9.1 COUNTING PRINCIPLES

9.1.1 Multiplication principle (the rule of product)

If a set A contains n elements, a set B contains m elements then there are $n \cdot m$ different ordered pairs (a, b) where a belongs to A and b belongs to B.

Example 9.1

How many two-digit even natural numbers are there?

Solution:

Two-digit even natural numbers can be represented as ordered pairs (a, b), where $a \in \{1,2,3,4,5,6,7,8,9\} = A$ and $b \in \{0,2,4,6,8\} = B$. The number of elements in the set A is $n = 9$ and the number of elements in the set B is $m = 5$. Thus, the total number of two-digit even natural numbers is

$$9 \cdot 5 = 45.$$

Note that, since ordered triples (a, b, c) can be represented as ordered pairs $\left((a, b), c\right)$, we can extend the multiplication principle to the case of three sets.

9.1.2 Multiplication principle (three and more sets)

If a set A contains n elements, a set B contains m elements and a set C contains p elements, then there are $n \cdot m \cdot p$ different ordered triples (a, b, c), where a belongs to A, b belongs to B and c belongs to C. The similar rule holds for four, five and more sets.

Example 9.2

How many three-digit odd natural numbers are there?

Solution:

Three-digit odd natural numbers can be represented as ordered triples (a, b, c), where $a \in \{1,2,3,4,5,6,7,8,9\} = A$, $b \in \{0,1,2,3,4,5,6,7,8,9\} = B$ and $c \in \{1,3,5,7,9\} = C$. The number of elements in the set A is $n = 9$, the number of elements in the set B is $m = 10$ and the number of elements in the set C is $p = 5$. Thus, the total number of three-digit odd natural numbers is

$$9 \cdot 10 \cdot 5 = 450.$$

9.1.3 Addition principle (the rule of sum)

If a set A contains n elements, a set B contains m elements and $A \cap B$ contains l elements, then $A \cup B$ contains $n + m - l$ elements.

Example 9.3

Alice bought at the grocery store: apples, plums, bananas, blueberries and oranges. Her husband John bought, in a different store, blackberries, gooseberries, grapes, plums, pears, grapefruits and bananas. How many different types of fruits will they eat today at home?

Solution:

We have

$n = 5,\ m = 7,\ l = 2.$

Hence, Alice and her husband have

$$m + n - l = 5 + 7 - 2 = 10$$

different fruits.

Definition: Permutations

A permutation is any arrangement of the elements of a finite set in a definite order.

The number of permutations

The number of all permutations of $n -$ elements set is

$$P_n = n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

where $n!$ is called the factorial of n.

Example 9.4

In how many ways can we arrange five different books on a shelf.

Solution:

We must calculate the number of all permutations of 5 – elements set. This number is equal to

$$P_5 = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

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Exercise 9.1

In how many ways can seven different ships dock in a harbour?

Solution:

5040.

9.1.4 Permutations of n elements taken k at time

The number of sequences of length k without repetitions whose elements are taken from $n-$ elements set $(k \leq n)$ at once is

$$P(n,k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

Sequences of length k without repetitions whose elements are taken from $n-$ elements set at once are often called *permutations of* n *elements taken* k *at once* or *variation without repetition* or $k-$ *permutations of* n.

Example 9.5

In how many ways can five people be seated in a row of eleven chairs.

Solution:

The number N of permutations of 11 elements taken 5 at once should be calculated. Thus,

$n = 11,\ k = 5$ and

$$N = P(11{,}5) = \frac{11!}{(11-5)!} = \frac{11!}{6!} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 55440.$$

Example 9.6

There are nine free places to dock in a harbour. In how many ways can six different ships dock there?

Solution:

The number N of permutations of 9 elements taken 6 at once should be calculated. Thus,

$n = 9,\ k = 6$ and

$$N = P(9{,}6) = \frac{9!}{(9-6)!} = \frac{9!}{3!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 60480.$$

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9.1.5 Permutations with repetition

The number of different sequences of length k that can be formed from $n-$ elements set, when repetitions are allowed is

$$n^k.$$

Different sequences of length k that can be formed from $n-$ elements set are often called *permutations with repetitions* or *variations with repetitions.*

In the following we will assume that any arrangement of letters forms a word.

Example 9.7

How many three-letter words can be arranged from the letters a, b, c, d?

Solution:

The number N of different sequences (words) of length 3 taken from $4-$ element set $\{a, b, c, d\}$, when repetitions are allowed, should be calculated. The phrase "repetitions are allowed" means that each letter can be taken more than once to form a word. Thus,

$$n = 4,\ k = 3 \qquad \text{and} \qquad N = 4^3 = 64.$$

Example 9.8

How many four-letter words can be arranged from the letters a, b, c?

Solution:

The number N of different sequences (words) of length 4 taken from $3-$ element set $\{a, b, c\}$, when repetitions are allowed, should be calculated. Thus,

$$n = 3,\ k = 4 \qquad \text{and} \qquad N = 3^4 = 81$$

Remark In some problems, the formula for the number of all permutations with repetitions, is not easy to apply. It is easier to use the multiplication principle.

Example 9.9

In how many ways can five ships from a country A dock in three different harbours in a country B?

Solution:

1st way:

- Let harbours in a country B be denoted by h1, h2, h3.
- Each ship has three possibilities to choose: to dock in h1, in h2 or in h3.
- Using the multiplication principle, we see that:
- two ships have $3 \cdot 3 = 9$ possibilities to choose,
- three ships have $3 \cdot 3 \cdot 3 = 27$ posibilities to choose,
- four ships have $3 \cdot 3 \cdot 3 \cdot 3 = 81$ posibilities to choose.
- Finally, five ships have

$$N = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243$$

possibilities to choose.

2nd way:

We have sequences of length $k = 5$ (five ships) that consist of numbers from $3-$ elements set $\{1,2,3\}$ (harbours) $(n = 3)$. Repetitions are allowed since two or more ships can choose the same harbour. Thus,

$$N = 3^5 = 243.$$

Definition: Combinations

An $k-$ element subset of a set with n elements $(k \leq n)$ is called a combination of n elements taken k at once.

9.1.6 The number of combinations

The number of all possibilities to choose a subset of k elements from a set of n elements (the order being irrelevant) is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

where $\binom{n}{k}$ is binomial coefficient.

Example 9.10

Anne, Beth, Charlie and Donald are trying to win two tickets to the Bahamas.

a) What are all the combinations of winners?
b) What is a number of all possibilities to choose winners?

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c) If Anne and Charlie are chosen, does it matter who got the first, and who got the second ticket?

Solution:

a) All the combinations of winners are:

$$\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{B,D\},\{C,D\},$$

where A stands for Anne, B stands for Beth etc.

b) The number of all possibilities to choose winners is six since it is the number of all combinations of winners. On the other hand, by using the formula we get the same number.

$$N = C(4,2) = \binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4!}{2!\,2!} = \frac{4\cdot 3}{2\cdot 1} = 6.$$

c) No, it does not matter since sets (subsets) have no order. We have $\{A,C\} = \{C,A\}$.

Remark

The following relations for binomial coefficients are useful in calculations:

- $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n(n-1)\cdots(k+1)}{(n-k)!}$
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$

Exercise 9.2

1. Compute $\binom{4000}{3998}$.

2. How many subsets of a set with four thousand elements have two elements?

Solution:

1. 7 998 000

2. 7 998 000

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Example 9.11

How many different committees of four members can be formed out of a club with ten members?

Solution:

The committee is an $4-$ element subset of an $10-$ element set of club members. Thus, the number N of different committees is equal to $C(10,4)$. We have

$$N = C(10,4) = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210.$$

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