9.6 BINOMIAL DISTRIBUTION AND POISSON DISTRIBUTION

Definition: Bernoulli experiment with n trials

The rules for a *Bernoulli experiment*:

1. The experiment is repeated a fixed number of times (n times).
2. Each trial has only two possible outcomes, ``success" and ``failure". The possible outcomes are the same for each trial.
3. The probability of success remains the same for each trial. We use p for the probability of success (on each trial) and $q = 1 - p$ for the probability of failure.
4. The trials are independent (the outcome of previous trials has no influence on the outcome of the next trial).

We are interested in the random variable X where X stands for the number of successes. Note the possible values of X are $0,1,2,\ldots,n$.

Our next goal is to calculate the probability distribution for the random variable X, where X counts the number of successes in a Bernoulli experiment with n trials.

Definition: binomial distribution

If X is the number of successes in a Bernoulli experiment with n independent trials, where the probability of success is p in each trial and the probability of failure is then $q = 1 - p$, then

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \ \ k = 0,1,2,\ldots,n.$$

Example 9.25

A basketball player takes 4 independent free throws. Every time, the probability of scoring a point is 0.6, and each point scored is recorded. Here each free throw is a trial and trials are assumed to be independent. Each trial has two outcomes: point (success) or no point (failure). The probability of success is $p = 0{,}6$ and the probability of failure is $q = 1 - p = 0{,}4$. We are interested in the variable X which counts the number of successes in 4 trials. This is an example of a Bernoulli experiment with $n = 4$ trials.

Solution:

$n = 4, \qquad p = 0.6, \qquad q = 0.4$

$$P(X = 0) = \binom{4}{0} \cdot (0.6)^0 \cdot (0.4)^4 = (0.4)^4 = 0.0256$$

$$P(X = 1) = \binom{4}{1} \cdot (0.6)^1 \cdot (0.4)^3 = 4 \cdot 0.6 \cdot (0.4)^3 = 0.1536$$

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$P(X=2)=\binom{4}{2}\cdot(0.6)^2\cdot(0.4)^2=6\cdot(0.6)^2\cdot(0.4)^2=0.3456$

$P(X=3)=\binom{4}{3}\cdot(0.6)^3\cdot(0.4)^1=4\cdot(0.6)^3\cdot 0.4=0.3456$

$P(X=4)=\binom{4}{4}\cdot(0.6)^4\cdot(0.4)^0=(0.6)^4=0.1296$

Finally

x_i	0	1	2	3	4
p_i	0.0256	0.1536	0.3456	0.3456	0.1296

Definition: Poisson distribution

The *Poisson distribution* with parameter $\lambda > 0$ is given by

$$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!},\qquad k=0,1,2,3,\ldots$$

The main property of Poisson distribution is that if X follows a Poisson distribution with parameter $\lambda > 0$ then $E(X)=V(X)=\lambda$.

Remark

The Poisson distribution is a probability distribution for the number of events that occur randomly and independently in a fixed interval of time (or space). If the mean number of events per interval is λ then the probability of observing k events in this interval is given by the Poisson distribution.

Example 9.26

Suppose that the average number of some events in a fixed time interval is 2. Find the probability that we will observe today five events in this time interval.

Solution:

The random variable X follows a Poisson distribution with mean 2, find $P(X=5)$. Since $\lambda=2$, we have

$$P(X=5)=e^{-2}\frac{2^5}{5!}\approx 0.03609.$$

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