9.5 DISCRETE PROBABILITY DISTRIBUTIONS

Definition: random variable

A real valued function X defined on the sample space Ω is called a *random variable*.

Remark:

We write $X: \Omega \to \mathbb{R}$ and $X(\omega)$ is a value of X for the event ω in Ω.

Example 9.21

Suppose that we are tossing a coin. Then, sample space is $\Omega = \{H, T\}$, where H represents head and T represents tail. We define a random variable $X: \Omega \to \mathbb{R}$ by

$$X(H) = 1 \quad \text{and} \quad X(T) = 0.$$

Example 9.22

Suppose that we are rolling a dice. Then, sample space is $\Omega = \{1,2,3,4,5,6\}$, where the numbers represent the numbers on the dice. We define a random variable $X: \Omega \to \mathbb{R}$ by

$$X(1) = 1,\ X(2) = 2,\ X(3) = 3,\ X(4) = 4,\ X(5) = 5,\ X(6) = 6.$$

We simply say that X is a number of dots rolled.

Remark

If a coin, we are tossing in Example 9.21 is fair the probability that we get a head is $\frac{1}{2}$ and the probability that we get a tail is $\frac{1}{2}$. We write this:

$$P(H) = P(\{\omega \in \Omega: X(\omega) = 1\}) = \frac{1}{2}$$

and

$$P(T) = P(\{\omega \in \Omega: X(\omega) = 0\}) = \frac{1}{2}.$$

For simplicity we will write $P(X = x) = P(\{\omega \in \Omega: X(\omega) = x\})$.

Similarly, we understand the notation: $P(X < x),\ P(X > x),\ P(X \leq x),\ P(X \geq x)$.

Definition: cumulative distribution function

The cumulative distribution function (distribution function) of a random variable $\boldsymbol{X}$ is defined by $F(x) = P(X \leq x)$ for every real number x.

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Properties of the cumulative distribution function

1. For every real x: $0 \leq F(x) \leq 1$
2. $\lim_{x\to-\infty} F(x) = 0, \lim_{x\to\infty} F(x) = 1$
3. $\lim_{x\to a^+} F(x) = F(a)$
4. F is nondecreasing

Definition: discrete random variable

A random variable is called *discrete* if it takes on with positive probability either a finite or at most a countably infinite set of discrete values i.e., there exists a finite sequence $x_1, \dots, x_n$ such that

(1)

$$P(X = x_i) = p_i, \quad \sum_{i=1}^{n} p_i = 1, \quad p_1, \dots, p_n \text{ positive numbers}$$

or there exists an infinite sequence $x_1, x_2, \dots$ such that

(2)

$$P(X = x_i) = p_i, \quad \sum_{i=1}^{\infty} p_i = 1, \quad p_1, p_2, \dots \text{ positive numbers.}$$

Definition: probability distribution

The formula (1) or resp. (2) is called probability distribution of discrete random variable.

Moreover, (1) can be written in the form

$X = x_i$	x_1	x_2	…	x_n
$P\ (X = x_i)$	p_1	p_2	…	p_n

or simply

x_i	x_1	x_2	…	x_n
p_i	p_1	p_2	…	p_n

Remark: In Example 9.21 if the coin is fair, we have

x_i	0	1
p_i	0.5	0.5

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Definition: Expected value

The *expected value* (*expectation* or *mean value)* of a discrete random variable X is defined by

$$E(X) = \sum_i x_i P(X = x_i).$$

Linearity of the expected value

If $E(X) < \infty$ and $E(y) < \infty$, then for any reals a, b

$$E(aX + bY) = aE(X) + bE(Y).$$

Expected value of a function of X.

Let $Y = g(X)$, where g is a function of discrete random variable X. Then Y is a discrete random variable with expectation

$$E(Y) = \sum_i g(x_i) P(X = x_i).$$

Definition: Variance, standard deviation

The *variance* of a discrete random variable X is defined by

$$V(X) = E\big(X - E(X)\big) = E(X^2) - E(X)^2.$$

The *standard deviation* of X is defined by

$$\sigma(X) = \sqrt{V(X)}.$$

Example 9.23

A random variable X has the following distribution

$X = x_i$	-2	0	2	3
$P(X = x_i)$	0.2	0.4	0.1	0.3

Compute: $E(X)$, $E(X^2)$, $V(X)$ and $\sigma(X)$. Sketch the graph of a cumulative distribution function.

Solution:

$$E(X) = (-2) \cdot (0.2) + 0 \cdot (0.4) + 2 \cdot (0.1) + 3 \cdot (0.3) = 0.7$$
$$E(X^2) = (-2)^2 \cdot (0.2) + 0^2 \cdot (0.4) + 2^2 \cdot (0.1) + 3^2 \cdot (0.3) = 3.9$$

$V(X) = 3.9 - (0.7)^2 = 3.41$
$\sigma(X) = \sqrt{3.41} \approx 1.8466$

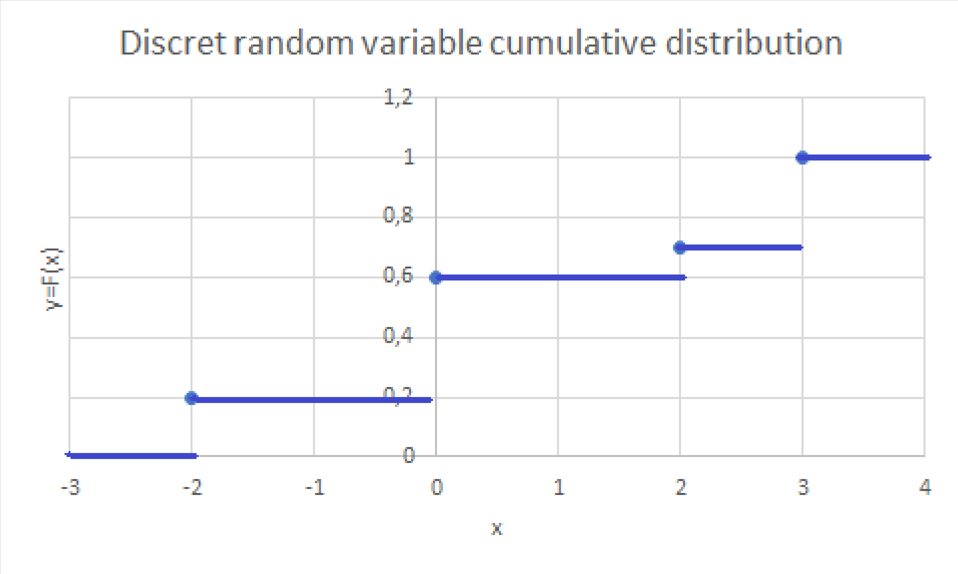

Figure 9.2. *Illustration to Example 9.23*

Remark

A random variable $Y = X^2$ in the above example has the following distribution:

$Y = y_i$	0	4	9
$P(Y = y_i)$	0.4	0.3	0.3

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = 0.4$$

$$P(Y = 4) = P(X^2 = 4) = P(X = 2 \text{ or } X = -2) = P(X = -2) + P(X = 2)$$
$$= 0.2 + 0.1 = 0.3.$$

Note that events $\{X = -2\}$ and $\{X = 2\}$ are disjoint, hence the probability of its sum is equal to the sum of its probabilities.

$$P(Y = 9) = P(X^2 = 9) = P(X = 3) = 0.3$$

Note that $P(X = -3) = 0.$

By using the distribution of $Y = X^2$ we can compute $E(X^2)$ the other way as follows,

$$E(X^2) = E(Y) = 0 \cdot 0.4 + 4 \cdot 0.3 + 9 \cdot 0.3 = 3.9.$$

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Example 9.24

Suppose that we are rolling two dice. We define a random variable X as the sum of the numbers obtained. Find the probability distribution of X. Compute expected value, variance and standard deviation of a random variable X.

Solution:

The sample space of the experiment is

$$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5)(6,6)\}$$

The number of elements of Ω is 36.

Using the natural definition of probability, we have:

$$P(X=2)=P(\{(1,1)\})=\frac{1}{36}$$

$$P(X=3)=P(\{(1,2),(2,1)\})=\frac{2}{36}=\frac{1}{18}$$
$$P(X=4)=P(\{(1,3),(2,2),(3,1)\})=\frac{3}{36}=\frac{1}{12}$$

$$P(X=5)=P(\{(1,4),(2,3),(3,2),(4,1)\})=\frac{4}{36}=\frac{1}{9}$$

$$P(X=6)=P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\})=\frac{5}{36}$$

$$P(X=7)=P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\})=\frac{6}{36}=\frac{1}{6}$$

$$P(X=8)=P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\})=\frac{5}{36}$$

$$P(X=9)=P(\{(3,6),(4,5),(5,4),(6,3)\})=\frac{4}{36}=\frac{1}{9}$$

$$P(X=10)=P(\{(4,6),(5,5),(6,4)\})=\frac{3}{36}=\frac{1}{12}$$

$$P(X=11)=P(\{(5,6),(6,5)\})=\frac{2}{36}=\frac{1}{18}$$

$$P(X=12)=P(\{(6,6)\})=\frac{1}{36}.$$

The probability distribution of X written in a table is:

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$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

The expected value $E(X)$ of X is computed as:

$$E(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} = 7$$

The variance of X is computed in two steps. First, we compute the expected value of X^2.

$$E(X^2) = 2^2 \cdot \frac{1}{36} + 3^3 \cdot \frac{1}{18} + 4^2 \cdot \frac{1}{12} + 5^2 \cdot \frac{1}{9} + 6^2 \cdot \frac{5}{36} + 7^2 \cdot \frac{1}{6} + 8^2 \cdot \frac{5}{36} + 9^2 \cdot \frac{1}{9} + 10^2 \cdot \frac{1}{12} + 11^2 \cdot \frac{1}{18} + 12^2 \cdot \frac{1}{36} = \frac{329}{6} \approx 54.83$$

Finally, we compute the variance,

$$V(X) = E(X^2) - E(X)^2 \approx 54.83 - 7^2 = 5.83.$$

The standard deviation is

$$\sigma(X) = \sqrt{V(X)} \approx 2.42.$$

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