m(A) = a_0 \cdot A^0 + a_1 \cdot A^1 + a_2 \cdot A^2 + \cdots + a_m \cdot A^m$$

where

$$\begin{aligned} &A^0 = I_n, \\ &A^1 = A, \\ &A^2 = A \cdot A, \\ &A^3 = A^2 \cdot A \overset{\text{a)}}{=} A \cdot A^2, \\ &\vdots \\ &A^m = A^{m-1} \cdot A \overset{\text{a)}}{=} A \cdot A^{m-1}. \end{aligned}$$

It can be noticed that $P_m(A)$ is also a real square matrix, and of the same order as the matrix A.

Example 2.16

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $P_3(x) = 3x^3 + 2x^2 + 2x + 3$, determine $P_3(A)$.

Solution:

$$P_3(A) = 3A^3 + 2A^2 + 2A + 3I_2$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P_3(A) &= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4I_2 \end{aligned}$$

Example 2.17

A polynomial $P_3(x) = x^3 - x^2 - 2x$ and a matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ are given.

Prove the following: $P_3(A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

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Solution:

$$P_3(A) = A^3 - A^2 - 2A$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$P_3(A) = \begin{bmatrix} 5 & -6 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

2.4. DETERMINANT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a real matrix of order n.

The determinant of a matrix A is a number which can be joined to that matrix and is marked by

$$\det A \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

If $\boldsymbol{A} = [\boldsymbol{a_{11}}]$, then $\mathbf{det}\,\boldsymbol{A} = \boldsymbol{a_{11}}$.

If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}$, then $\mathbf{det}\,\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{vmatrix} = \boldsymbol{a_{11}a_{22} - a_{21}a_{12}}$.

2.5. DETERMINANT OF THE MATRIX OF ORDER $\boldsymbol{n \geq 3}$

The minor of the element a_{ij} of the matrix A is determinant of the matrix that is formed from the matrix A by deleting its ith row and jth column. We denote that number by M_{ij}.

Example 2.18

For $A = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$ is

$$M_{11} = -4\,, M_{12} = 2\,, M_{21} = 1\,, M_{22} = 3.$$

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Example 2.19

For $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 0 \\ 4 & 6 & 8 \end{bmatrix}$ is

$$M_{11} = \begin{vmatrix} 7 & 0 \\ 6 & 8 \end{vmatrix} = 56\,, M_{12} = \begin{vmatrix} 5 & 0 \\ 4 & 8 \end{vmatrix} = 40\,, M_{13} = \begin{vmatrix} 5 & 7 \\ 4 & 6 \end{vmatrix} = 2\,, M_{21} = \begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} = -4, \ldots$$

The algebraic complement or *cofactor* of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$ is marked $\boldsymbol{A_{ij}}$. That is a number defined by the formula:

$$\boldsymbol{A_{ij} = (-1)^{i+j} M_{ij}}.$$

Example 2.20

For $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 0 \\ 4 & 6 & 8 \end{bmatrix}$ is

$$A_{11} = M_{11} = 56\,, A_{12} = -M_{12} = -40\,, A_{13} = M_{13} = 2\,, A_{21} = -M_{21} = 4, \ldots$$

2.6. LAPLACE EXPANSION FOR THE DETERMINANT

The following formulas are applied for a real matrix $\boldsymbol{A = [a_{ij}]}$ of order $\boldsymbol{n \geq 2}$:

$$\det A = \sum_{k=1}^{n} a_{ik} A_{ik}$$ (Laplace expansion by the $\boldsymbol{i}$th row),

$$\det A = \sum_{k=1}^{n} a_{kj} A_{kj}$$ (Laplace expansion by the $\boldsymbol{j}$th column).

Example 2.21

Determine $\det A$ using Laplace expansion:

a) by the 1st row
b) by the 2nd column

if $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 0 \\ 4 & 6 & 8 \end{bmatrix}$.

Solution: $n = 3$

a)

$$\det A = \sum_{k=1}^{3} a_{1k} A_{1k} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 3 \cdot 56 + 1 \cdot (-40) + 2 \cdot 2 = 132$$

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b)

$$
\begin{aligned}
\det A = \sum_{k=1}^{3} a_{k2}A_{k2} &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\
&= 1 \cdot (-1)^{1+2} \begin{vmatrix} 5 & 0 \\ 4 & 8 \end{vmatrix} + 7 \cdot (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 4 & 8 \end{vmatrix} + 6 \cdot (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 5 & 0 \end{vmatrix} \\
&= -40 + 112 + 60 = 132
\end{aligned}
$$

Determinant properties:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be real square matrices.

1) If a row (column) of the matrix $\boldsymbol{A}$ contains only zeros, then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

2) If the matrix $\boldsymbol{A}$ has $\mathbf{2}$ proportional rows (columns), then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

3) If a row (column) of the matrix $\boldsymbol{A}$ is a linear combination of the remaining rows (columns) of the matrix $\boldsymbol{A}$, then $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$.

4) The determinant of the unit matrix is $\mathbf{1}$.

5) If $\boldsymbol{A}$ is a triangular or diagonal matrix, then its determinant is equal to the product of the elements of its main diagonal.

6) If the elements of another row (column) of the matrix $\boldsymbol{A}$ are added to a row (column) of the matrix $\boldsymbol{A}$ multiplied by a number, $\mathbf{det}\,\boldsymbol{A}$ does not change.

7) By changing the place of two rows (columns) of the matrix $\boldsymbol{A}$, $\mathbf{det}\,\boldsymbol{A}$ changes the sign.

8) If the elements of a row (column) of a matrix $\boldsymbol{A}$ are multiplied by the number $\boldsymbol{\lambda}$, the result is a square matrix $\boldsymbol{C}$ with a determinant

$$\mathbf{det}\,\boldsymbol{C} = \boldsymbol{\lambda} \cdot \mathbf{det}\,\boldsymbol{A}.$$

9) $\mathbf{det}\,\boldsymbol{A} = \mathbf{det}\,\boldsymbol{A}^{\boldsymbol{T}}$.

10) $\mathbf{det}(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathbf{det}\,\boldsymbol{A} \cdot \mathbf{det}\,\boldsymbol{B}$.

Example 2.22

Calculate $\det A$, using the properties of the determinant, if $A = \begin{bmatrix} 5 & 4 & 5 & 4 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix}$.

Solution:

Method 1 (Laplace expansion by the 1st column):

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$$\det A=\left|\begin{array}{cccc}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{array}\right|\begin{array}{l}R_1-5R_2\\ \\ R_3-2R_2\\ \\ \end{array}\overset{6)}{=}\left|\begin{array}{cccc}0&-6&0&-16\\1&2&1&4\\0&-1&3&-4\\0&2&1&1\end{array}\right|=\sum_{k=1}^{4}a_{k1}A_{k1}=a_{21}A_{21}=$$

$$=1\cdot(-1)^{2+1}\left|\begin{array}{ccc}-6&0&-16\\-1&3&-4\\2&1&1\end{array}\right|\begin{array}{l} \\ R_2-3R_3\\ \\ \end{array}\overset{6)}{=}-\left|\begin{array}{ccc}-6&0&-16\\-7&0&-7\\2&1&1\end{array}\right|=-\sum_{k=1}^{3}a_{k2}A_{k2}=-a_{32}A_{32}=$$

$$=-1\cdot(-1)^{3+2}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}\overset{8)}{=}-7\begin{vmatrix}-6&-16\\1&1\end{vmatrix}\overset{8)}{=}14\begin{vmatrix}3&8\\1&1\end{vmatrix}=14(3-8)=-70.$$

Method 2 (Laplace expansion by the 4th row):

$$\det A=\begin{vmatrix}5&4&5&4\\1&2&1&4\\2&3&5&4\\0&2&1&1\end{vmatrix}\overset{6)}{=}\begin{vmatrix}5&-6&5&-1\\1&0&1&3\\2&-7&5&-1\\0&0&1&0\end{vmatrix}=\sum_{k=1}^{4}a_{4k}A_{4k}=a_{43}A_{43}=1\cdot(-1)^{4+3}\begin{vmatrix}5&-6&-1\\1&0&3\\2&-7&-1\end{vmatrix}\overset{6)}{=}$$

$$\overset{6)}{=}-\begin{vmatrix}5&-6&-16\\1&0&0\\2&-7&-7\end{vmatrix}=-\sum_{k=1}^{3}a_{2k}A_{2k}=-a_{21}A_{21}=-1\cdot(-1)^{2+1}\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=\begin{vmatrix}-6&-16\\-7&-7\end{vmatrix}=-70.$$

2.7. INVERSE MATRIX

Let A be a real square matrix of order n.

The matrix A is a *regular* if $\det A\neq 0$. The matrix A is *singular* if $\det A=0$.

Only regular matrix has an inverse matrix.

For a regular matrix A of order n there is a unique matrix B such that

$$A\cdot B=B\cdot A=I_n,$$

where I_n is a unit matrix of order n. The matrix B is also a regular matrix of order n, marked by A^{-1}, and is called *the inverse matrix* of the matrix A.

2.8. DETERMINING THE INVERSE MATRIX BY CALCULATING DETERMINANTS

The inverse matrix $\boldsymbol{A^{-1}}$ of a regular matrix $\boldsymbol{A}$ of order $\boldsymbol{n}$ can be determined by the formula:

$$\boldsymbol{A^{-1}=\frac{1}{\det A}\begin{bmatrix}A_{11}&A_{21}&\cdots&A_{n1}\\A_{12}&A_{22}&\cdots&A_{n2}\\\vdots&\vdots&\ddots&\vdots\\A_{1n}&A_{2n}&\cdots&A_{nn}\end{bmatrix}},$$

where $\boldsymbol{A_{ij}}$ is the cofactor of the element $\boldsymbol{a_{ij}}$ of the matrix $\boldsymbol{A}$.

This formula is usually applied only for small values of n $(n = 2$ and $n = 3)$ as for determining A^{-1} it is necessary to calculate even n^2 determinants of order $n - 1$.

Example 2.23

Determine the inverse matrix of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by calculating the determinants.

Solution: $n = 2$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{1 \cdot 4 - 3 \cdot 2}\begin{bmatrix} (-1)^{1+1} \cdot 4 & (-1)^{2+1} \cdot 2 \\ (-1)^{1+2} \cdot 3 & (-1)^{2+2} \cdot 1 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Example 2.24

Using the same method, find the inverse matrix of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution: $n = 3$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\det A = 2\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 2$$

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot 1 = 1$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot (-1) = 1$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \cdot (1 + 1) = 2$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot 0 = 0$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot (-2) = -2$$

$$A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -1 \cdot 2 = -2$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 1 \cdot 0 = 0$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = -1 \cdot 0 = 0$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot (-2) = -2$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

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Determination of the inverse matrix using the Gauss-Jordan method

Elementary transformations (on rows or columns) of the real matrix are:

a) replacing the position of any two rows (columns),

b) multiplying any row (column) by a number other than zero,

c) adding any row (column) to another row (column).

If $\boldsymbol{A}$ is a regular matrix of order $\boldsymbol{n} > \mathbf{3}$, Gauss-Jordan method is often used to find $\boldsymbol{A}^{-\mathbf{1}}$.

Proceed as follows:

1) form $\boldsymbol{n} \times \mathbf{2}\boldsymbol{n}$ matrix $\overline{\boldsymbol{A}} = [\boldsymbol{A}|\boldsymbol{I}_{\boldsymbol{n}}]$,

2) transform the matrix $\overline{\boldsymbol{A}}$ into the matrix $[\boldsymbol{I}_{\boldsymbol{n}}|\boldsymbol{B}]$, using elementary transformations exclusively on rows,

Then $\boldsymbol{A}^{-\mathbf{1}} = \boldsymbol{B}$.

Example 2.25

Calculate the inverse matrix of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: $n = 4$

$$\bar{A} = [A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_4 \\ R_2 + R_4 \\ R_3 - R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I_4|B] \Rightarrow A^{-1} = B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.26

Calculate the inverse matrix of the matrix $A = \begin{bmatrix} 5 & 4 & 5 & 4 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix}$ using

Gauss-Jordan method.

Solution: $n = 4$

$$\bar{A} = [A|I_4] = \left[\begin{array}{cccc|cccc} 5 & 4 & 5 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & 3 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \\ R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 5 & 4 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & 3 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 - 2R_1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & -16 & 1 & -5 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_3 \\ R_2 \\ \\ \end{array} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 & -2 & 1 & 0 \\ 0 & -6 & 0 & -16 & 1 & -5 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 6 & 0 & 16 & -1 & 5 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 6R_2 \\ R_4 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 18 & -8 & -1 & -7 & 6 & 0 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 36 & -16 & -2 & -14 & 12 & 0 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7 & -4 & 0 & -3 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 7 & -7 & 0 & -4 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 3R_3 \\ \\ R_4 - 7R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & -140 & 14 & -46 & -12 & 36 \end{array}\right] \begin{array}{l} \\ \\ \\ -R_4/140 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & -140 & 14 & -46 & -12 & 36 \end{array}\right] \begin{array}{l} \\ \\ \\ -R_4/140 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -137 & 14 & -45 & -12 & 35 \\ 0 & 1 & 0 & 61 & -6 & 20 & 5 & -15 \\ 0 & 0 & 1 & 19 & -2 & 6 & 2 & -5 \\ 0 & 0 & 0 & 1 & -7/70 & 23/70 & 6/70 & -18/70 \end{array}\right] \begin{array}{l} R_1 + 137R_4 \\ R_2 - 61R_4 \\ R_3 - 19R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 21/70 & 1/70 & -18/70 & -16/70 \\ 0 & 1 & 0 & 0 & 7/70 & -3/70 & -16/70 & 48/70 \\ 0 & 0 & 1 & 0 & -7/70 & -17/70 & 26/70 & -8/70 \\ 0 & 0 & 0 & 1 & -7/70 & 23/70 & 6/70 & -18/70 \end{array}\right] = [I_4|B]$$

Therefore,

$$A^{-1} = B = \begin{bmatrix} 21/70 & 1/70 & -18/70 & -16/70 \\ 7/70 & -3/70 & -16/70 & 48/70 \\ -7/70 & -17/70 & 26/70 & -8/70 \\ -7/70 & 23/70 & 6/70 & -18/70 \end{bmatrix} = \frac{1}{-70}\begin{bmatrix} -21 & -1 & 18 & 16 \\ -7 & 3 & 16 & -48 \\ 7 & 17 & -26 & 8 \\ 7 & -23 & -6 & 18 \end{bmatrix}.$$

2.9. MATRIX EQUATIONS

Matrix equations are equations in which matrices are used and at least one of the matrices is unknown.

To solve such equation, we need to find all the matrices for which that equation is valid.

How to solve the equations

$$AX = B \text{ and } YA = B$$

in which A and B are known, and X and Y are unknown real matrices?

Assume that A is a regular matrix of order n.
Then A^{-1} exists and, by multiplying the first equation by A^{-1} on the left, we get

$$A^{-1}\cdot \big/ AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$\left(A^{-1}A\right)X = A^{-1}B$$
$$I_nX = A^{-1}B$$
$$X = A^{-1}B.$$

Multiplying the second equation by A^{-1} on the left results in

$$YA = B \big/ \cdot A^{-1}$$
$$(YA)A^{-1} = BA^{-1}$$
$$Y\left(AA^{-1}\right) = BA^{-1}$$
$$YI = BA^{-1}$$
$$Y = BA^{-1}.$$

Note that in general does not have to be $X = Y$ because the matrix multiplication is generally not commutative.

Example 2.27

Solve the equation $AX - B = A^2X - I_2$ if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -85 & -100 \\ -186 & -215 \end{bmatrix}$.

Solution:

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$$\begin{aligned}
&AX-B=A^2X-I_2\\
&AX-A^2X=B-I_2\\
&\left(A-A^2\right)^{-1}\cdot/\left(A-A^2\right)X=B-I_2\\
&I_2X=\left(A-A^2\right)^{-1}\left(B-I_2\right)\\
&X=\left(A-A^2\right)^{-1}\left(B-I_2\right)
\end{aligned}$$

if $A-A^2$ is a regular matrix.

$$A^2=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix} \qquad A-A^2=\begin{bmatrix}-6 & -8\\ -12 & -18\end{bmatrix}\Rightarrow \det(A-A^2)=12\neq 0$$

$$B-I_2=\begin{bmatrix}-86 & -100\\ -186 & -216\end{bmatrix} \qquad (A-A^2)^{-1}=\frac{1}{12}\begin{bmatrix}-18 & 8\\ 12 & -6\end{bmatrix}$$

$$X=(A-A^2)^{-1}(B-I_2)=\frac{1}{12}\begin{bmatrix}-18 & 8\\ 12 & -6\end{bmatrix}\begin{bmatrix}-86 & -100\\ -186 & -216\end{bmatrix}=\frac{1}{12}\begin{bmatrix}60 & 72\\ 84 & 96\end{bmatrix}=\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$$

Example 2.28

Solve the equation $AX^{-1}B+C=AX^{-1}$ if

$$A=\begin{bmatrix}2 & 1 & 1\\ 0 & 0 & -1\\ 0 & 1 & -1\end{bmatrix},B=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 2\\ 2 & 0 & 1\end{bmatrix}\text{ and }C=\begin{bmatrix}0 & 1 & 2\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Solution:

It can be proven that for regular matrices A and B is valid:

$$(AB)^{-1}=B^{-1}A^{-1}$$

if both products, AB and $B^{-1}A^{-1}$, exist.

$$\begin{aligned}
&A^{-1}\cdot/AX^{-1}B+C=AX^{-1}\\
&\underbrace{\left(A^{-1}A\right)}_{=I_3}X^{-1}B+A^{-1}C=\underbrace{\left(A^{-1}A\right)}_{=I_3}X^{-1}\\
&X\cdot/X^{-1}B+A^{-1}C=X^{-1}\\
&\underbrace{\left(XX^{-1}\right)}_{=I_3}B+X\left(A^{-1}C\right)=\underbrace{XX^{-1}}_{=I_3}\\
&B+X\left(A^{-1}C\right)=I_3\\
&X\left(A^{-1}C\right)=I_3-B/\cdot\left(A^{-1}C\right)^{-1}\\
&X=\left(I_3-B\right)\left(A^{-1}C\right)^{-1}=\left(I_3-B\right)C^{-1}A
\end{aligned}$$

if the matrix X is regular. (Namely, it is easy to notice that the matrices A and C are regular.)

$$I_3-B=\begin{bmatrix}0 & -1 & -2\\ -1 & 1 & -2\\ -2 & 0 & 0\end{bmatrix} \qquad C^{-1}=\frac{1}{-2}\begin{bmatrix}1 & -1 & -2\\ -2 & 0 & 4\\ 0 & 0 & -2\end{bmatrix}$$

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$$\begin{aligned}X = (I_3 - B)C^{-1}A &= \begin{bmatrix} 0 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & 0 & 0 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} 1 & -1 & -2 \\ -2 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} 4 & 2 & 2 \\ -6 & 7 & -14 \\ -4 & 2 & -8 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 3 & -3.5 & 7 \\ 2 & -1 & 4 \end{bmatrix}\end{aligned}$$

The equation $AX + XB = C$ cannot be solved using an inverse matrix. The unknown matrix X is located to the right of matrix A, and to the left of matrix B. As the multiplication of matrices is generally not commutative, on the left side of the equation it is not possible to extract, as a common factor, the matrix X. Namely,

$$\begin{aligned} AX + XB &\neq AX + BX = (A + B)X, \\ AX + XB &\neq XA + XB = X(A + B). \end{aligned}$$

It is not difficult to notice that an equation $AX + XB = C$ only makes sense if all the matrices that appear in it are square. So, we know that the order of the matrix X is equal to the order of the matrix A.

How to solve such an equation is shown in the following example.

Example 2.29

Solve the equation $\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} X + X \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 16 & 21 \end{bmatrix}$.

Solution:

X is the matrix of the 2nd order, i.e., $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. The matrix X with unknown elements is included in the equation, and the result is:

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 16 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 4x_{11} + x_{21} & 4x_{12} + x_{22} \\ -3x_{11} + 2x_{21} & -3x_{12} + 2x_{22} \end{bmatrix} + \begin{bmatrix} x_{11} + 5x_{12} & 3x_{11} + 7x_{12} \\ x_{21} + 5x_{22} & 3x_{21} + 7x_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 16 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 5x_{11} + 5x_{12} + x_{21} & 3x_{11} + 11x_{12} + x_{22} \\ -3x_{11} + 3x_{21} + 5x_{22} & -3x_{12} + 3x_{21} + 9x_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 16 & 21 \end{bmatrix}.$$

Two matrices of the same dimensions can be equal if and only if their elements, located in the same positions in the matrices, are equal. Therefore, it has to be

$$\begin{cases} 5x_{11} & + & 5x_{12} & + & x_{21} & = & -4 \\ 3x_{11} & + & 11x_{12} & + & x_{22} & = & -1 \\ -3x_{11} & + & 3x_{21} & + & 5x_{22} & = & 16 \\ -3x_{12} & + & 3x_{21} & + & 9x_{22} & = & 21 \end{cases}.$$

We have obtained a system of linear equations that is easy to solve.

From equation 1: $x_{21} = -4 - 5x_{11} - 5x_{12}$.

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From equation 2: $x_{22} = -1 - 3x_{11} - 11x_{12}$.

By including in the third and fourth equations, the result is:

$$\begin{aligned} -3x_{11} + 3(-4 - 5x_{11} - 5x_{12}) + 5(-1 - 3x_{11} - 11x_{12}) = 16 \\ -3x_{12} + 3(-4 - 5x_{11} - 5x_{12}) + 9(-1 - 3x_{11} - 11x_{12}) = 21 \end{aligned}$$

$$\left.\begin{array}{rcrcr} -33x_{11} & - & 70x_{12} & = & 33 \\ -42x_{11} & - & 117x_{12} & = & 42 \end{array}\right\} \Rightarrow x_{11} = -1\,, x_{12} = 0$$

$$\begin{aligned} x_{21} &= -4 - 5x_{11} - 5x_{12} = -4 - 5\cdot(-1) - 5\cdot 0 = 1 \\ x_{22} &= -1 - 3x_{11} - 11x_{12} = -1 - 3\cdot(-1) - 11\cdot 0 = 2. \end{aligned}$$

Therefore, the matrix $X = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ is the (only) solution of this equation.

Similar to the equation in *Example 3*, equations of the form $AX = B$ and $YA = B$ with unknowns X and Y, in which the matrix A is neither regular nor square, are solved.

2.10. MATRIX RANK

A single-column real matrix is also called *a column vector* (or shorter, a *vector*).

A single-row real matrix is also called *a row vector*.

Example 2.30

$$C = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

is a vector of dimension 3 because it has 3 components: $1{,}4$ and 7.

Vector C is *a zero vector* if all its components are equal to zero.

Analogously, a zero-row vector is defined.

The zero row (column) vector is marked by O.

A non-zero row (column) vector is a row (column) vector for which at least one component is different than zero.

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Example 2.31

Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ can be written in the form

$$A = [C_1 \quad C_2],$$

where $C_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ are column vectors,

or in the form

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

where $R_1 = [1 \quad 2]\,, R_2 = [3 \quad 4]$ and $R_3 = [5 \quad 6]$ are row vectors.

A linear combination of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is any vector C defined by the formula

$$C = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n\,,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers.

Analogously, a linear combination of row vectors is defined.

Notice that from $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ follows $C = 0$.

The opposite does not need to be valid, i.e., if $C = O$, then it does not necessarily have to be

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

It is said that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is *linearly independent*, i.e., that the vectors $C_1, C_2, \dots, C_n$ are *linearly independent*, if from $C = O$ necessarily follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

It is said that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is *linearly dependent* if it is not linearly independent, i.e., if from $C = O$ does not necessarily follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

i.e., that at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ can be different than zero.

If $C = O$ and for example $\alpha_1 \neq 0$, the result is

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = O \Leftrightarrow \alpha_1 C_1 = -\alpha_2 C_2 - \cdots - \alpha_n C_n \Leftrightarrow C_1 = \beta_2 C_2 + \cdots + \beta_n C_n\,,$$

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where $\beta_2 = -\frac{\alpha_2}{\alpha_1}, \dots, \beta_n = -\frac{\alpha_n}{\alpha_1}$ are well defined numbers because $\alpha_1 \neq 0$.

Vector C_1 is written in the form of a linear combination of vectors $C_2, \dots, C_n$.

It can be concluded that a set of vectors $C_1, C_2, \dots, C_n$ of the same dimensions is linearly dependent if and only if at least one of these vectors can be represented as a linear combination of the remaining vectors of that set.

Theorem:

In each real matrix, the maximum number of linearly independent column vectors is equal to the maximum number of linearly independent row vectors. This number is called *the rank of the matrix* $\boldsymbol{A}$ and is marked by $\boldsymbol{r}(\boldsymbol{A})$.

The rank of the real matrix does not change if elementary transformations are performed on the matrix.

We say that two real matrices of the same dimensions, A and B, are *equivalent* if one can be transformed from the other by applying finally many elementary transformations.

In such case we write $A \sim B$.

It means that equivalent matrices have the same rank.

The rank of the real matrix A is determined using *the Gauss method* – by elementary transformations the matrix A is transformed into an equivalent matrix B in which all elements below the diagonal determined by the elements a $b_{11}, b_{22}, \dots$ are equal to zero.

Note that, by applying this procedure, real square matrices are transformed into upper triangular matrices.

In applying this procedure, the following should be taken into account:

1) If one component of a non-zero row vector is equal to zero, and the same component of another non-zero row vector is different from zero, then those two row vectors are linearly independent. The same is also true for vector columns.

2) Each null row vector reduces the rank of the matrix by 1 which is obvious because every set of row-vectors containing a null row vector is linearly dependent. The same is true for column vectors.

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Example 2.32

Determine the rank of the matrix $A=\begin{bmatrix}0 & 1 & 2 & 7\\ -1 & 0 & 3 & 2\\ 2 & 3 & 0 & 17\end{bmatrix}$.

Solution:

$$A=\begin{bmatrix}0 & 1 & 2 & 7\\ -1 & 0 & 3 & 2\\ 2 & 3 & 0 & 17\end{bmatrix}\begin{matrix}R_2\\ R_1\\ \\ \end{matrix}\sim\begin{bmatrix}-1 & 0 & 3 & 2\\ 0 & 1 & 2 & 7\\ 2 & 3 & 0 & 17\end{bmatrix}\begin{matrix}\\ \\ R_3+2R_1\end{matrix}\sim\begin{bmatrix}-1 & 0 & 3 & 2\\ 0 & 1 & 2 & 7\\ 0 & 3 & 6 & 21\end{bmatrix}\begin{matrix}\\ \\ R_3-3R_2\end{matrix}$$

$$\sim\begin{bmatrix}-1 & 0 & 3 & 2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 0 & 0\end{bmatrix}\Rightarrow r(A)=2$$

Some remarks about the solution:

We can write matrix A in the form $A=\begin{bmatrix}R_1\\ R_2\\ R_3\end{bmatrix}$, where $R_1=[0 \quad 1 \quad 2 \quad 7]$, $R_2=[-1 \quad 0 \quad 3 \quad 2]$, $R_3=[2 \quad 3 \quad 0 \quad 17]$ are row vectors of dimension 4.

It is not difficult to notice that each of the sets $\{R_1,R_2\}$, $\{R_1,R_3\}$, $\{R_2,R_3\}$ is linearly independent. For example, R_1 and R_2 are linearly independent because the first component of R_1 is the number 0, while the first component of R_2 is the number -1.

Let us check this using the definition of linear independence. It should be proven that from

$$\alpha_1R_1+\alpha_2R_2=O$$

necessarily follows that $\alpha_1=\alpha_2=0$.

$$\alpha_1R_1+\alpha_2R_2=O$$

$$\alpha_1[0 \quad 1 \quad 2 \quad 7]+\alpha_2[-1 \quad 0 \quad 3 \quad 2]=[0 \quad 0 \quad 0 \quad 0]$$

$$[0 \quad \alpha_1 \quad 2\alpha_1 \quad 7\alpha_1]+[-\alpha_2 \quad 0 \quad 3\alpha_2 \quad 2\alpha_2]=[0 \quad 0 \quad 0 \quad 0]$$

$$[-\alpha_2 \quad \alpha_1 \quad 2\alpha_1+3\alpha_2 \quad 7\alpha_1+2\alpha_2]=[0 \quad 0 \quad 0 \quad 0]\Leftrightarrow\begin{cases} -\alpha_2=0\\ \alpha_1=0\\ 2\alpha_1+3\alpha_2=0\\ 7\alpha_1+2\alpha_2=0\end{cases}$$

$$\Leftrightarrow \alpha_1=\alpha_2=0$$

The result is that $r(A)=2$, so it can be concluded that the set $\{R_1,R_2,R_3\}$ is linearly dependent (because otherwise the rank of the matrix would be 3, not 2). Let us check that too!

According to the definition, it should be proven that from

$$\beta_1R_1+\beta_2R_2+\beta_3R_3=O$$

does not necessarily follow that $\beta_1 = \beta_2 = \beta_3 = 0$.

$$\beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 = O$$

$$\beta_1[0 \quad 1 \quad 2 \quad 7] + \beta_2[-1 \quad 0 \quad 3 \quad 2] + \beta_3[2 \quad 3 \quad 0 \quad 17] = [0 \quad 0 \quad 0 \quad 0]$$

$$[-\beta_2 + 2\beta_3 \quad \beta_1 + 3\beta_3 \quad 2\beta_1 + 3\beta_2 \quad 7\beta_1 + 2\beta_2 + 17\beta_3] = [0 \quad 0 \quad 0 \quad 0]$$

$$\Leftrightarrow \begin{cases} \quad -\beta_2 + 2\beta_3 = 0 \\ \beta_1 \quad + 3\beta_3 = 0 \\ 2\beta_1 + 3\beta_2 \quad = 0 \\ 7\beta_1 + 2\beta_2 + 17\beta_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = -3\beta_3 \\ \beta_2 = 2\beta_3 \\ \beta_3 \text{ is any real number} \end{cases}$$

It can be seen that for $\beta_3 = 0$ we have $\beta_1 = \beta_2 = \beta_3 = 0$. However, this is not the only possibility.

Thus, e.g., $\beta_3 = 1 \Rightarrow \beta_1 = -3\,, \beta_2 = 2$.

Therefore:

$$-3R_1 + 2R_2 + R_3 = O \Leftrightarrow R_3 = 3R_1 - 2R_2.$$

It is proven that a row vector R_3 can be written in a form of a linear combination of row vectors R_1 and R_2.

Exercise (for better understanding of the topic)

Show that for any three real numbers k_1, k_2 and k_3 different from zero and any $a, b, c \in \mathbb{R}$ the upper triangular matrix $A = \begin{bmatrix} k_1 & a & b \\ 0 & k_2 & c \\ 0 & 0 & k_3 \end{bmatrix}$ has a rank 3.

Solution:

$$R_1 = [k_1 \quad a \quad b]\,, R_2 = [0 \quad k_2 \quad c]\,, R_3 = [0 \quad 0 \quad k_3]$$

$$\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3 = O$$

$$\alpha_1[k_1 \quad a \quad b] + \alpha_2[0 \quad k_2 \quad c] + \alpha_3[0 \quad 0 \quad k_3] = [0 \quad 0 \quad 0]$$

$$[k_1\alpha_1 \quad a\alpha_1 + k_2\alpha_2 \quad b\alpha_1 + c\alpha_2 + k_3\alpha_3] = [0 \quad 0 \quad 0] \Leftrightarrow \begin{cases} k_1\alpha_1 = 0 \\ a\alpha_1 + k_2\alpha_2 = 0 \\ b\alpha_1 + c\alpha_2 + k_3\alpha_3 = 0 \end{cases}$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

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This is the so-called lower-triangular system which is simply solved by determining, from the first equation, α_1, then from the second α_2, and finally from the third equation α_3.

A more detailed description of the Gauss method

The rank of the $m \times n$ real matrix A needs to be determined.

If $a_{11} = 0$, then using elementary transformations, the matrix A is transformed into the matrix A_1 in which $a_{11} \neq 0$. If such matrix A_1 does not exist, then the matrix A is a null-matrix with the rank $r(A) = 0$. Otherwise, i.e., if such a matrix A_1 exists, using elementary transformations, the matrix A_1 is transformed into the matrix A_1' in which $a_{21} = a_{31} = \cdots = a_{m1} = 0$. Then, using elementary transformations (but not on the 1st row nor on the 1st column of the matrix A_1'), the matrix A_1' is transformed into the matrix A_2 in which $a_{22} \neq 0$. If such a matrix A_2 does not exist, then $r(A) = r(A_1') = 1$. Otherwise, using elementary transformations (but not on the 1st row nor on the 1st column of the matrix A_2) the matrix A_2 is transformed into the matrix A_2' in which $a_{32} = a_{42} = \cdots = a_{m2} = 0$. Then, using elementary transformations (but not on the first two rows nor on the first two columns of the matrix A_2'), the matrix A_2' is transformed into the matrix A_3 in which $a_{33} \neq 0$. If such a matrix A_3 does not exist, then $r(A) = r(A_2') = 2$. Otherwise, using elementary transformations (but not on the first two rows nor on the first two columns of the matrix A_3), the matrix A_3 is transformed into the matrix A_3' in which $a_{43} = a_{53} = \cdots = a_{m3} = 0$. Then, using elementary transformations (but not on the first three rows nor on the first three columns of the matrix A_3'), the matrix A_3' is transformed into the matrix A_4 in which $a_{44} \neq 0$. If such a matrix A_4 does not exist, then $r(A) = r(A_3') = 3$. Etc.

Sometimes it is convenient to put 1s on the diagonal, which can always be easily achieved, although it is sometimes difficult to avoid calculating with fractions. So, with the usage of elementary transformations, 1 is put on the position a_{11}. The new elements of the new matrix are marked in the same way as the elements of the matrix A. Then the 1st row is multiplied by $-a_{21}, -a_{31}, \dots, -a_{m1}$ and is added to the 2nd row, the 3rd row, …, the mth row. In this way, a new matrix is created (with the same element labels as in the matrix A) in which all the elements in the 1st column below the element a_{11} are equal to zero. After that, neither the 1st row nor the 1st column are going to be changed in elementary transformations. Then, 1 is put on the position a_{22}. The new elements of the new matrix are marked in the same way as the elements of the matrix A. Then the 2nd row is multiplied by $-a_{32}, -a_{42}, \dots, -a_{m2}$ and is added to the 3rd row, the 4th row, …, the mth row. In this way, a new matrix is created (with the same element labels as in the matrix A) in which all the elements in the 1st column below the element a_{11} and all the elements in the 2nd column below the element a_{22} are equal to zero. After that, neither the first two rows nor the first two columns are going to be changed in elementary transformations. Etc.

At the end of the procedure, if A is not a null-matrix (each null-matrix has a rank 0), we get:

a) the matrix $B = \left[\frac{B_1|B_3}{B_2}\right]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, and B_2 is a null-matrix, so $r(A) = r(B) = r(B_1) =$ order of B_1,

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or

b) the matrix $B = \left[\frac{B_1}{B_2}\right]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, and B_2 is a null-matrix, so $r(A) = r(B) = r(B_1) =$ order of $B_1 = n$ (which is possible only when $m > n$),

or

c) the matrix $B = [B_1|B_3]$, where B_1 is the upper triangular matrix that has no zeros on the main diagonal, then $r(A) = r(B) = r(B_1) =$ order of $B_1 = m$ (which is possible only when $m < n$),

or

d) the upper triangular matrix B that has no zeros on the main diagonal, so

$r(A) = r(B) = r(B_1) =$ order of $B_1 = m = n$ (which is possible only when $m = n$).

Notice that in any case $0 \leq r(A) \leq \min\{m, n\}$.

Thus, the rank of the matrix A, which is equivalent to the matrix B, in one of the previous 4 cases is easily determined.

In the *Example 3.3*, the following result is obtained

$$A \sim \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\frac{B_1|B_3}{B_2}\right], \text{ where } B_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix}.$$

Therefore, $r(A) = r(B) = r(B_1) =$ order of $B_1 = 2$. (case a)).

$$\text{Thus, } A_1 = \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 2 & 3 & 0 & 17 \end{bmatrix}, A_1' = A_2 = \begin{bmatrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 7 \\ 0 & 3 & 6 & 21 \end{bmatrix}.$$

Example 2.33

Determine the rank of the matrix $A = \begin{bmatrix} 9 & 20 & 6 \\ 10 & 9 & -5 \\ 8 & 31 & 17 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 9 & 20 & 6 \\ 10 & 9 & -5 \\ 8 & 31 & 17 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -11 & -11 \\ 10 & 9 & -5 \\ 8 & 31 & 17 \end{bmatrix} \begin{matrix} \\ R_2 - 10R_1 \\ R_3 - 8R_1 \end{matrix} \sim \begin{bmatrix} 1 & -11 & -11 \\ 0 & 119 & 105 \\ 0 & 119 & 105 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -11 & -11 \\ 0 & 119 & 105 \\ 0 & 0 & 0 \end{bmatrix}}_{=B} = \left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1 = \begin{bmatrix} 1 & -11 \\ 0 & 119 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} -11 \\ 105 \end{bmatrix}$.

Then $r(A) = r(B) = r(B_1) =$ order of $B_1 = 2$.

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Example 2.34

Determine the rank of the matrix $A=\begin{bmatrix}1&1&1\\1&m&1\\1&m^2&m^2\end{bmatrix}$ depending on the real parameter m.

Solution:

$$A=\begin{bmatrix}1&1&1\\1&m&1\\1&m^2&m^2\end{bmatrix}\begin{matrix}\\R_2-R_1\\R_3-R_1\end{matrix}\sim\begin{bmatrix}1&1&1\\0&m-1&0\\0&m^2-1&m^2-1\end{bmatrix};$$

$$m=1\Rightarrow A\sim\underbrace{\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}}_{=B}=\left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=[1]\,,B_2=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\,,B_3=[1\quad 1]$.
Thus, $r(A)=r(B)=r(B_1)=$ order of $B_1=1$.

$$m\neq 1\Rightarrow A\sim\begin{bmatrix}1&1&1\\0&m-1&0\\0&(m-1)(m+1)&(m-1)(m+1)\end{bmatrix}\begin{matrix}\\R_2/(m-1)\\R_3/(m-1)\end{matrix}$$
$$\sim\begin{bmatrix}1&1&1\\0&1&0\\0&m+1&m+1\end{bmatrix}\begin{matrix}\\\\R_3-(m+1)R_2\end{matrix}\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&m+1\end{bmatrix};$$

$$m=-1\Rightarrow A\sim\underbrace{\begin{bmatrix}1&1&1\\0&1&0\\0&0&0\end{bmatrix}}_{=B}=\left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}\,,B_2=[0\quad 0\quad 0]\,,B_3=\begin{bmatrix}1\\0\end{bmatrix}$;
then $r(A)=r(B)=r(B_1)=\text{order of }B_1=2$.

$$m\notin\{-1,1\}\Rightarrow A\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&m+1\end{bmatrix}\begin{matrix}\\\\R_3/(m+1)\end{matrix}\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow r(A)=r(B)=3\ \text{(case d))}.$$

Therefore,

$$m=1\Rightarrow r(A)=1,$$
$$m=-1\Rightarrow r(A)=2,$$
$$m\notin\{-1,1\}\Rightarrow r(A)=3.$$

Example 2.35

For what value of the real parameter t is the rank of the matrix $A=\begin{bmatrix}t&1&1&1\\1&t&1&1\\1&1&t&1\\1&1&1&t\end{bmatrix}$ equal to 3?

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Solution:

$$A=\begin{bmatrix}t&1&1&1\\1&t&1&1\\1&1&t&1\\1&1&1&t\end{bmatrix}\begin{matrix}R_2\\R_1\\ \\ \end{matrix}\sim\begin{bmatrix}1&t&1&1\\t&1&1&1\\1&1&t&1\\1&1&1&t\end{bmatrix}\begin{matrix} \\R_2-tR_1\\R_3-R_1\\R_4-R_1\end{matrix}\sim\begin{bmatrix}1&t&1&1\\0&1-t^2&1-t&1-t\\0&1-t&t-1&0\\0&1-t&0&t-1\end{bmatrix};$$

case 1: $t=1$

$$A\sim\underbrace{\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{=B}=\begin{bmatrix}B_1|B_3\\\hline B_2\end{bmatrix},$$

where $B_1=[1]\,,B_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\,,B_3=[1\quad 1\quad 1];$
then $r(A)=r(B)=r(B_1)=$ order of $B_1=1$.

case 2: $t\neq 1$

$$A\sim\begin{bmatrix}1&t&1&1\\0&(1-t)(1+t)&1-t&1-t\\0&1-t&-(1-t)&0\\0&1-t&0&-(1-t)\end{bmatrix}\begin{matrix} \\R_2/(1-t)\\R_3/(1-t)\\R_4/(1-t)\end{matrix}\sim\begin{bmatrix}1&t&1&1\\0&1+t&1&1\\0&1&-1&0\\0&1&0&-1\end{bmatrix}\begin{matrix} \\R_3\\R_2\\ \end{matrix}$$

$$\sim\begin{bmatrix}1&t&1&1\\0&1&-1&0\\0&1+t&1&1\\0&1&0&-1\end{bmatrix}\begin{matrix} \\ \\R_3-(1+t)R_2\\R_4-R_2\end{matrix}\sim\begin{bmatrix}1&t&1&1\\0&1&-1&0\\0&0&2+t&1\\0&0&1&-1\end{bmatrix}\begin{matrix} \\ \\R_4\\R_3\end{matrix}$$

$$\sim\begin{bmatrix}1&t&1&1\\0&1&-1&0\\0&0&1&-1\\0&0&2+t&1\end{bmatrix}\begin{matrix} \\ \\ \\R_4-(2+t)R_3\end{matrix}\sim\begin{bmatrix}1&t&1&1\\0&1&-1&0\\0&0&1&-1\\0&0&0&3+t\end{bmatrix};$$

$$t=-3\Rightarrow A\sim\begin{bmatrix}1&-3&1&1\\0&1&-1&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}\Rightarrow r(A)=3.$$

Thus, for $t=-3$

$$r(A)=3.$$

Example 2.36

Determine the rank of the matrix $A=\begin{bmatrix}1&-2&3&-1&-1&-2\\2&-1&1&0&-2&-2\\-2&-5&8&-4&3&-1\\6&0&-1&2&-7&-5\\-1&-1&1&-1&2&1\end{bmatrix}$.

Solution:

$$A=\begin{bmatrix}1&-2&3&-1&-1&-2\\2&-1&1&0&-2&-2\\-2&-5&8&-4&3&-1\\6&0&-1&2&-7&-5\\-1&-1&1&-1&2&1\end{bmatrix}\begin{matrix}\\R_2-2R_1\\R_3+2R_1\\R_4-6R_1\\R_5+R_1\end{matrix}\sim\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&-9&14&-6&1&-5\\0&12&-19&8&-1&7\\0&-3&4&-2&1&-1\end{bmatrix}\begin{matrix}\\ \\R_3+3R_2\\R_4-4R_2\\R_5+R_2\end{matrix}$$

$$\sim\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&0&-1&0&1&1\\0&0&1&0&-1&-1\\0&0&-1&0&1&1\end{bmatrix}\begin{matrix}\\ \\ \\R_4+R_3\\R_5-R_3\end{matrix}\sim\underbrace{\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&0&-1&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}}_{=B}=\left[\frac{B_1|B_3}{B_2}\right],$$

where $B_1=\begin{bmatrix}1&-2&3\\0&3&-5\\0&0&-1\end{bmatrix}$, $B_2=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$, $B_3=\begin{bmatrix}-1&-1&-2\\2&0&2\\0&1&1\end{bmatrix}$;

then $r(A)=r(B)=r(B_1)=\text{order of }B_1=3$.

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2.11. SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

A system of m linear algebraic equations with unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases} \tag{1}$$

can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{2}$$

or

$$A \cdot X = B,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is *a coefficient matrix*, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is *a vector of unknowns*, and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is *a vector of free terms*.

The system (1) is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$. If at least one of the scalars $b_i\,, i = 1,2, \ldots, m$, is different than zero, it is said that the system is *inhomogeneous*.

Every homogeneous system has at least *trivial solution*, i.e., solution in which

$$x_1 = x_2 = \cdots = x_n = 0.$$

It is said that X is a nontrivial solution of a system if at least one component of the vector X is different than zero.

An augmented matrix $\tilde{A}$ for the system (1) is also defined.

$$\tilde{A} = [A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Kronecker-Capelli theorem:

The system $(\mathbf{1})$ has at least one solution if and only if the rank $r(A)$ of the coefficient matrix A is equal to the rank $r(\tilde{A})$ of the augmented matrix $\tilde{A}$.

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a) System in which $m = n = r(A) = r(\tilde{A})$ $(\det A \neq 0)$ has unique solution and can be solved by:

I. Gauss method (Note 3),

II. an inverse matrix

$$A \cdot X = B \Rightarrow X = A^{-1} \cdot B \,, \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

III. Cramer’s rule

$$x_i = \frac{D(x_i)}{\det A} \,, \quad i = 1,2, \dots, n,$$

where $D(x_i)$ is the determinant, obtained from $\det A$, by replacing its ith column with the column of free terms, i.e.

$$D(x_i) = \left| \begin{array}{cccc|c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{array} \right| .$$

b) System in which $r(A) = r(\tilde{A}) = r < n$ has infinitely many solutions and can be solved by the Gauss method (Note 3). r linearly independent equations with r unknowns are selected. Those r unknowns are calculated depending on $n - r$ of the remaining unknowns (so-called *parameters*).

It is said that this type of the system is *an indefinite system*.

Note 1:

A homogeneous system in which the number of equations coincides with the number of unknowns has also a non-trivial solution if and only if $\det A = 0$.

Note 2:

If we apply elementary transformations, but only on the rows of matrix $\tilde{A}$, an augmented matrix of the system, that is equivalent to the system (1), is obtained.

It can easily be seen that equivalent systems have the same solutions (if solutions exist).

Note 3:

Any system of linear equations can be solved by the Gauss method of eliminating variables by reducing the system (1) to an equivalent system with an upper triangular matrix. An extension of the Gauss method is the Gauss-Jordan method where the system matrix is reduced to a unit matrix from which directly provides the solution.

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Note 4: For a system that has no solution, we say that it is *incompatible*, *impossible*, or *inconsistent*. The system is impossible if and only if $r(A) \neq r(\tilde{A})$.

Then there is no vector X such that $A \cdot X = B$.

Example 2.37

Solve the system of linear equations

$$\begin{cases} 2x_1 & + & 3x_2 & + & 5x_3 & = & 10 \\ 3x_1 & + & 7x_2 & + & 4x_3 & = & 3 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 3 \end{cases}$$

a) by Cramer's rule,
b) by inverse matrix,
c) by Gauss method,
d) by Gauss-Jordan method.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 1 & 2 & 2 \end{bmatrix}, \tilde{A} = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 10 \\ 3 & 7 & 4 & 3 \\ 1 & 2 & 2 & 3 \end{array}\right], X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix};$$

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 7 & 4 \\ 1 & 2 & 2 \end{vmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \end{matrix} = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} R_2 + R_1 = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 2 \end{vmatrix}$$
$$= 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \neq 0;$$

a)

$$D(x_1) = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 10 & 3 & 5 \\ 3 & 7 & 4 \\ 3 & 2 & 2 \end{vmatrix} \begin{matrix} R_1 - 3R_3 \\ \\ \end{matrix} = \begin{vmatrix} 1 & -3 & -1 \\ 3 & 7 & 4 \\ 3 & 2 & 2 \end{vmatrix} \begin{matrix} \\ R_2 - R_3 \\ R_3 - 3R_1 \end{matrix}$$
$$= \begin{vmatrix} 1 & -3 & -1 \\ 0 & 5 & 2 \\ 0 & 11 & 5 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 5 & 2 \\ 0 & 1 & 1 \end{vmatrix} R_2 - 2R_3 = \begin{vmatrix} 1 & -3 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = 3 \Rightarrow x_1 = \frac{D(x_1)}{\det A} = \frac{3}{1} = 3,$$

$$D(x_2) = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = \begin{vmatrix} 2 & 10 & 5 \\ 3 & 3 & 4 \\ 1 & 3 & 2 \end{vmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \end{matrix} = \begin{vmatrix} 0 & 4 & 1 \\ 0 & -6 & -2 \\ 1 & 3 & 2 \end{vmatrix} R_2 + 2R_1$$
$$= \begin{vmatrix} 0 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2 \Rightarrow x_2 = \frac{D(x_2)}{\det A} = \frac{-2}{1} = -2$$

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$$D(x_3)=\begin{vmatrix}a_{11}&a_{12}&b_1\\a_{21}&a_{22}&b_2\\a_{31}&a_{32}&b_3\end{vmatrix}=\begin{vmatrix}2&3&10\\3&7&3\\1&2&3\end{vmatrix}\begin{matrix}R_1-2R_3\\R_2-3R_3\\ \end{matrix}=\begin{vmatrix}0&-1&4\\0&1&-6\\1&2&3\end{vmatrix}\begin{matrix}\\R_2+R_1\\ \end{matrix}$$

$$=\begin{vmatrix}0&-1&4\\0&0&-2\\1&2&3\end{vmatrix}=1\cdot(-1)^{3+1}\cdot\begin{vmatrix}-1&4\\0&-2\end{vmatrix}=2\Rightarrow x_3=\frac{D(x_3)}{\det A}=\frac{2}{1}=2$$

b)

$$A_{11}=(-1)^{1+1}\cdot\begin{vmatrix}7&4\\2&2\end{vmatrix}=6\,,A_{21}=(-1)^{2+1}\cdot\begin{vmatrix}3&5\\2&2\end{vmatrix}=4\,,A_{31}=(-1)^{3+1}\cdot\begin{vmatrix}3&5\\7&4\end{vmatrix}=-23$$

$$A_{12}=(-1)^{1+2}\cdot\begin{vmatrix}3&4\\1&2\end{vmatrix}=-2\,,A_{22}=(-1)^{2+2}\cdot\begin{vmatrix}2&5\\1&2\end{vmatrix}=-1\,,A_{32}=(-1)^{3+2}\cdot\begin{vmatrix}2&5\\3&4\end{vmatrix}=7$$

$$A_{13}=(-1)^{1+3}\cdot\begin{vmatrix}3&7\\1&2\end{vmatrix}=-1\,,A_{23}=(-1)^{2+3}\cdot\begin{vmatrix}2&3\\1&2\end{vmatrix}=-1\,,A_{33}=(-1)^{3+3}\cdot\begin{vmatrix}2&3\\3&7\end{vmatrix}=5$$

$$X=A^{-1}\cdot B=\begin{bmatrix}6&4&-23\\-2&-1&7\\-1&-1&5\end{bmatrix}\cdot\begin{bmatrix}10\\3\\3\end{bmatrix}=\begin{bmatrix}3\\-2\\2\end{bmatrix}\Rightarrow x_1=3\,,x_2=-2\,,x_3=2$$

c)

$$\tilde{A}=\left[\begin{array}{ccc|c}2&3&5&10\\3&7&4&3\\1&2&2&3\end{array}\right]\begin{matrix}R_3\\ \\R_1\end{matrix}\sim\left[\begin{array}{ccc|c}1&2&2&3\\3&7&4&3\\2&3&5&10\end{array}\right]\begin{matrix}\\R_2-3R_1\\R_3-2R_1\end{matrix}\sim\left[\begin{array}{ccc|c}1&2&2&3\\0&1&-2&-6\\0&-1&1&4\end{array}\right]\begin{matrix}\\ \\R_3+R_2\end{matrix}$$

$$\sim\left[\begin{array}{ccc|c}1&2&2&3\\0&1&-2&-6\\0&0&-1&-2\end{array}\right]$$

equation 3: $-x_3=-2\Rightarrow x_3=2$

equation 2: $x_2-2x_3=-6\Rightarrow x_2=2x_3-6=2\cdot2-6=-2$

equation 1: $x_1+2x_2+2x_3=3\Rightarrow x_1=3-2x_2-2x_3=3-2\cdot(-2)-2\cdot2=3$

d)

$$\tilde{A}\sim\left[\begin{array}{ccc|c}1&2&2&3\\0&1&-2&-6\\0&0&-1&-2\end{array}\right]\begin{matrix}R_1-2R_2\\ \\-R_3\end{matrix}\sim\left[\begin{array}{ccc|c}1&0&6&15\\0&1&-2&-6\\0&0&1&2\end{array}\right]\begin{matrix}R_1-6R_3\\R_2+2R_3\\ \end{matrix}\sim\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-2\\0&0&1&2\end{array}\right]$$

$$\Rightarrow x_1=3\,,x_2=-2\,,x_3=2$$

Example 2.38

Solve the system of linear equations

$$\begin{cases} 3x_1 & + & 4x_2 & = & 11 \\ 4x_1 & + & 3x_2 & = & 10 \end{cases}$$

by Cramer’s rule and by the inverse matrix.

Solution:

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

By Cramer’s rule:

$$\det A = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9 - 16 = -7$$

$$D(x_1) = \begin{vmatrix} 11 & 4 \\ 10 & 3 \end{vmatrix} = 33 - 40 = -7 \Rightarrow x_1 = \frac{D(x_1)}{\det A} = 1$$

$$D(x_2) = \begin{vmatrix} 3 & 11 \\ 4 & 10 \end{vmatrix} = 30 - 44 = -14 \Rightarrow x_2 = \frac{D(x_2)}{\det A} = 2$$

By the inverse matrix:

$$A_{11} = (-1)^{1+1} \cdot 3 = 3 \qquad A_{21} = (-1)^{2+1} \cdot 4 = -4$$

$$A_{12} = (-1)^{1+2} \cdot 4 = -4 \qquad A_{22} = (-1)^{2+2} \cdot 3 = 3$$

$$X = A^{-1}B = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 10 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 33 - 40 \\ -44 + 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, x_2 = 2$$

Example 2.39

Find all solutions of the following system.

$$\begin{cases} 4x_1 & + & 3x_2 & - & 3x_3 & - & 3x_4 & + & 3x_5 & = & 2 \\ 2x_1 & + & x_2 & - & x_3 & - & x_4 & - & x_5 & = & 0 \\ 7x_1 & + & 5x_2 & - & 5x_3 & - & 5x_4 & + & 4x_5 & = & 3 \\ x_1 & + & x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & 1 \end{cases}$$

Solution:

The system cannot be solved by Cramer's rule or by the invers matrix because $n > m \geq r(A)$.

We will solve it by applying Gauss method.

$$\tilde{A} = \left[\begin{array}{ccccc|c} 4 & 3 & -3 & -3 & 3 & 2 \\ 2 & 1 & -1 & -1 & -1 & 0 \\ 7 & 5 & -5 & -5 & 4 & 3 \\ 1 & 1 & -1 & -1 & 2 & 1 \end{array}\right] \begin{array}{l} R_4 \\ \\ R_1 \\ R_3 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -1 & -1 & 0 \\ 4 & 3 & -3 & -3 & 3 & 2 \\ 7 & 5 & -5 & -5 & 4 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 7R_1 \end{array}$$

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$$\sim\left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 & -5 & -2 \\ 0 & -1 & 1 & 1 & -5 & -2 \\ 0 & -2 & 2 & 2 & -10 & -4 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 & -5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow r(A) = r(\tilde{A}) = 2$$

Therefore, the system has (at least) one solution.

The last matrix, i.e., the matrix $\left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 & -5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$, is an augmented matrix of the system that has the same solutions as the defined system. Note that in this system we actually have only two equations. Namely, the rows with all zeros, i.e., the last two rows represent equations that are valid for each selection of unknowns $x_1, x_2, \dots, x_5$. So, only the first 2 rows, i.e., two equations, are observed. Since there are 5 unknowns and two equations, 3 unknowns can be chosen in any way (because $5 - 2 = 3$), and the remaining two are determined using the selected 3. Unknowns that can be chosen in any way are called *system parameters* and are labelled in a special way.

Accordingly,

$$\text{the number of parameters} = n - r(A) = 5 - 2 = 3.$$

Let the parameters be unknowns x_3, x_4 and x_5. The following labels are introduced:

$$x_3 = \alpha\,, x_4 = \beta\,, x_5 = \gamma.$$

The system

$$\begin{cases} x_1 + x_2 - x_3 - x_4 + 2x_5 = 1 \\ - x_2 + x_3 + x_4 - 5x_5 = -2 \end{cases}$$

with the augmented matrix $\left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 1 & -5 & -2 \end{array}\right]$ is solved starting from the end, i.e., from the second equation.

equation 2: $-x_2 + \alpha + \beta - 5\gamma = -2 \Leftrightarrow x_2 = 2 + \alpha + \beta - 5\gamma$

equation 1: $x_1 + 2 + \cancel{\alpha} + \cancel{\beta} - 5\gamma \cancel{-\alpha} \cancel{-\beta} + 2\gamma = 1 \Leftrightarrow x_1 = 3\gamma - 1.$

All system solutions are all arranged fives $(3\gamma - 1, 2 + \alpha + \beta - 5\gamma, \alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary parameters. Therefore, it is written

$$X = \begin{bmatrix} 3\gamma - 1 \\ 2 + \alpha + \beta - 5\gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

Example 2.40

$$\begin{cases} 6x_1 + 2x_2 - 2x_3 + 5x_4 + 7x_5 = 0 \\ 9x_1 + 4x_2 - 3x_3 + 8x_4 + 9x_5 = 0 \\ 6x_1 + 6x_2 - 2x_3 + 7x_4 + x_5 = 0 \\ 3x_1 + 4x_2 - x_3 + 4x_4 - x_5 = 0 \end{cases}$$

Solution:

A system in which **each** equation on the right side of the equality has zero is called a **homogeneous system**. Such a system obviously always has a solution (in which all unknowns are equal to zero). Such a solution is called a trivial solution. However, this solution does not have to be the only one.

When determining the rank of a matrix of a homogeneous system, it is not necessary to write zeros on the right-hand sides because they do not change by applying elementary transformations.

$$A = \begin{bmatrix} 6 & 2 & -2 & 5 & 7 \\ 9 & 4 & -3 & 8 & 9 \\ 6 & 6 & -2 & 7 & 1 \\ 3 & 4 & -1 & 4 & -1 \end{bmatrix} \begin{array}{l} R_4 \\ R_3 \\ R_1 \\ R_2 \end{array} \sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 6 & 6 & -2 & 7 & 1 \\ 6 & 2 & -2 & 5 & 7 \\ 9 & 4 & -3 & 8 & 9 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 0 & -2 & 0 & -1 & 3 \\ 0 & -6 & 0 & -3 & 9 \\ 0 & -8 & 0 & -4 & 12 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 3R_2 \\ R_4 - 4R_2 \end{array} \sim \begin{bmatrix} 3 & 4 & -1 & 4 & -1 \\ 0 & -2 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow r(A) = r(\tilde{A}) = 2$$

the number of parameters$= n - r(A) = 5 - 2 = 3$

$$x_3 = \alpha\,, x_4 = \beta\,, x_5 = \gamma$$

equation 2:

$$-2x_2 - x_4 + 3x_5 = 0 \Rightarrow x_2 = \frac{3x_5 - x_4}{2} = \frac{3\gamma - \beta}{2}$$

equation 1:

$$3x_1 + 4x_2 - x_3 + 4x_4 - x_5 = 0 \Rightarrow x_1 = \frac{x_3 + x_5 - 4x_2 - 4x_4}{3} = \frac{\alpha - 2\beta - 5\gamma}{3}$$

All system solutions are all arranged fives $\left(\frac{\alpha-2\beta-5\gamma}{3}, \frac{3\gamma-\beta}{2}, \alpha, \beta, \gamma\right)$, where $\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary parameters.

Therefore,

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$$X = \begin{bmatrix} \dfrac{\alpha - 2\beta - 5\gamma}{3} \\ \dfrac{3\gamma - \beta}{2} \\ \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

We do not always have to take for the parameters the unknowns x_3, x_4 and x_5.

For example, if x_1, x_2 and x_5 are chosen parameters:

$$x_1 = u_1 \,, x_2 = u_2 \text{ and } x_5 = u_3 \,,$$

then, the result of the 2nd equation is

$$-2u_2 - x_4 + 3u_3 = 0 \Leftrightarrow x_4 = 3u_3 - 2u_2 \,,$$

and by including this result into the 1st equation, results in

$$3u_1 + 4u_2 - x_3 + 4(3u_3 - 2u_2) - u_3 = 0 \Leftrightarrow x_3 = 3u_1 - 4u_2 + 11u_3.$$

Therefore, in this case we have a simpler notation of the solution:

$$X = \begin{bmatrix} u_1 \\ u_2 \\ 3u_1 - 4u_2 + 11u_3 \\ 3u_3 - 2u_2 \\ u_3 \end{bmatrix}.$$

Of course, if $r(A) = r(\tilde{A}) = n$, then $n - r(A) = 0$, so no unknown can be a parameter, i.e., the system has a unique solution.

Example 2.41

$$\begin{cases} 4x & - & 4y & + & z & = & 8 \\ 6x & - & 3y & - & 2z & = & 21 \\ -x & + & 3y & + & 7z & = & 4 \end{cases}$$

Solution:

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & -4 & 1 & 8 \\ 6 & -3 & -2 & 21 \\ -1 & 3 & 7 & 4 \end{array}\right] \begin{array}{l} R_3 \\ R_1 \\ R_2 \end{array} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 4 & -4 & 1 & 8 \\ 6 & -3 & -2 & 21 \end{array}\right] \begin{array}{l} \\ R_2 + 4R_1 \\ R_3 + 6R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 8 & 29 & 24 \\ 0 & 15 & 40 & 45 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_3 \\ R_3/5 \end{array} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 1 & 18 & 3 \\ 0 & 3 & 8 & 9 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \sim \left[\begin{array}{ccc|c} -1 & 3 & 7 & 4 \\ 0 & 1 & 18 & 3 \\ 0 & 0 & -46 & 0 \end{array}\right]$$

$$\Rightarrow r(A) = r(\tilde{A}) = n = 3$$

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$$\Rightarrow \begin{cases} \text{equation 3:} & z = 0, \\ \text{equation 2:} & y = 3, \\ \text{equation 1:} & -x + 3y = 4 \Rightarrow x = 5. \end{cases}$$

Example 2.42

A system of linear equations is given

$$\begin{cases} 4x_1 & - & 2x_2 & + & 5x_3 & + & 3x_4 & = & 0 \\ 3x_1 & + & 6x_2 & + & 5x_3 & - & 4x_4 & = & 0 \\ 3x_1 & + & 3x_2 & + & px_3 & - & 1.5x_4 & = & 0 \\ x_1 & + & 4x_2 & + & 2x_3 & - & 3x_4 & = & 0 \end{cases},$$

where $p \in \mathbb{R}$. It is necessary to determine a p for which an arranged four $(-6,1,4,2)$ is the solution of a given system. Is it possible to choose a p so that the system has a unique solution?

Solution:

$$A = \begin{bmatrix} 4 & -2 & 5 & 3 \\ 3 & 6 & 5 & -4 \\ 3 & 3 & p & -1.5 \\ 1 & 4 & 2 & -3 \end{bmatrix} \begin{matrix} R_4 \\ \\ \\ R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 3 & 6 & 5 & -4 \\ 3 & 3 & p & -1.5 \\ 4 & -2 & 5 & 3 \end{bmatrix} \begin{matrix} \\ \\ 2R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 3 & 6 & 5 & -4 \\ 6 & 6 & 2p & -3 \\ 4 & -2 & 5 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 6R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & -6 & -1 & 5 \\ 0 & -18 & 2p - 12 & 15 \\ 0 & -18 & -3 & 15 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \\ R_4 - 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & -6 & -1 & 5 \\ 0 & 0 & 2p - 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 cases.

1) $\quad 2p - 9 = 0 \Leftrightarrow p = \frac{9}{2}; \qquad r(A) = 2$

$$\text{the number of parameters} = n - r(A) = 4 - 2 = 2$$

$$x_2 = \alpha \,, x_4 = \beta$$

equation 2: $\quad -6x_2 - x_3 + 5x_4 = 0 \Rightarrow x_3 = 5x_4 - 6x_2 = 5\beta - 6\alpha$

equation 1: $\quad x_1 + 4x_2 + 2x_3 - 3x_4 = 0 \Rightarrow x_1 = 3x_4 - 4x_2 - 2x_3 = 8\alpha - 7\beta$

All system solutions are all arranged fours $(8\alpha - 7\beta, \alpha, 5\beta - 6\alpha, \beta)$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary parameters.

If $x_2 = \alpha = 1$ and $x_4 = \beta = 2$, then

$$x_1 = 8\alpha - 7\beta = -6,$$
$$x_3 = 5\beta - 6\alpha = 5.$$

Accordingly, the point $(-6,1,4,2)$ is the solution of the system for $p=\frac{9}{2}$.

2) $\quad 2p-9\neq 0 \Leftrightarrow p\neq\frac{9}{2}$; $\quad r(A)=3$

$$\text{the number of parameters} = n-r(A)=4-3=1$$

$$x_2=5\alpha$$

equation 3: $\quad \underbrace{(2p-9)}_{\neq 0}x_3=0\Rightarrow x_3=0$

equation 2: $\quad -6x_2-x_3+5x_4=0\Rightarrow x_4=\frac{6x_2+x_3}{5}=6\alpha$

equation 1: $\quad x_1+4x_2+2x_3-3x_4=0\Rightarrow x_1=3x_4-4x_2-2x_3=-2\alpha$

Note that the unknown x_3 (in case 2) cannot be a parameter (it must be exactly equal to zero for equation 3 to be valid).

All system solutions are all arranged fours $(-2\alpha,5\alpha,0{,}6\alpha)$, where $\alpha\in\mathbb{R}$ is an arbitrary parameter.

It can be concluded that there is no $p\in\mathbb{R}$ for which the system has a unique solution.

Example 2.43

Prove that the system

$$\left\{\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & & & + & 2x_4 & = & 2\\ 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & 0\\ -x_1 & - & x_2 & + & 2x_3 & + & 2x_4 & = & 2\\ -5x_1 & - & 4x_2 & + & 5x_3 & + & 7x_4 & = & 5 \end{array}\right.$$

is impossible.

Solution:

$$\tilde{A}=\left[\begin{array}{cccc|c} 2 & 2 & 0 & 2 & 2\\ 2 & 1 & 1 & -1 & 0\\ -1 & -1 & 2 & 2 & 2\\ -5 & -4 & 5 & 7 & 5 \end{array}\right]\begin{matrix} R_1/2\\ \\ \\ \\ \end{matrix}\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1\\ 2 & 1 & 1 & -1 & 0\\ -1 & -1 & 2 & 2 & 2\\ -5 & -4 & 5 & 7 & 5 \end{array}\right]\begin{matrix} \\ R_2-2R_1\\ R_3+R_1\\ R_4+5R_1 \end{matrix}$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1\\ 0 & -1 & 1 & -3 & -2\\ 0 & 0 & 2 & 3 & 3\\ 0 & 1 & 5 & 12 & 10 \end{array}\right]\begin{matrix} \\ \\ \\ R_4+R_2 \end{matrix}\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1\\ 0 & -1 & 1 & -3 & -2\\ 0 & 0 & 2 & 3 & 3\\ 0 & 0 & 6 & 9 & 8 \end{array}\right]\begin{matrix} \\ \\ \\ R_4-3R_3 \end{matrix}$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 & -2 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \Rightarrow r(A) = 3 \neq 4 = r(\tilde{A}).$$

The last line in the last matrix is an equation for which no choice of unknowns can be satisfactory. Namely, for any $x_1, x_2, x_3, x_4 \in \mathbb{R}$ is

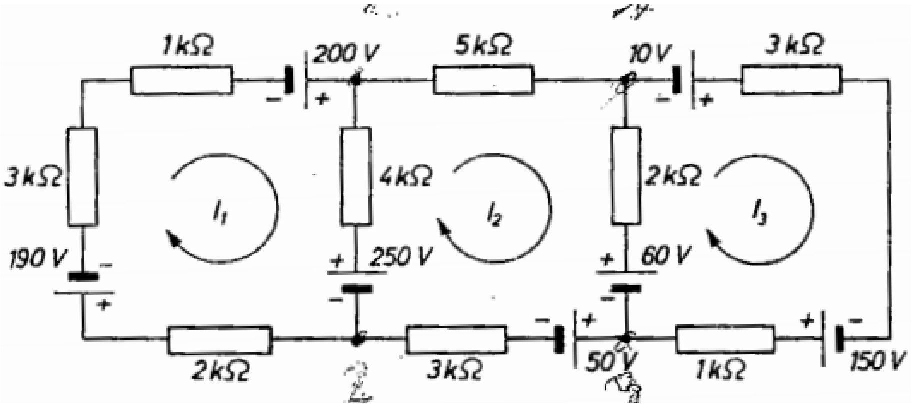
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0,$$

and $0 \neq -1$.

2.12. SOME EXAMPLES OF MATRIX APPLICATION

Example 2.44

The figure below shows one electrical circuit, such as an electrical circuit on a ship. We want to determine the current strength in all branches. It will be shown how this problem is reduced to the problem of solving a system of three linear equations with 3 unknowns, using the method of contour currents.



The circuit has 6 branches and 4 nodes, so it is necessary to select 3 independent contours.

For the selected contours I_1, I_2 and I_3 the contour currents have been drawn in the clockwise direction.

The equations of contour currents are:

$$I_1(3000 + 1000 + 4000 + 2000) - I_2 \cdot 4000 = -190 + 200 - 250,$$

$$I_2(4000 + 5000 + 2000 + 3000) - I_1 \cdot 4000 - I_3 \cdot 2000 = 250 - 60 - 50,$$

$$I_3(2000 + 3000 + 1000) - I_2 \cdot 2000 = 60 + 10 + 150.$$

After addition and shortening, the following system is obtained.

$$\begin{cases} 10I_1 & - & 4I_2 & + & 0I_3 & = & -0.24 \\ -4I_1 & + & 14I_2 & - & 2I_3 & = & 0.14 \\ 0I_1 & - & 2I_2 & + & 6I_3 & = & 0.22 \end{cases}$$

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Its coefficient matrix $A=\begin{bmatrix}10 & -4 & 0\\ -4 & 14 & -2\\ 0 & -2 & 6\end{bmatrix}$ is regular. Using Cramer's rule, the following is found:

$$I_1=\frac{D_1}{\det A}\,,I_2=\frac{D_2}{\det A}\ \text{ and }\ I_3=\frac{D_3}{\det A}\,,$$

where

$$\det A=\begin{vmatrix}10 & -4 & 0\\ -4 & 14 & -2\\ 0 & -2 & 6\end{vmatrix}=704,$$

$$D_1=\begin{vmatrix}-0.24 & -4 & 0\\ 0.14 & 14 & -2\\ 0.22 & -2 & 6\end{vmatrix}=-14.08,$$

$$D_2=\begin{vmatrix}10 & -0.24 & 0\\ -4 & 0.14 & -2\\ 0 & 0.22 & 6\end{vmatrix}=7.04,$$

$$D_3=\begin{vmatrix}10 & -4 & -0.24\\ -4 & 14 & 0.14\\ 0 & -2 & 0.22\end{vmatrix}=28.16.$$

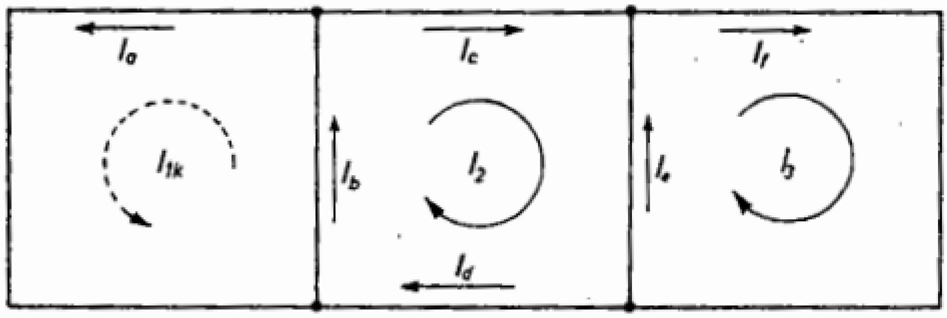
Therefore,

$$I_1=-20\,mA\,,I_2=10\,mA\ \text{ and }\ I_3=40\,mA.$$

Currents in some branches can now be calculated.

As $I_1<0$, the presumed direction of that current was incorrect.

After correcting the direction of I_1



the currents I_a, I_b, I_c, I_d, I_e and I_f are as follows:

$$I_a=I_{1k}=20\,mA,$$

$$I_b=I_{1k}+I_2=30\,mA,$$

$$I_c = I_2 = 10\,mA,$$

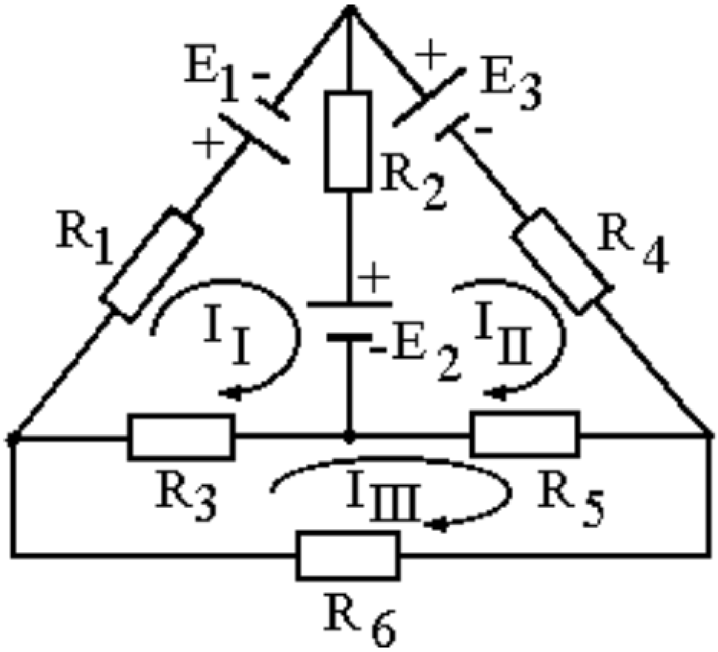
$$I_d = I_2 = 10\,mA,$$

$$I_e = I_3 - I_2 = 30\,mA,$$

$$I_f = I_3 = 40\,mA.$$

Example 2.45

The contour current method should be used to determine the strength of the electric current through all the resistors in the figure.



It is known that:

$$E_1 = 10\,V\,, E_2 = 20\,V\,, E_3 = 5\,V\,, R_1 = 20\,\Omega\,, R_2 = 5\,\Omega\,, R_3 = 25\,\Omega\,, R_4 = 15\,\Omega\,,$$
$$R_5 = 30\,\Omega\,, R_6 = 10\,\Omega.$$

The equations of contour currents are:

$$\begin{aligned} -E_1 - E_2 &= (R_1 + R_2 + R_3)\cdot I_I - R_2\cdot I_{II} - R_3\cdot I_{III}\,, \\ E_2 - E_3 &= -R_2\cdot I_I + (R_2 + R_4 + R_5)\cdot I_{II} - R_5\cdot I_{III}\,, \\ 0 &= -R_3\cdot I_I - R_5\cdot I_{II} + (R_3 + R_5 + R_6)\cdot I_{III}\,. \end{aligned}$$

By entering and arranging data, the following system is obtained.

$$\begin{cases} 10I_I & - & I_{II} & - & 5I_{III} & = & -6 \\ -I_I & + & 10I_{II} & - & 6I_{III} & = & 3 \\ -5I_I & - & 6I_{II} & + & 13I_{III} & = & 0 \end{cases}$$

The coefficient matrix $A = \begin{bmatrix} 10 & -1 & -5 \\ -1 & 10 & -6 \\ -5 & -6 & 13 \end{bmatrix}$ is regular. Using Cramer's rule, the following is found:

$$I_I = \frac{D_I}{\det A}\,, I_{II} = \frac{D_{II}}{\det A} \text{ and } I_{III} = \frac{D_{III}}{\det A}\,,$$

where

$$\det A = \begin{vmatrix} 10 & -1 & -5 \\ -1 & 10 & -6 \\ -5 & -6 & 13 \end{vmatrix} = 617,$$

$$D_I = \begin{vmatrix} -6 & -1 & -5 \\ 3 & 10 & -6 \\ 0 & -6 & 13 \end{vmatrix} = -435,$$

$$D_{II} = \begin{vmatrix} 10 & -6 & -5 \\ -1 & 3 & -6 \\ -5 & 0 & 13 \end{vmatrix} = 57,$$

$$D_{III} = \begin{vmatrix} 10 & -1 & -6 \\ -1 & 10 & 3 \\ -5 & -6 & 0 \end{vmatrix} = -141.$$

Therefore,

$$I_I = -0.71\ A\,, I_{II} = 92\ mA \text{ and } I_{III} = -0.23\ A.$$

The negative sign of the first and third contour currents means that the direction for these two contour currents is incorrectly assumed, and the direction should be drawn correctly in the diagram. The amounts of these currents are correct. The strengths of the currents through the individual resistors are determined from the circuit with the corrected direction of the contour currents.

$$\begin{aligned} I_{R_1} &= I_I = 0.71\ A \\ I_{R_2} &= I_I + I_{II} = 0.802\ A \\ I_{R_3} &= I_I - I_{III} = 0.48\ A \\ I_{R_4} &= I_{II} = 0.092\ A \\ I_{R_5} &= I_{II} + I_{III} = 0.32\ A \\ I_{R_6} &= I_{III} = 0.23\ A \end{aligned}$$

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Applications of matrix multiplication in geometry and computer graphics.

Example 2.46 Symmetry of an object with respect to the line

The coordinate axes can be chosen so that the equation of that line is $x = 0$.

The point A with coordinates (x_A, y_A) is unambiguously associated with the vector $\begin{bmatrix} x_A \\ y_A \end{bmatrix}$ so it can be written $A = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$. If $S_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then

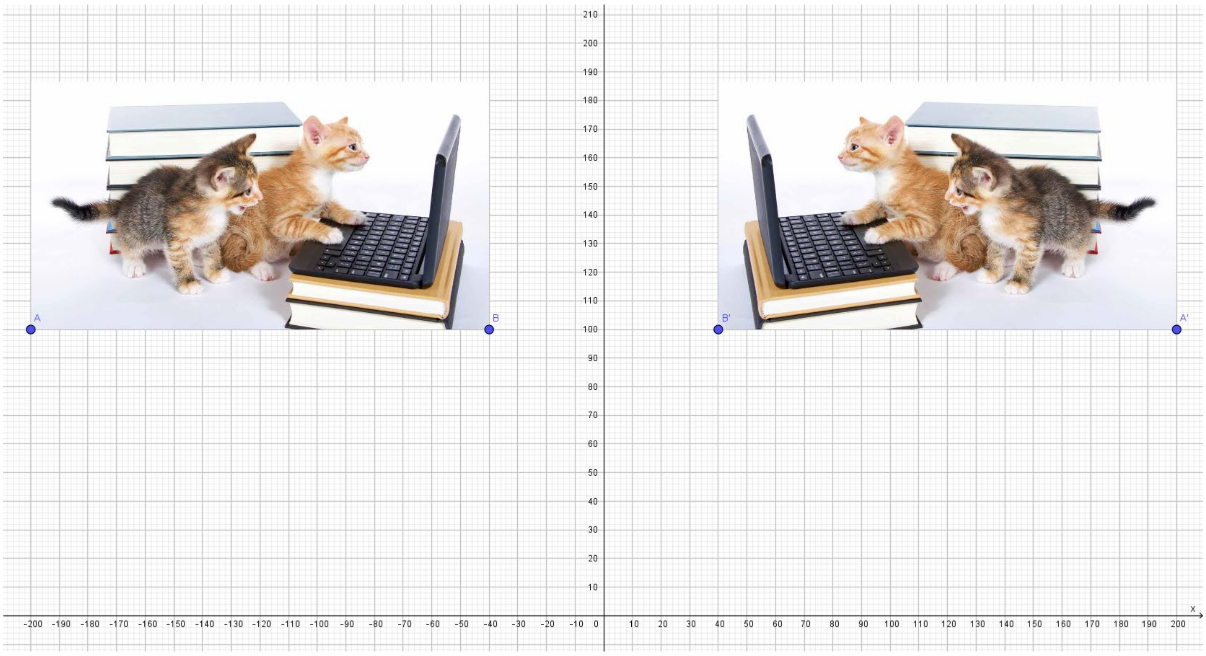
$$S_y \cdot A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} -x_A \\ y_A \end{bmatrix}.$$

So, it is said that by the action of the matrix S_y on the point A with coordinates (x_A, y_A), the point A' with coordinates $(-x_A, y_A)$ is obtained. The point A' is symmetric to the point A with respect to the straight line $x = 0$ and that's why S_y is called the symmetry matrix (with respect to the line $x = 0$).

The action of the matrix S_y on any object O in plane xy creates an object O' that is symmetrical to the object O with respect to the line $x = 0$.

S_y acts on the photograph (i.e., on each point of the photograph) above the line segment $\overline{AB}$.

This creates a photograph above the line segment $\overline{A'B'}$.



Example 2.47 Rotation of an object in the positive direction for an angle $\boldsymbol{\alpha}$ around a point

Coordinate axes can be chosen so that they intersect exactly at the point $(0,0)$ around which the rotation takes place. If $A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$, where r and φ are polar coordinates of the point A.

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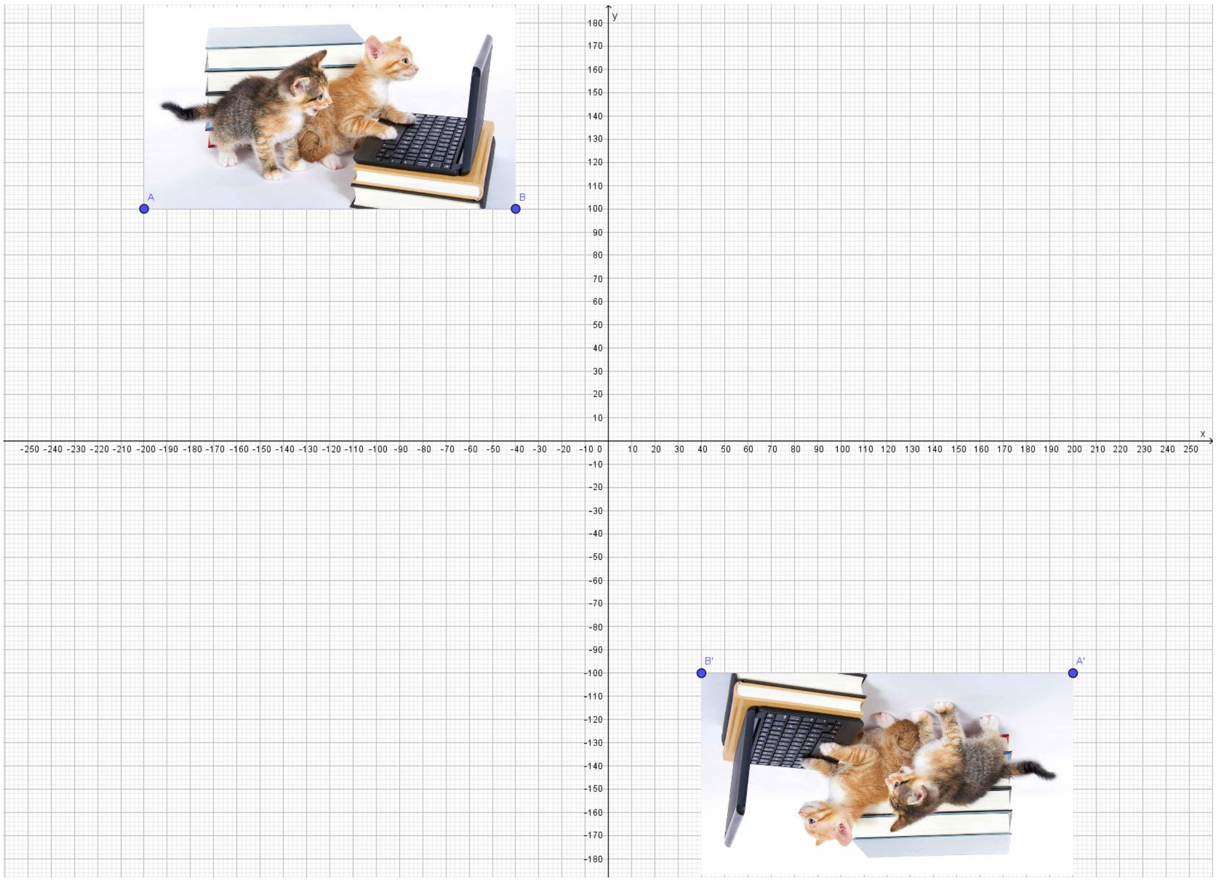
If $R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then

$$R_\alpha \cdot A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix} = \begin{bmatrix} r(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi) \\ r(\sin\alpha\cos\varphi + \cos\alpha\sin\varphi) \end{bmatrix} = \begin{bmatrix} r\cos(\alpha+\varphi) \\ r\sin(\alpha+\varphi) \end{bmatrix}$$

so, by the action of the matrix R_α on the point A with the coordinates $(r\cos\varphi\,, r\sin\varphi)$, the point A' is obtained with coordinates $(r\cos(\alpha+\varphi)\,, r\sin(\alpha+\varphi))$. Note that the point A' is obtained by rotating the point A in the positive direction by the angle α around the origin. R_α is therefore called the rotation matrix in the positive direction for the angle α.

By acting of the matrix R_α on any object O in the plane xy the object O' is obtained, and each point of the object O' is obtained by rotating the corresponding point of the object O in the positive direction by the angle α around the origin.

$R_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ rotates the photograph (i.e., each point of the photograph) above the line segment $\overline{AB}$ in the positive direction by the angle $\pi = 180°$ around the origin. This creates an image below the line segment $\overline{A'B'}$.



Images on websites or obtained with a mobile phone camera are called digital photos. Such images are matrices. Suppose that in the details of one such image we have discovered that it is an image of the dimensions 1200×640. Then that image is a 640×1200 real matrix whose elements are pixels. A pixel is the smallest graphic element composed of a single colour. If the image contains

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only black and white colour, then the only elements of the matrix are the numbers 0 and 1. These numbers determine the colour of each pixel: zero represents black and one represents white. Digital photos that have only two colours are called binary images.

If the selected image is a black and white image, then it is also a 640×1200 real matrix, but its elements are integers between 0 and 255. Zero is again black (the colour of the minimum intensity), 255 is white (the colour of the maximum intensity), and numbers $1 - 254$ represent shades of grey from the darkest represented by number 1 to the lightest represented by number 254.

Colour images are created by overlapping of (i.e., by adding) three matrices - red, green, and blue.

The elements, i.e., the pixels, of the red matrix are arranged triplets $(r, 0,0)$, where r is the integer between 0 and 255. $(0,0,0)$ is black, $(255,0,0)$ is red, and the triplets $(r, 0,0)$ when r growing from 1 to 254 represent shades of red from a darker colour to a lighter colour.

The elements of the green matrix are arranged triplets $(0, g, 0)$, where g is the integer between 0 and 255. $(0,0,0)$ is black, $(0,255,0)$ is green, and the triplets $(0, g, 0)$ when g growing from 1 to 254 represent shades of green from a darker colour to a lighter colour.

The elements of the blue matrix are arranged triplets $(0,0, b)$, where b is the integer between 0 and 255. $(0,0,0)$ is black, $(0,0,255)$ is blue, and the triplets $(0,0, b)$ when b growing from 1 to 254 represent shades of blue from a darker colour to a lighter colour.

This type of a colour system is called an RGB system. In the RGB system, it is possible to display

$256^3 = 2^{24} = 16777216$ different colours.

The white colour of this system is a triplet

$$(255,255,255) = (255,0,0) + (0,255,0) + (0,0,255).$$

The shades of grey show triples of the shape (a, a, a), where a is the integer between 1 and 254.

Digital image processing and matrix operations

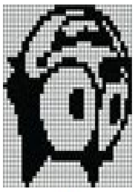
As digital photographs can be displayed using matrices, the question arises how operations on the elements of the matrix affect the corresponding photograph. This is shown in the following example.

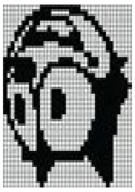
Example 2.48


A


B


C


D


E


F


G

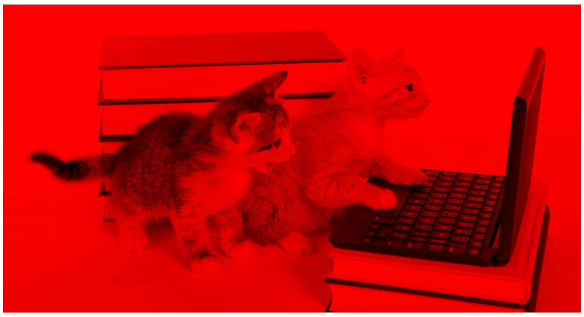

H

If $A = [a_{i,j}]$ is the given matrix (binary image) of the dimensions 35×35 then:

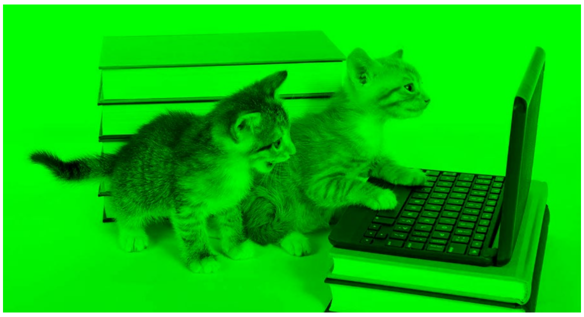
$$B = [b_{i,j}] = [a_{j,i}] = A^T, \qquad C = [c_{i,j}] = [a_{i,35-j+1}], \qquad D = [d_{i,j}] = [a_{35-i+1,j}],$$

$$E = [e_{i,j}] = [a_{35-i+1,35-j+1}], \qquad G = [g_{i,j}] = [a_{j,35-i+1}], \qquad H = [h_{i,j}] = [a_{35-j+1,i}].$$

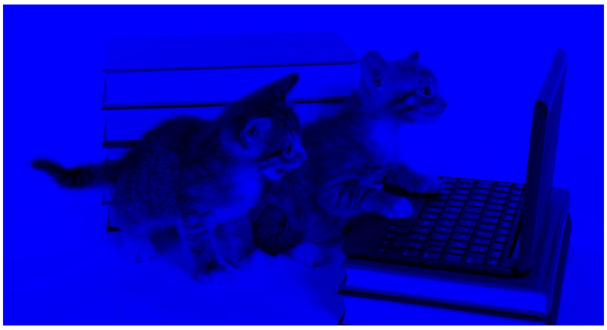
Example 2.49 Digital colour photography created by summing its red, green, and blue matrix



+



+



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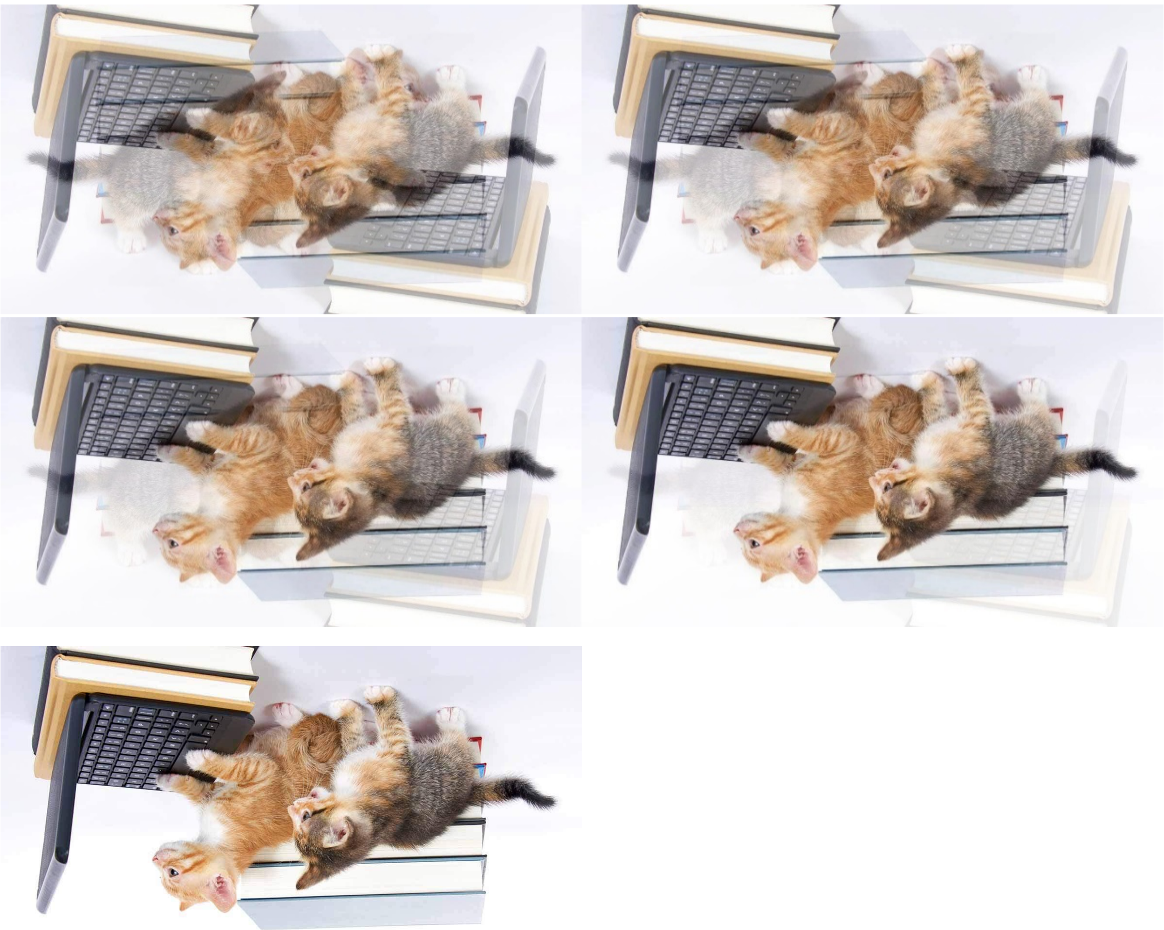
Example 5 The effect of switching from one image to another image often used in PowerPoint presentations, slides, and slide shows

Take two digital colour photos of the same dimensions. The first photograph is represented by the matrix A, and the second by the matrix Z in the RGB system. For each number $t \in [0,1]$ the matrix $M(t) = (1-t)A + tZ$ is defined.

Notice that $M(0) = A$ and $M(1) = Z$. The larger $t \in (0,1)$, the less the matrix $M(t)$ resembles the matrix A, and is more similar to the matrix Z.



Example 6 Application of the singular value decomposition of the matrix in image approximation

The real matrix U of order n is orthogonal if: $U^T U = I_n$.

If A is any $m \times n$ real matrix, it can be proven that A can be written in the form of a product of three matrices as follows: $A = USV^T$,

where U is the orthogonal matrix of order m, V is the orthogonal matrix of order n,

and $S = [s_{i,j}]$ $m \times n$ real matrix for whose elements is valid

$$s_{i,j} = 0 \text{ when } i \neq j$$
$$s_{1,1} \geq s_{2,2} \geq \cdots \geq s_{k,k} \geq 0 \text{ with a label } k = \min\{m, n\}.$$

Such notation of the matrix A is called *a singular decomposition* of that matrix.

It is briefly said and written:

USV^T is SVD (singular value decomposition) of the matrix A.

If the columns of the matrix U are in the order $u_1, u_2, \dots, u_m$ and the columns of the matrix V in the order $v_1, v_2, \dots, v_n$, it can easily be checked that the following is valid:

$$USV^T = \sum_{l=1}^{k} s_{l,l} u_l v_l^T .$$

Therefore,

$$A = \sum_{l=1}^{k} A_l ,$$

where $A_l = s_{l,l} u_l v_l^T$ is the $m \times n$ real matrix for each $l \in \{1, \dots, k\}$.

As $s_{1,1} \geq s_{2,2} \geq \cdots \geq s_{k,k}$, it is obvious that

$$\lim_{r \to k} \sum_{l=1}^{r} A_l = \sum_{l=1}^{k} A_l = A.$$

Accordingly, the closer r is to k, the matrix

$$C = \sum_{l=1}^{r} A_l$$

is better approximation of the matrix A.

Suppose that a space probe is programmed to send to a laboratory on Earth large amounts of black-and-white images of dimensions 1000×1000. For each such image, the probe would have to send $1000 \cdot 1000 = 1000000$ pixels (i.e., 1000000 numbers, one number for each pixel). However, before sending, the computer in the probe makes an SVD of each image, i.e., of the real matrix A. Then it takes the first 40 columns of the matrix U ($40 \cdot 1000 = 40000$ numbers), the first 40 columns of the matrix V ($40 \cdot 1000 = 40000$ numbers) and numbers $s_{1,1}, s_{2,2}, \dots, s_{40,40}$ (40 numbers), i.e., the total of $40000 + 40000 + 40 = 80040$ numbers, which are, instead of the elements of the matrix A, sent to Earth. So, instead of 1000000 numbers per image, the probe sends only 80040 numbers per image to Earth. On Earth, then, for each image, the matrix $\sum_{l=1}^{40} A_l$ is determined which is an approximation of the image, i.e., the matrix A. In that way, it is possible to obtain each image faster with a loss of quality.

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Figure 2.1 Black and white image of the dimensions 1000×1000 (k=1000)

Figure 2.2 Approximation of the same image (r=40)

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Figure 2.3 Image in colour of the dimensions 1200×640 (k=640)

Figure 2.4 Approximation of the same image (r=50)

Example 7 Removing interference (noise) from the image

Noise is defined as an unwanted random signal. Such a signal is mixed with the useful signal and affects its quality. Median filter is a technique used in digital processing to reduce the impact of noise in order to increase quality. Let us briefly describe what a median filter does. In all possible ways, 3 adjacent rows and 3 adjacent columns of the image (matrix A) are selected. The intersection of the selected rows and columns is the matrix B of order 3. The elements of the

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matrix B are arranged in an ascending order. The central element of the matrix B, i.e., the element located in the second row and the second column, is replaced by the fifth member (the so-called median) of the obtained sequence.

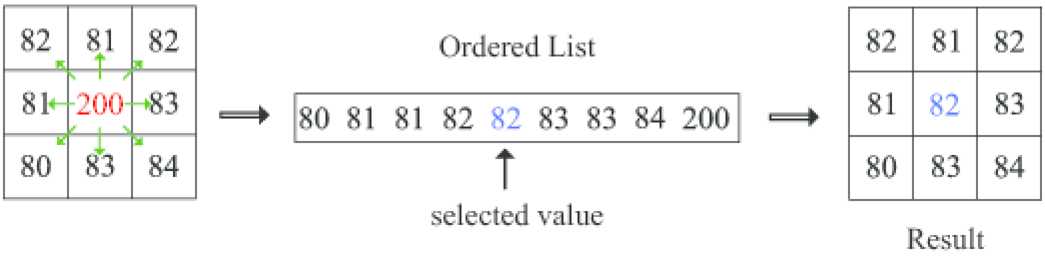

Figure 2.5 Black and white image with noise

Figure 2.6 The same image after the application of a median filter

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2.13. EXERCISES

Task 2.1

Determine the type of the default matrix and the required element:

1. $B=\begin{bmatrix}55 & 44\end{bmatrix};\ b_{12}$

2. $C=\begin{bmatrix}48 & 18\\ 6 & 0\\ -6 & 5\\ 18 & -15\end{bmatrix};\ c_{31}$

3. $A=\begin{bmatrix}v_{11} & v_{12} & v_{13} & \cdots & v_{1q}\\ v_{21} & v_{22} & v_{23} & \cdots & v_{2q}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ v_{p1} & v_{p2} & v_{p3} & \cdots & v_{pq}\end{bmatrix},\ a_{22}$

4. $E=\begin{bmatrix}d & d & d & d & d\end{bmatrix};\ e_{1r}\ \left(\text{za svaki } r\in\{1,2,\ldots,5\}\right)$

Task 2.2

Find the values of x, y, z and w from the following equation

$$\begin{bmatrix}x-y & x-z\\ y-w & w\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 6\end{bmatrix}.$$

Task 2.3

1. If matrices are $A=\begin{bmatrix}-1 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & 0 & -1\\ 5 & -1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}1 & x & w\\ r & z & 4\end{bmatrix}$. Determine the type of the matrix:

a) $A+B$,

b) $B-C$,

c) $A-B+C$,

d) $\frac{1}{2}B$,

e) $2A-B$,

f) $2A-4C$,

g) $3B^T$,

h) $2A^T-C^T$.

Task 2.4

$A=\begin{bmatrix}44.2 & 0 & 12.2\\ 1.5 & -2.35 & 5.6\end{bmatrix}$, $B=\begin{bmatrix}5.4 & 0\\ 1.4 & 7.8\\ 5.6 & 6.6\end{bmatrix}$ and $C=\begin{bmatrix}-10 & -20 & -30\\ 10 & 20 & 30\end{bmatrix}$. Calculate:

a) $A-C$,

b) $C-A$,

c) $1.1B$,

d) $-0.2B$,

e) $A^T+4.2B$,

f) $(A+2.3C)^T$,

g) $(2.1A-2.3C)^T$,

h) $(A-C)^T-B$.

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Task 2.5

Determine the matrix $F = AB$ if

a) $A = \begin{bmatrix} 0 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 4 & 2 \end{bmatrix}$,

b) $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

c) $A = \begin{bmatrix} 1.2 & 1.3 & 1.1 & 1.1 \\ 3.4 & 4.4 & 2.3 & 1.1 \\ 2.3 & 0 & -2.2 & 1.1 \\ 1.1 & 2.3 & 3.4 & -1.2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2.2 & 9.8 \\ -3.4 & -4.8 & -4.2 \\ 3.4 & 5.6 & 1 \\ -2.1 & 0 & -3.3 \end{bmatrix}$,

d) $A = \begin{bmatrix} 1 & -7 & 0 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & -2 & 1 & -1 \\ 1 & 1 & -7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}$.

Task 2.6

If the matrix A $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $P_4(x) = -x^4 + 3x^3 + 2x^2 - 2x$, determine $P_4(A)$.

Task 2.7

If the matrix B is $B = \begin{bmatrix} -4 & -4 & -1 \\ 3 & 3 & 1 \\ -4 & -4 & 0 \end{bmatrix}$. Calculate $Q_3(B)$ if $Q_3(x) = 4x^3 + 3x^2 - 2x - 1$.

Task 2.8 Calculate:

a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix}$, b) $\begin{vmatrix} 1 & 3 & -5 & -7 \\ 3 & -5 & -2 & -1 \\ 1 & 3 & 0 & -2 \\ 3 & -5 & 1 & 4 \end{vmatrix}$, c) $\begin{vmatrix} 1 & 3 & 1 & 6 \\ 1 & 2 & 0 & 7 \\ -1 & 3 & -15 & 18 \\ 2 & -1 & 5 & 6 \end{vmatrix}$,

d) $\begin{vmatrix} 0 & -c & b & -x \\ c & 0 & -a & -y \\ -b & a & 0 & -z \\ x & y & z & 0 \end{vmatrix}$.

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Task 2.9 $n \in \mathbb{R}$

a) $A=\begin{bmatrix}1 & 2 & 3 & \cdots & n\\ 0 & 1 & 2 & \cdots & n-1\\ 0 & 0 & 1 & \cdots & n-2\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$, b) $B=\begin{bmatrix}1 & n & n & \cdots & n\\ n & 2 & n & \cdots & n\\ n & n & 3 & \cdots & n\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ n & n & n & \cdots & n\end{bmatrix}$.

Task 2.10 Determine all $x \in \mathbb{R}$ for which

$$A=\begin{bmatrix}2 & 1 & 3 & 1\\ 2 & 2-x^2 & 3 & 1\\ 1 & 3 & 5 & 2\\ 1 & 3 & 9-x^2 & 2\end{bmatrix}$$

is singular.

Task 2.11 Find A^{-1} if

a) $A=\begin{bmatrix}5 & 9\\ 3 & 4\end{bmatrix}$, b) $A=\begin{bmatrix}-1 & -2\\ -3 & -4\end{bmatrix}$,

c) $A=\begin{bmatrix}5 & 1 & -2\\ -2 & 0 & 5\\ 7 & 2 & 8\end{bmatrix}$, d) $A=\begin{bmatrix}3 & 1 & 10\\ 2 & 5 & 4\\ -1 & -3 & 1\end{bmatrix}$,

e) $A=\begin{bmatrix}5 & -5 & 5 & 6\\ 2 & 1 & 1 & 2\\ -1 & -1 & 0 & 1\\ 5 & 1 & 2 & 1\end{bmatrix}$, f) $A=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$.

Task 2.12 Solve the matrix equation $(AXB)^{-1}=(X^{-1}+A)A^{-1}$

$A=\begin{bmatrix}2 & -3 & 1\\ 3 & -5 & -1\\ 3 & -4 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$.

Task 2.13

Solve the matrix equation $(XA+C)(AX+2AB)^{-1}=A^{-1}$ if

$A=\begin{bmatrix}-1 & 2 & 3\\ 0 & 3 & -1\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}$ i $C=\begin{bmatrix}1 & -1 & 2\\ -1 & 0 & 2\\ 0 & 0 & 2\end{bmatrix}$.

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Task 2.14 Solve the equation:

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{bmatrix} \cdot X \cdot \begin{bmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{bmatrix}.$$

Task 2.15 Solve the matrix equation $AX + 2B = C + BX$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & -3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Task 2.16 Solve the matrix equation $AXB^{-1} = I - A$,

$$A = \begin{bmatrix} 3 & 3 & 2 \\ -4 & 1 & -4 \\ -3 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Task 2.17 Solve the matrix equation $(A+3I)(X-I) = B, \quad I$

$$A = \begin{bmatrix} -2 & 5 & -2 \\ 2 & 8 & 0 \\ -1 & -5 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 21 & 1 \\ 2 & 50 & -2 \\ 1 & -22 & 0 \end{bmatrix}.$$

Task 2.18 Determine the rank of the following matrices:

a) $A = \begin{bmatrix} 0 & 0 & -3 & 8 & -2 \\ -3 & 6 & 12 & 16 & -34 \\ 1 & -2 & -9 & 8 & 8 \end{bmatrix}$,

b) $B = \begin{bmatrix} 3 & -7 & -1 & 3 & 2 \\ 1 & -6 & 1 & 7 & 7 \\ 9 & 2 & -11 & -1 & -7 \\ -5 & -4 & 7 & -7 & -3 \end{bmatrix}$,

c) $C = \begin{bmatrix} 1 & 1 & 1 & 0 & -2 & 0 \\ -2 & -1 & -1 & -1 & 1 & -1 \\ -3 & -2 & -2 & -1 & 3 & -1 \\ -8 & -2 & -2 & -6 & -2 & -6 \\ 3 & -1 & -1 & 4 & 6 & 4 \end{bmatrix}$,

d) $D = \begin{bmatrix} -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$.

Task 2.19 Solve the system using Cramer's rule and

$$\begin{aligned} 3x_1 + x_2 + 3x_3 &= 2 \\ -2x_1 + 2x_2 - 2x_3 &= 2 \\ 3x_2 + 3x_3 &= 2 \end{aligned}$$

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Task 2.20 Can the system

$$\begin{array}{rcrcr} 4x_1 & - & 3x_2 & = & 8 \\ 4x_1 & - & x_2 & = & 4 \\ 2x_1 & + & x_2 & = & -1 \\ & - & x_2 & = & 2 \end{array}$$

be solved using Cramer's rule?

Note: The rank is determined first. The result is $n = r(A) = r(\tilde{A}) = 2$ and a new system (equivalent to the default) with a system matrix C and an extended matrix system of the $\tilde{C}$ u in which $m = n = r(C) = r(\tilde{C}) = 2,$ this system can be solved using Cramer's rule.

Task 2.21 Calculate all system solutions

$$\begin{array}{rcrcrcrcrcr} 6x_1 & + & x_2 & + & 3x_3 & - & 4x_4 & - & 4x_5 & = & 3 \\ -2x_1 & + & x_2 & + & x_3 & - & 2x_4 & - & 2x_5 & = & 1 \\ 3x_1 & - & x_2 & + & x_3 & & & & & = & 0 \\ 3x_1 & & & - & x_3 & + & x_4 & + & x_5 & = & 0 \end{array}$$

Is vector $X^* = \frac{1}{10}\begin{bmatrix} 11 \\ 100 \\ 0 \\ 34 \\ 0 \end{bmatrix}$ solution to the system?

Task 2.22 How many solutions does the system have?

$$\begin{array}{rcrcrcrcrcr} 2x_1 & - & x_2 & - & x_3 & + & 2x_4 & + & x_5 & = & 1 \\ 6x_1 & + & x_2 & - & 5x_3 & + & 9x_4 & - & 3x_5 & = & -2 \\ 2x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & + & x_5 & = & 0 \\ & - & x_2 & + & 3x_3 & - & x_4 & + & 3x_5 & = & 2 \\ & - & 4x_2 & - & 8x_3 & - & 2x_4 & & & = & 1 \end{array} \quad ?$$

Task 2.23 Calculate all system solutions.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & - & 2x_5 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & - & x_5 & = & 1 \\ 3x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & - & 3x_5 & = & 1 \\ 4x_1 & + & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & 3 \\ 3x_1 & - & x_2 & - & x_3 & + & 4x_4 & + & 6x_5 & = & 4 \end{array}$$

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Task 2.24 For which $p \in \mathbb{R}$ is system

$$\begin{array}{rcrcrcr} -2x_1 & + & x_2 & - & 2x_3 & = & 3 \\ 2x_1 & + & 2x_2 & + & x_3 & = & -2 \\ 4x_1 & + & px_2 & + & 4x_3 & = & -7 \end{array}$$

impossible?

Is there $p \in \mathbb{R}$ for which a system has infinite solutions?

What is the value of $p \in$ □ for vector $X = \begin{bmatrix} 1/3 \\ -1/3 \\ -2 \end{bmatrix}$ to be the solution to that system?

Answers:

Task 2.1

1. B is type 1×2; $b_{12} = 44$.
2. C is type 4×2; $c_{31} = -6$.
3. A is type $p \times q$; $a_{22} = v_{22}$.
4. E is type 1×5; $e_{1r} = d$.

Task 2.2 $x = y = z = w = 6$.

Task 2.3

a) $A + B = \begin{bmatrix} 2 & 1 & -1 \\ 5 & -3 & 2 \end{bmatrix}$,

b) $B - C = \begin{bmatrix} 2 & -x & -1-w \\ 5-r & -1-z & -3 \end{bmatrix}$,

c) $A - B + C = \begin{bmatrix} -3 & 1+x & 1+w \\ -5+r & -1+z & 4 \end{bmatrix}$,

d) $\frac{1}{2}B = \begin{bmatrix} 1.5 & 0 & -0.5 \\ 2.5 & -0.5 & 0.5 \end{bmatrix}$,

e) $2A - B = \begin{bmatrix} -5 & 2 & 1 \\ -5 & -3 & 1 \end{bmatrix}$,

f) $2A - 4C = \begin{bmatrix} -6 & 2-4x & -4w \\ -4r & -4-4z & -14 \end{bmatrix}$,

g) $3B^T = \begin{bmatrix} 9 & 15 \\ 0 & -3 \\ -3 & 3 \end{bmatrix}$,

h) $2A^T - C^T = \begin{bmatrix} -3 & -r \\ 2-x & -4-z \\ -w & -2 \end{bmatrix}$.

Task 2.4

a) $A - C = \begin{bmatrix} 54.2 & 20 & 42.2 \\ -8.5 & -22.35 & -24.4 \end{bmatrix}$,

b) $C - A = \begin{bmatrix} -54.2 & -20 & -42.2 \\ 8.5 & 22.35 & 24.4 \end{bmatrix}$,

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c) $1.1B=\begin{bmatrix}5.94 & 0\\ 1.54 & 8.58\\ 6.16 & 7.26\end{bmatrix}$, d) $-0.2B=\begin{bmatrix}-1.08 & 0\\ -0.28 & -1.56\\ -1.12 & -1.32\end{bmatrix}$,

e) $A^T+4.2B=\begin{bmatrix}66.88 & 1.5\\ 5.88 & 30.41\\ 35.72 & 33.32\end{bmatrix}$, f) $\left(A+2.3C\right)^T=\begin{bmatrix}21.2 & 24.5\\ -46 & 43.65\\ -56.8 & 74.6\end{bmatrix}$,

g) $\left(2.1A-2.3C\right)^T=\begin{bmatrix}115.82 & -19.85\\ 46 & -50.94\\ 94.62 & -57.24\end{bmatrix}$, h) $\left(A-C\right)^T-B=\begin{bmatrix}48.8 & -8.5\\ 18.6 & -30.15\\ 36.6 & -31\end{bmatrix}$.

Task 2.5

a) $F=\begin{bmatrix}4 & 1\\ 7 & 4\end{bmatrix}$, b) F does not exist,

c) $F=\begin{bmatrix}-1.79 & 2.56 & 3.77\\ -6.05 & -0.76 & 13.51\\ -7.49 & -7.26 & 16.71\\ 7.36 & 10.42 & 8.48\end{bmatrix}$, d) $F=\begin{bmatrix}-21\\ 15\\ -3\\ -10\end{bmatrix}$.

Task 2.6 $P_4\left(A\right)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

Task 2.7 $Q_3\left(B\right)=\begin{bmatrix}-1 & 0 & 2\\ 1 & 0 & -2\\ 4 & 4 & -1\end{bmatrix}$.

Task 2.8 a) 0, b) -140, c) 40, d) $\left(ax+by+cz\right)^2$.

Task 2.9 <u>Instruction</u>: Apply the property of determinant 5).

a) $\det A=1$,

b)

$$\det B=\begin{vmatrix}1 & n & n & \cdots & n & n\\ n & 2 & n & \cdots & n & n\\ n & n & 3 & \cdots & n & n\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ n & n & n & \cdots & n-1 & n\\ n & n & n & \cdots & n & n\end{vmatrix}\begin{matrix}R_1-R_n\\ R_2-R_n\\ R_3-R_n\\ \vdots\\ R_{n-1}-R_n\\ \end{matrix}=\begin{vmatrix}1-n & 0 & 0 & \cdots & 0 & 0\\ 0 & 2-n & 0 & \cdots & 0 & 0\\ 0 & 0 & 3-n & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \left(n-1\right)-n & 0\\ n & n & n & \cdots & n & n\end{vmatrix}=$$

$$=\left(1-n\right)\left(2-n\right)\cdots\left(\left(n-1\right)-n\right)n=\left(-1\right)^{n-1}n!$$

Task 2.10 Matrix A is singular if $x \in \{-2,-1,1,2\}$.

Task 2.11

a) $A^{-1} = \frac{1}{7}\begin{bmatrix} -4 & 9 \\ 3 & -5 \end{bmatrix}$, b) $A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$,

c) $A^{-1} = \frac{1}{9}\begin{bmatrix} -10 & -12 & 5 \\ 51 & 54 & -21 \\ -4 & -3 & 2 \end{bmatrix}$, d) $A^{-1} = \frac{1}{35}\begin{bmatrix} 17 & -31 & -46 \\ -6 & 13 & 8 \\ -1 & 8 & 13 \end{bmatrix}$,

e) $A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & -8 & 26 & 14 \\ 2 & 8 & -19 & -9 \\ 10 & 16 & -63 & -29 \\ -2 & 0 & 15 & 5 \end{bmatrix}$, f) $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Task 2.11

$$\begin{aligned}
&(AXB)^{-1} = \left(X^{-1} + A\right)A^{-1} \\
&B^{-1}X^{-1}A^{-1} = \left(X^{-1} + A\right)A^{-1} / \cdot A \\
&B^{-1}X^{-1} = X^{-1} + A / \cdot X \\
&B^{-1} = I + AX \\
&A^{-1} \cdot / B^{-1} - I = AX \\
&A^{-1}\left(B^{-1} - I\right) = X;
\end{aligned}$$

$$A^{-1} = \begin{bmatrix} 29 & -11 & -8 \\ 18 & -7 & -5 \\ -3 & 1 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{33}\begin{bmatrix} 6 & -8 & 5 \\ 6 & 3 & -6 \\ 3 & 7 & 8 \end{bmatrix}$$

$$B^{-1} - I = \frac{1}{33}\begin{bmatrix} 6 & -8 & 5 \\ 6 & 3 & -6 \\ 3 & 7 & 8 \end{bmatrix} - \frac{1}{33}\begin{bmatrix} 33 & 0 & 0 \\ 0 & 33 & 0 \\ 0 & 0 & 33 \end{bmatrix} = \frac{1}{33}\begin{bmatrix} -27 & -8 & 5 \\ 6 & -30 & -6 \\ 3 & 7 & -25 \end{bmatrix}$$

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$$X = A^{-1}\left(B^{-1}-I\right) = \begin{bmatrix} 29 & -11 & -8 \\ 18 & -7 & -5 \\ -3 & 1 & 1 \end{bmatrix} \frac{1}{33} \begin{bmatrix} -27 & -8 & 5 \\ 6 & -30 & -6 \\ 3 & 7 & -25 \end{bmatrix} = \frac{1}{33} \begin{bmatrix} -873 & 42 & 411 \\ -543 & 31 & 257 \\ 90 & 1 & -46 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{291}{11} & \frac{14}{11} & \frac{137}{11} \\ -\frac{181}{11} & \frac{31}{33} & \frac{257}{33} \\ \frac{30}{11} & \frac{1}{33} & -\frac{46}{33} \end{bmatrix}.$$

Task 2.13

$$\left(XA+C\right)\left(AX+2AB\right)^{-1} = A^{-1} / \cdot \left(AX+2AB\right)$$

$$XA+C = A^{-1}\left(AX+2AB\right)$$

$$XA+C = X+2B$$

$$XA-X = 2B-C$$

$$X\left(A-I\right) = 2B-C / \cdot \left(A-I\right)^{-1}$$

$$X = \left(2B-C\right)\left(A-I\right)^{-1};$$

$$2B = \begin{bmatrix} 4 & -2 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad 2B-C = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A-I = \begin{bmatrix} -2 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(A-I\right)^{-1} = \frac{1}{-4} \begin{bmatrix} 2 & -2 & -8 \\ 0 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$X = \left(2B-C\right)\left(A-I\right)^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & 2 & 11 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -1.5 & 1 & 5.5 \\ -0.5 & -0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Task 2.14

$$X=\begin{bmatrix}2&-3&1\\4&-5&2\\5&-7&3\end{bmatrix}^{-1}\begin{bmatrix}2&0&-2\\18&12&9\\23&15&11\end{bmatrix}\begin{bmatrix}9&7&6\\1&1&2\\1&1&1\end{bmatrix}^{-1}=$$

$$=\begin{bmatrix}-1&2&-1\\-2&1&0\\-3&-1&2\end{bmatrix}\begin{bmatrix}2&0&-2\\18&12&9\\23&15&11\end{bmatrix}\frac{1}{-2}\begin{bmatrix}-1&-1&8\\1&3&-12\\0&-2&2\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2&-2&-2\\-2&-4&-6\\-4&-6&-2\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&3\\2&3&1\end{bmatrix}.$$

Task 2.15

$$AX+2B=C+BX$$
$$AX-BX=C-2B$$
$$\left(A-B\right)^{-1}\cdot\big/\left(A-B\right)X=C-2B$$
$$X=\left(A-B\right)^{-1}\left(C-2B\right);$$

$$A-B=\begin{bmatrix}2&0&6\\0&-2&2\\0&0&-1\end{bmatrix}\qquad\left(A-B\right)^{-1}=\frac{1}{4}\begin{bmatrix}2&0&12\\0&-2&-4\\0&0&-4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0&6\\0&-1&-2\\0&0&-2\end{bmatrix}$$

$$2B=\begin{bmatrix}-2&4&-6\\0&8&4\\0&0&4\end{bmatrix}\qquad C-2B=\begin{bmatrix}5&-4&7\\0&-6&0\\0&0&-3\end{bmatrix}$$

$$X=\left(A-B\right)^{-1}\left(C-2B\right)=\frac{1}{2}\begin{bmatrix}1&0&6\\0&-1&-2\\0&0&-2\end{bmatrix}\begin{bmatrix}5&-4&7\\0&-6&0\\0&0&-3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}5&-4&-11\\0&6&6\\0&0&6\end{bmatrix}=$$

$$=\begin{bmatrix}2.5&-2&-5.5\\0&3&3\\0&0&3\end{bmatrix}.$$

Task 2.16

$$AXB^{-1}=I-A\big/\cdot B$$
$$A^{-1}\cdot\big/AX=\left(I-A\right)B$$
$$X=A^{-1}\left(I-A\right)B$$
$$X=\left(A^{-1}-I\right)B;$$

$$A^{-1}=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}\qquad A^{-1}-I=\begin{bmatrix}0&11&-14\\0&-4&4\\-1&-12&14\end{bmatrix}$$

$$X=\left(A^{-1}-I\right)B=\begin{bmatrix}0 & 11 & -14\\ 0 & -4 & 4\\ -1 & -12 & 14\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 1 & -2 & 0\\ 0 & -1 & 1\end{bmatrix}=\begin{bmatrix}11 & -8 & -14\\ -4 & 4 & 4\\ -13 & 10 & 12\end{bmatrix}.$$

Task 2.17

$$\left(A+3I\right)^{-1}\cdot/\left(A+3I\right)\left(X-I\right)=B$$

$$X-I=\left(A+3I\right)^{-1}B$$

$$X=I+\left(A+3I\right)^{-1}B;$$

$$A+3I=\begin{bmatrix}1 & 5 & -2\\ 2 & 11 & 0\\ -1 & -5 & 1\end{bmatrix} \qquad \left(A+3I\right)^{-1}=\begin{bmatrix}-11 & -5 & -22\\ 2 & 1 & 4\\ -1 & 0 & -1\end{bmatrix}$$

$$X=I+\left(A+3I\right)^{-1}B=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}-11 & -5 & -22\\ 2 & 1 & 4\\ -1 & 0 & -1\end{bmatrix}\begin{bmatrix}-3 & 21 & 1\\ 2 & 50 & -2\\ 1 & -22 & 0\end{bmatrix}=\begin{bmatrix}2 & 3 & -1\\ 0 & 5 & 0\\ 2 & 1 & 0\end{bmatrix}.$$

Task 2.18 a) $R(A)=2,$ b) $R(B)=3,$ c) $R(C)=2,$ d) $R(D)=6.$

Task 2.19 $X=\dfrac{1}{12}\begin{bmatrix}10\\ 15\\ -7\end{bmatrix}.$

Task 2.20 Yes, it can. $X=\begin{bmatrix}0.5\\ -2\end{bmatrix}.$

Task 2.21 $X=\begin{bmatrix}1/7\\ u_1+u_2\\ u_1+u_2-3/7\\ u_1-6/7\\ u_2\end{bmatrix},\quad u_1,u_2\in\square$. The vector X^* is not the solution because it does not meet the 3rd and 4th equation of the system.

Task 2.22 The system has no solution (therefore, it is an impossible system) because

$$r\left(A\right)=3\neq 4=r\left(\tilde{A}\right).$$

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Task 2.23 $$X=\begin{bmatrix}\frac{2}{3}-u_3-u_2\\ u_2-u_1+3u_3\\ u_1\\ u_2\\ u_3+\frac{1}{3}\end{bmatrix},\quad u_1,u_2,u_3\in\mathbb{R}.$$

Task 2.24 The system is impossible for $p=-2$. There is no $p\in\mathbb{R}$ for which the system has infinite number of solutions.. Namely, for $p\neq-2$ is $m=n=r(A)=r(\tilde{A})=3$ the system has a unique solution.

Vector $X=\begin{bmatrix}1/3\\ -1/3\\ -2\end{bmatrix}$ is a solution to the system for $p=1$.

2.14. MATRIX - QUIZ

Circle the answer that you think the best complements the statement.

1 If the matrix A is $A=\begin{bmatrix} 2 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & -3 & 5 \\ -2 & 1 & 6 \end{bmatrix}$, then is $6A+7B=$

○ $\begin{bmatrix} 40 & 3 & 35 \\ -20 & 31 & 30 \end{bmatrix}$ ○ $\begin{bmatrix} -40 & -3 & 35 \\ -20 & 31 & 30 \end{bmatrix}$ ○ $\begin{bmatrix} 40 & -3 & 35 \\ -20 & 31 & 30 \end{bmatrix}$ ○ $\begin{bmatrix} 40 & -3 & 35 \\ 20 & 31 & 30 \end{bmatrix}$

2. $\begin{vmatrix} 3 & 4 & 5 \\ -1 & 0 & 7 \\ 2 & 4 & -3 \end{vmatrix} =$

● 60 ○ - 60 ○ -4 ○ 4

3. A matrix has 72 elements. The number of rows in this matrix cannot be:

○ 4 ○ 5 ○ 6 ○ 7

4. Use the Cramer's rule to solve the system

$$\begin{array}{rcrcr} 0.5x & + & 1.4y & = & 9 \\ 0.3x & - & 0.2y & = & 0.2 \end{array}$$

The solution of this system is a vector:

○ $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ ○ $\begin{bmatrix} -4 \\ -5 \end{bmatrix}$ ○ there is no solution ○ $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

5. If the matrix A is $A=\begin{bmatrix} 6 & 3 & 9 & 9 & -1 \\ 8 & -8 & 9 & 0 & 8 \\ -7 & -4 & -7 & 6 & 6 \\ 8 & 1 & 9 & -7 & 9 \end{bmatrix}$, then is $a_{32}=$

○ 9 ○ -7 ○ -4 ○ 6