9.4 COMPLETE SYSTEM OF EVENTS. BAYES' RULE

Definition: Independent events

Two events A and B are called *independent* (statistically independent) if

$$P(A \cap B) = P(A)P(B).$$

Remark

Physically independent experiments are assumed statistically independent, e.g., successive throws of a coin, a die, etc.

Example 9.17

Suppose that two balls are drawn with replacement (the first ball is replaced before the second is drawn) at random from a bag containing 4 red and 3 black balls. What is the probability that both balls selected are red?

Solution:

Let A be the event of drawing a red ball the first time, and B the event of drawing a red ball the second time. The sample space Ω consists of all ordered pairs (x, y), where both x and y denote elements of a set of 7 outcomes (4 red, 3 black).

The multiplication rule implies that Ω contains $7 \cdot 7 = 49$ elements. Since $A \cap B$ consists of the ordered pairs of the form (red, red), there is $4 \cdot 4 = 16$ elements in $A \cap B$. Therefore,

$$P(A \cap B) = \frac{16}{49} = \frac{4}{7} \cdot \frac{4}{7} = P(A)P(B).$$

Hence, events A and B are independent.

Example 9.18

Suppose that two balls are drawn without replacement (the first ball is not replaced before the second is drawn) at random from a bag containing 4 red and 3 black balls.

Any one of 7 balls may be selected on the first draw. One of only 6 balls may be chosen on the second draw (no replacement).

Therefore, there are $7 \cdot 6 = 42$ elements in Ω (ordered pairs without repetitions).

Let A be the event of drawing a red ball the first time, and B the event of drawing a red ball the second time.

Co-funded by the
Erasmus+ Programme
of the European Union

There is of course $4 \cdot 6 = 24$ elements of the set A and $6 \cdot 4 = 24$ elements of the set B. This gives

$$P(A) = P(B) = \frac{24}{42} = \frac{4}{7}.$$

We have $4 \cdot 3 = 12$ elements of $A \cap B$. Therefore, we have

$$P(A \cap B) = \frac{12}{42} = \frac{2}{7} \neq \frac{4}{7} \cdot \frac{4}{7} = P(A)P(B).$$

Hence, events A and B are dependent.

Remark

Since in Example 9.18 $P(A \cap B) = \frac{12}{42} = \frac{4}{7} \cdot \frac{3}{6}$ and $P(A) = \frac{4}{7}$ we can interpret the second factor $\frac{3}{6}$ as the probability that the second ball drawn is red under the condition that the first ball drawn was red. Denote this by $P(A|B)$.

The last remark suggests a general law

Conditional probability formula

Let $P(A)$ denote the probability of an event A, and $P(B|A)$ denote the probability of an event B after event A has occurred. If $P(A \cap B)$ is the probability that A and B occur, then

$$P(A \cap B) = P(A)P(B|A).$$

If $P(A) > 0$ we have *conditional probability formula*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

or if $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Remark

In the Example 9.18 $P(A|B) = \frac{3}{6} = \frac{1}{2}$ denotes the probability of an event A after event B has occurred. In the example the probability that the first ball drown was red under the condition that the second ball drawn is red.

$P(A|B)$ is called "the conditional probability of A given B", or "the probability of A under the condition B".

Remark:

If A and B are independent events, then $P(A|B) = P(A)$ if $P(B) > 0$.

Co-funded by the
Erasmus+ Programme
of the European Union

Exercise 9.3

Show that if A and B are independent events, then

A and $\bar{B}$ are independent events.

Solution:

$$\begin{aligned} P(A \cap \bar{B}) &= P\big(A \cap (\Omega \setminus B)\big) = P(A \setminus B) = P(A) - P(A \cap B) = P(A) - P(A)P(B) \\ &= P(A)\big(1 - P(B)\big) = P(A)P(\bar{B}) \end{aligned}$$

Exercise 9.4

Show that if A and B are independent events, then $\bar{A}$ and $\bar{B}$ are independent events.

Solution:

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= P\big((\Omega \setminus A) \cap (\Omega \setminus B)\big) = P\big(\Omega \setminus (A \cup B)\big) = 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B) \\ &= \big(1 - P(A)\big)\big(1 - P(B)\big) = P(\bar{A})P(\bar{B}). \end{aligned}$$

Bayes' rule

Suppose that $P(B) > 0.$ Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

We say that Ω is partitioned into cases $E_1, \dots, E_n$ if events $E_1, \dots, E_n$ are mutually disjoint sets and $\Omega = E_1 \cup \dots \cup E_n$.

Total Probability Formula

Suppose that the sample space Ω is partitioned into $E_1, \dots, E_n$ cases.

Then for any event A

$$P(A) = P(A|E_1)P(E_1) + \cdots + P(A|E_n)P(E_n).$$

Example 9.19

Three identical bowls are labelled $1, 2, 3$. First bowl contains 3 red and 3 blue marbles. Second bowl contains 4 red and 2 blue marbles. Third bowl contains 1 red and 5 blue. First, a bowl is randomly selected, and then a marble is randomly selected from the bowl.

What is the probability that a marble selected is blue?

Co-funded by the Erasmus+ Programme of the European Union

Solution:

E_i – an event that bowl number i is selected. We have $P(E_i) = \frac{1}{3}$.

A – an event that blue marble is selected. We have

$$P(A|E_1) = \frac{1}{2}, \qquad P(A|E_2) = \frac{1}{3}, \qquad P(A|E_3) = \frac{5}{6}$$

and

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + P(A|E_3)P(E_3) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{5}{6}\cdot\frac{1}{3} = \frac{5}{9}.$$

Given that the selected marble was blue, what is the probability that bowl 2 was selected?

Solution:

Using Bayes' rule, we get

$$P(E_2|A) = \frac{P(A|E_2)P(E_2)}{P(A)} = \frac{\frac{1}{3}\cdot\frac{1}{3}}{\frac{5}{9}} = \frac{1}{5}.$$

Example 9.20

A certain disease has an incidence rate of 1%. Suppose that for some diagnostic test the false negative rate is 10% and false positive rate is 2%. Compute the probability that a person, chosen at random from the population, who tests positive actually has the disease.

Solution:

Define the events for the person in question:
D: has the disease, T: test indicate disease (test is positive)
The complementary events are:
$\bar{D}$: has not the disease, $\bar{T}$: test does not indicate disease (test is negative).
Since disease has an incidence rate of 1%, we have

$$P(D) = 0.01$$

and for the complementary event

$$P(\bar{D}) = 1 - P(D) = 0.99.$$

Since the false negative rate is 10%, we have

$$P(\bar{T}|D) = 0.1.$$

Since the false positive rate is 2%, we have

$$P(T|D) = 0.02$$

We have to compute $P(D|T)$.

We use Bayes' rule

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)},$$

where

$$P(T|D) = 1 - P(\overline{T}|D) = 1 - 0.1 = 0.9.$$

By the total probability formula

$$P(T) = P(T|D)P(D) + P(T|\overline{D})P(\overline{D}) = 0.9 \cdot 0.01 + 0.02 \cdot 0.99 = 0.0288.$$

Finally,

$$P(D|T) = \frac{0.9 \cdot 0.01}{0.028} = 0.3125.$$

Only 31.25% of randomly selected people whose test is positive has in fact a disease.
The result seems surprising. The low incidence of the disease in the population is a critical variable here.

Exercise 9.5

In the above example compute the probability that a person, chosen at random from the population, who tests negative actually has not the disease.

Solution:

0.99897

Co-funded by the
Erasmus+ Programme
of the European Union