9.7 CONTINUOUS PROBABILITY DISTRIBUTIONS

Definition: continuous random variable

We call a random variable X *continuous*, if there exists a continuous (almost everywhere) nonnegative function f such that for any real number a

$$P(X \leq a) = \int_{-\infty}^{a} f(x)dx.$$

The function f is called the *probability density function* (*density function*) of a random variable X.

Remark

If a given function is continuous except for a finite number of points then it is continuous almost everywhere.

If $\boldsymbol{f}$ is a probability density function, then

$$\int_{-\infty}^{\infty} f(t)dt = 1.$$

Remark:

The *cumulative distribution function* (distribution function) of a continuous random variable $\boldsymbol{X}$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

for every real number $\boldsymbol{x}$.

Cumulative distribution function of continuous variable and probability

1 $P(X \geq a) = P(X > a) = 1 - F(a)$
2 $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = F(b) - F(a)$
3 $P(X = a) = 0$

Example 9.27

Given is a probability density function of a random variable X,

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & elswhere. \end{cases}$$

Find the cumulative distribution function F of a random variable X.

Solution:

We will use the formula for the cumulative distribution function. We must consider three cases:

1. $x < 0$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0$

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2. $0 \leq x \leq 1$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 1\,dt = 0 + x = x$

3. $x > 1$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{1} 1dt + \int_{1}^{x} 0dt = 0 + 1 + 0 = 1$

Finally, a distribution function is given by

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1 \\ 1, & x \geq 1. \end{cases}$$

Its graph is given below.

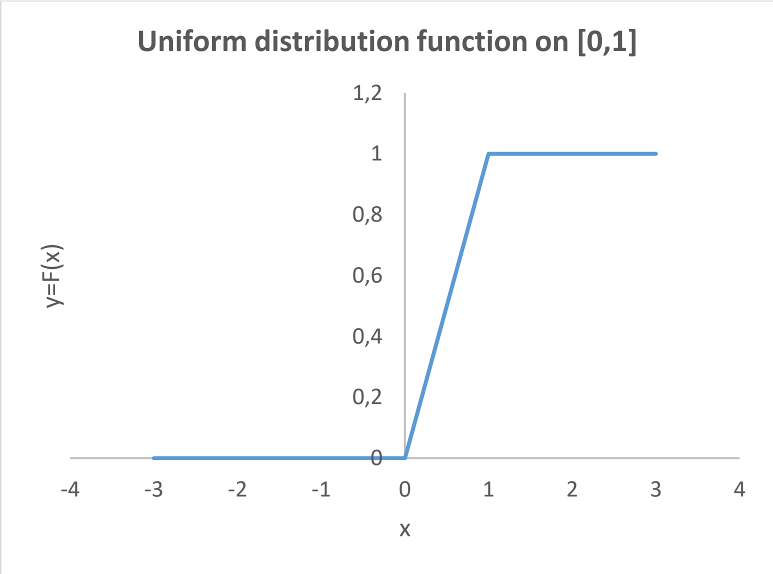

Figure 9.3. *Illustration for Example 9.27*

Definition: uniform random variable (continuous uniform distribution)

Uniform (rectangular) random variable X on the interval $[a, b]$ is a variable with the probability density

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}.$$

9.7.1 Cumulative distribution function of a uniform random variable

The cumulative distribution function of a uniform random variable $\boldsymbol{X}$ is given by

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{1}{b-a}, & a \leq x \leq b. \\ 1, & x > b \end{cases}$$

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Exercise 9.6

Find the cumulative distribution function of a uniform random variable X on the interval $[1,4]$.

Solution:

Since $a = 1,\ b = 4$, we have $b - a = 4 - 1 = 3$ and

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & x \in [1,4] \\ 1, & x > 4. \end{cases}$$

Definition: Expected value

The *expected value* (*expectation or mean value*) of a continuous random variable X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx,$$

where $\boldsymbol{f}$ is a probability density function of $\boldsymbol{X}$.

Similarly, as for discrete variables we have a linearity of $E(X)$.

If $E(X) < \infty$ and $E(y) < \infty$, then for any reals a, b

$$E(aX + bY) = aE(X) + bE(Y).$$

Expected value of a function of X

Let $Y = g(X)$, where g is a continuous function of a continuous random variable X with probability density function f. Then Y is a continuous random variable and it holds

$$E(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Similarly, as for discrete variables we define:

The *variance* of X

$$V(X) = E\big(X - E(X)\big) = E(X^2) - E(X)^2.$$

The *standard deviation* of X

$$\sigma(X) = \sqrt{V(X)}.$$

The standard deviation of X is also denoted by σ and variance by σ^2.

Example 9.28

Compute $E(X)$, $E(X^2)$, $V(X)$, $\sigma(X)$ for uniform random variable on the interval $[0,1]$.

Solution:

The density is given by

$f(x) = 1 \quad \text{for} \quad x \in [0,1] \quad \text{and} \quad f(x) = 0 \text{ elsewhere}.$

Thus, we have

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 xdx = \frac{1}{2}$$
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$
$$V(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$
$$\sigma(X) = \sqrt{\frac{1}{12}} \approx 0.2887.$$

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