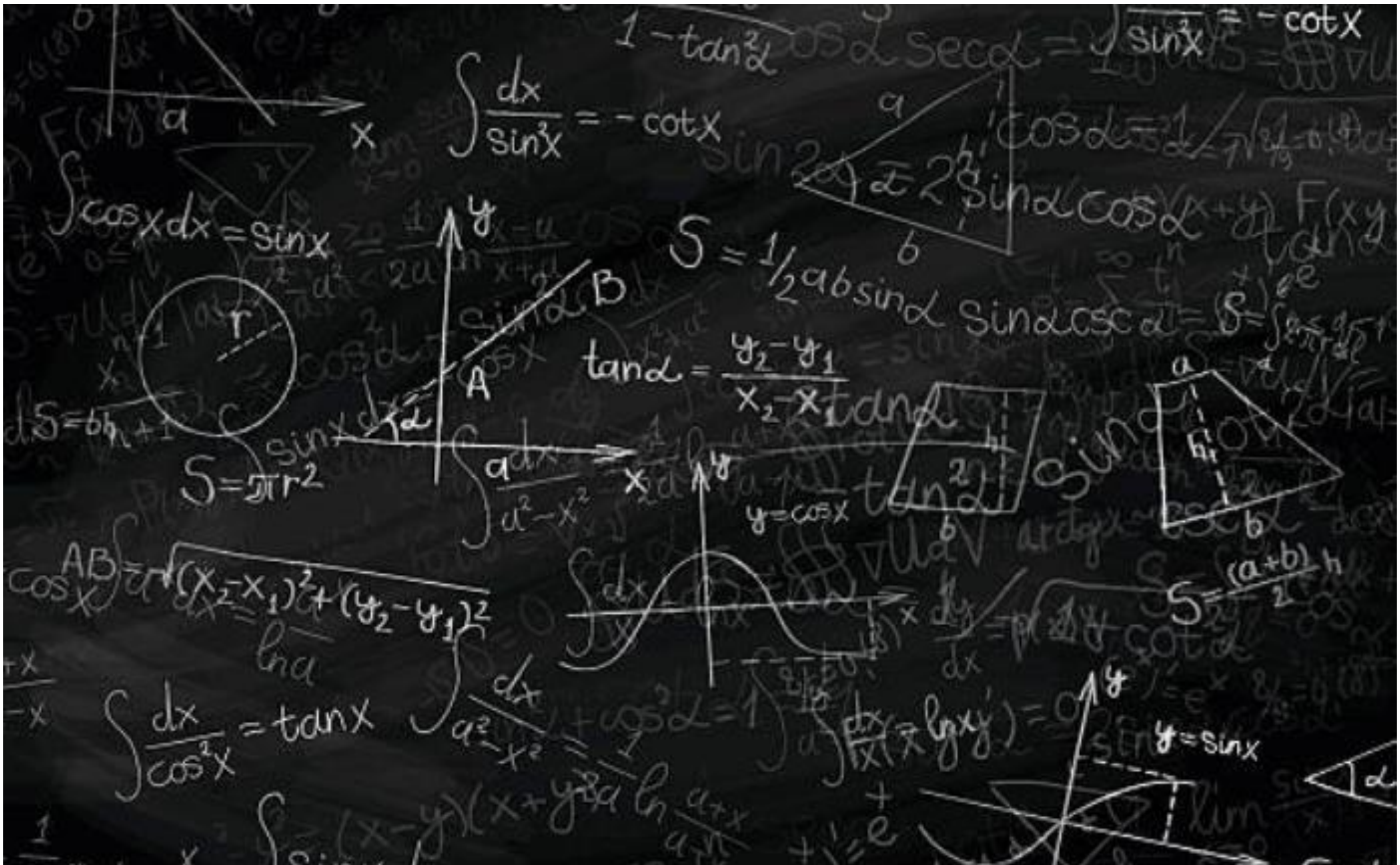
Innovative Approach in Mathematical Education for Maritime Students



Teacher's Manual

Curves

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MareMathics

Innovative Approach in Mathematical Education for Maritime Students

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https://marematics.pfst.hr/

Manual for teachers

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Reviewed by partners from Faculty of Maritime studies in Split

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Innovative Approach in Mathematical Education for Maritime Students

2019-1-HR01-KA203-061000

The Manual is the outcome of the collaborative work of all the Partners for the development of the MareMathics Project.

Partners in the project:

University of Split Faculty of Maritime Studies
CROATIA

Tallinn University of Technology, Estonian Maritime Academy
ESTONIA

Latvian Maritime Academy
LATVIA

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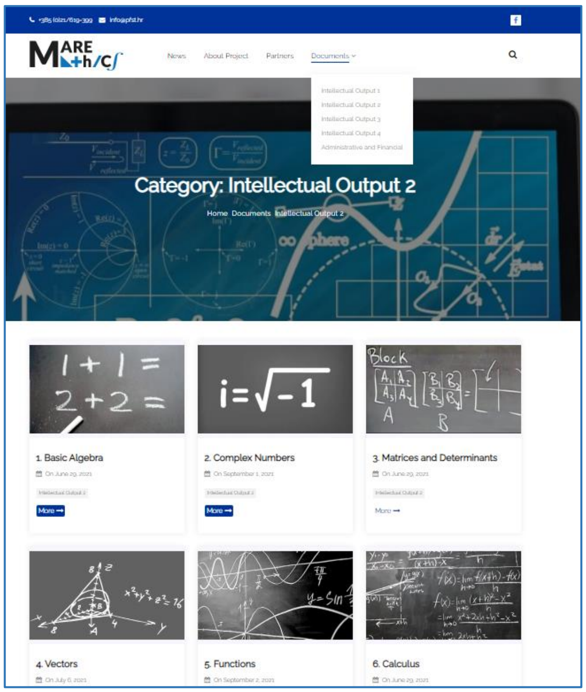
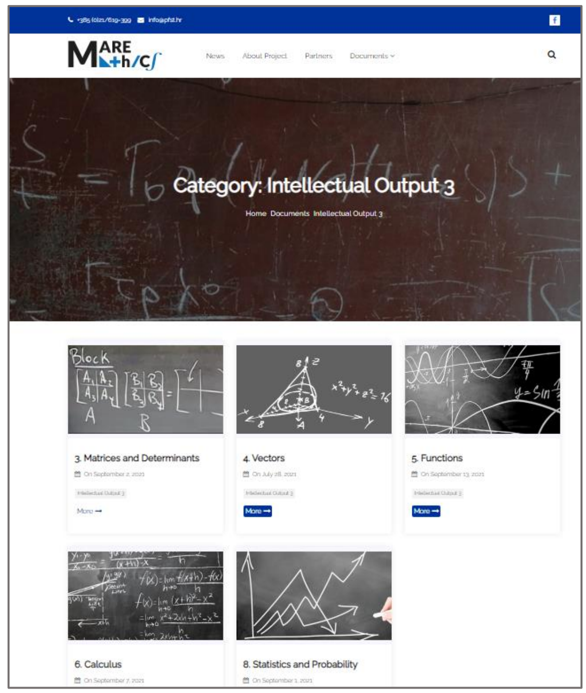
CONTENTS

CURVES: Teaching and Learning PLan

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The resources are picked from project ***MareMathics*** and available on the https://marematics.pfst.hr/.





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Lesson 1. The ellipse

Name of Unit	Workload	Handbook
The ellipse	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.11 Curves**

DETAILED DESCRIPTION

In this lesson, we study the symmetric oval-shaped curve known as the ellipse. We will use a geometric definition for an ellipse to derive its equation. Students will complete a reading assignment, answer discussion questions, complete an activity and take a quiz. The class activity can also be used as an online individual project.

AIM: At the end of this lesson students will be able to construct an ellipse and understand its graphical representation and write the equation of each when given appropriate information.

They will also be able to understand an ellipse as an extension of a circle. Understanding of the graphical representation and the extension from a circle to an ellipse will be evaluated by analyzing the answers students provide to the questions in each activity.

Learning Outcomes:

At the end of this lecture, each student should be able to

1. Graph ellipses centered at the origin.
2. Write equations of ellipses in standard form.
3. Graph ellipses not centered at the origin.
4. Solve applied problems involving ellipses.

Previous knowledge of mathematics:

- Properties of Segments (including definition, midpoint, perpendicular segments)
- Properties of Circles (including definition, radius, center)

Relatedness with solving problems in the maritime field:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in

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a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections.

Contents:

- **STANDARD FORM OF THE EQUATION OF AN ELLIPSE**
  - STANDARD FORMS OF THE EQUATIONS OF AN ELLIPSE
  - USING THE STANDARD FORM OF THE EQUATION OF AN ELLIPSE
- TRANSLATIONS OF ELLIPSES
- APPLICATIONS
- ECCENTRICITY
- PRACTICE EXERCISES

Assessment strategies:

Evaluating student’s activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation about ellipse
- Videos: Elipse and
  - Conic Section 3D Animation https://youtu.be/eTDaJ4ebK28
  - What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo
  - Intro to ellipses https://youtu.be/lvAYFUIEpFI (author: https://www.khanacademy.org/)
  - Ellipse standard equation from graph https://youtu.be/_JrQF8Rkaio (author: https://www.khanacademy.org/)
  - Ellipse graph from standard equation https://youtu.be/h5dIVNjVjXg (author: https://www.khanacademy.org/)
  - Foci of an ellipse from equation https://youtu.be/QR2vxfwiHAU (author: https://www.khanacademy.org/)
  - Eccentricity of an ellipse https://youtu.be/9hyQksn80ug
- GeoGebra https://www.geogebra.org/
- Worksheets
- Quiz: Ellipse
- Websites:
  - Ellipse , https://www.geogebra.org/m/d9Tz9khR
  - https://www.geogebra.org/m/qZ8aGDzR
  - Tim Brzezinski, Conic Sections, https://www.geogebra.org/m/D55ER2yN
  - Applications of Ellipses (pleacher.com) https://www.pleacher.com/mp/mlessons/calculus/appellip.html
  - Constructing an Ellipse https://www.geogebra.org/m/ZypVYe2n#material/sYNvM9bQ
  - Parabola (Locus)https://www.geogebra.org/m/nghvza2z

Lesson: The Ellipse

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use ellipses? Circles are all around you in everyday life, from tires on cars to buttons on coats, as well as on the tops of bowls, glasses, and water bottles. Ellipses are less common. One example is the orbits of planets, but you should be able to find the area of a circle or an ellipse, or the circumference of a circle, based on information given to you in a problem. Circles and ellipses are examples of conic sections, which are curves formed by the intersection of a plane with a cone.
10 min	Presentation Video: ellipse	Introduce the ellipse concept, definition of the ellipse	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	As the foci of an ellipse get closer together, what happens to the shape of the ellipse? What can we say about any two points lying on the same ellipse? In terms of axes of symmetry, how is an ellipse different from a circle? Is a circle actually an ellipse? Identifying the parts of the ellipse.
10 min	Presentation	Standard forms of the equation of an ellipse	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	How an equation depends upon the direction of the major axis, and the relationship between the distances from the center of the ellipse to each of the foci, major axis endpoint, and minor axis endpoint. Sketch the graph of an ellipse
20 min	Presentation Example 1,2,3	Using the standard form of the	Frontal and questioning Group work	Active listening and contributing to questions	Graphing an ellipse centered at the origin Determining the foci, vertices, sub-vertices, center, minor axis,

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		equation of an ellipse	Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Complete the worksheets	and major axis of an ellipse. Determining the equation of an ellipse by the given information.
15 min	Presentation Example 4	Translations of ellipses	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Equation of an ellipse under a vertical translation, is the equation of an ellipse under a vertical translation. Graph an ellipse with center not at the origin. Graphing an ellipse centered at (h, k). Converting the equation of an ellipse to standard form
10 min	Presentation Application examples: Example5	Application in engineering, maritime	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes.
15 min	Presentation Application examples: Example6	Eccentricity	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples		One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of eccentricity
5 min	Summary		Giving homework	Complete the quizzes View the video uploaded to OneDrive Complete the worksheets	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• Whiteboard • Lesson: https://maremathics.pfst.hr/wp-content/uploads/2022/04/IO2-5-Functions-11-1.pdf
Learning objectives	By the end of the lesson all students: • Graph ellipses centered at the origin • Write equations of ellipses in standard form • Graph ellipses not centered at the origin • Solve applied problems involving ellipses

A. The first section is a definition of the ellipse, define the standard form of the equation of an ellipse. Teacher presents and discusses with students' video: Ellipse and shows students that they can use GeoGebra to plot graphs of an ellipse. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
B. The second section is about horizontal and vertical translations of ellipses. Teacher presents and discusses with students' standard forms of equations of ellipses centered at (h, k) and shows students that they can use GeoGebra to plot graphs their graphs. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
C. The third section is a about some applications of ellipses. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
D. The fourth section is about measure ring the ovalness of an ellipse. Teacher presents and discusses with students a video4 and shows how to solve practice exercises. Teacher asks a student to solve the exercises
E. In the fifth section teacher introduces some practical applications. Teacher shows how to solve application exercises.

Students Activity

- Teacher asks students to exercise to be sure they understand how to do their task and how to use knowledge about the ellipse to solve real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.
- Students get homework – quizzes. They have to solve their tasks on QUIZIZZ platform.

APPENDIX 1: Exercises

Graph each ellipse and locate the foci.

1. $\frac{x^2}{16}+\frac{y^2}{4}=1;$
2. $\frac{x^2}{9}+\frac{y^2}{36}=1;$
3. $\frac{x^2}{25}+\frac{y^2}{64}=1;$
4. $\frac{x^2}{49}+\frac{y^2}{81}=1;$
5. $\frac{x^2}{\frac{9}{4}}+\frac{y^2}{\frac{25}{4}}=1;$
6. $x^2=1-4y^2;$
7. $25x^2+4y^2=100;$
8. $5x^2+16y^2=64;$
9. $7x^2=35-5y^2;$
10. $\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{25}{16}}=1.$

Find the standard form of the equation of each ellipse and give the location of its foci.

11. $.\,2x^2+9y^2+16x-90y+239=0;$
12. $5x^2+y^2-3x+40=0;$
13. $x^2+4y^2+10x-16y+25=0;$
14. $36x^2+4y^2-4y-44=0;$
15. $16x^2+100y^2+64x-300y-111=0;$
16. $2x^2+3y^2-4x-5y+1=0.$

Find the standard form of the equation of each ellipse satisfying the given conditions.

17. Foci: $(-5,0),(5,0)$; vertices: $(-8,0),(8,0)$;
18. Foci: $(-2,0),(2,0)$; vertices: $(-6,0),(6,0)$;
19. Foci: $(0,-4),(0,4)$; vertices: $(0,-7),(0,7)$;
20. Foci: $(0,-3),(0,3)$; y –intercepts: -3 and 3;
21. Foci: $(0,-2),(0,2)$; y –intercepts: -2 and 2;
22. Major axis horizontal with length 8; length of minor axis – 4; center: $(0,0)$;
23. Major axis horizontal with length 12; length of minor axis – 6; center: $(0,0)$;
24. Major axis vertical with length 10; length of minor axis – 4; center: $(-2,3)$;

25. Major axis vertical with length 20; length of minor axis – 10; center: $(2, -3)$;
26. Endpoints of major axis: $(7, 9)$ and $(7, 3)$; Endpoints of minor axis: $(5, 6)$ and $(9, 6)$;
27. Endpoints of major axis: $(2, 2)$ and $(8, 2)$; Endpoints of minor axis: $(5, 3)$ and $(5, 1)$.

Graph each ellipse and give the location of its foci

28. $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$;
29. $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$;
30. $(x+3)^2 + 4(y-2)^2 = 16$;
31. $(x-3)^2 + 9(y+2)^2 = 18$;
32. $\frac{(x-4)^2}{9} + \frac{(y+2)^2}{25} = 1$;
33. $\frac{(x-3)^2}{9} + \frac{(y+1)^2}{16} = 1$;
34. $\frac{x^2}{25} + \frac{(y-2)^2}{36} = 1$;
35. $\frac{(x-4)^2}{4} + \frac{y^2}{25} = 1$;
36. $9(x-1)^2 + 4(y+3)^2 = 36$;
37. $36(x+4)^2 + (y+3)^2 = 36$.

Convert each equation to standard form by completing the square on x and y. Then graph the ellipse and give the location of its foci.

38. $9x^2 + 25y^2 - 36x + 50y - 164 = 0$;
39. $4x^2 + 9y^2 - 32x + 36y + 64 = 0$;
40. $9x^2 + 16y^2 - 18x + 64y - 71 = 0$;
41. $x^2 + 4y^2 + 10x - 8y + 13 = 0$;
42. $4x^2 + y^2 + 16x - 6y - 39 = 0$;
43. $4x^2 + 25y^2 - 24x + 100y + 36 = 0$.

Find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

44. $\begin{cases} x^2 + y^2 = 1 \\ x^2 + 9y^2 = 9 \end{cases}$;
45. $\begin{cases} x^2 + y^2 = 25 \\ 25x^2 + y^2 = 25 \end{cases}$;
46. $\begin{cases} x^2 + y^2 = 25 \\ 25x^2 + y^2 = 25 \end{cases}$;
47. $\begin{cases} \frac{x^2}{25} + \frac{y^2}{9} = 1 \\ y = 3 \end{cases}$;
48. $\begin{cases} 4x^2 + y^2 = 4 \\ 2x - y = 2 \end{cases}$;
49. $\begin{cases} \frac{x^2}{4} + \frac{y^2}{36} = 1 \\ x = -2 \end{cases}$;
50. $\begin{cases} 4x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$.

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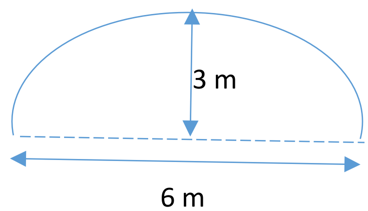
APPENDIX 2: Application exercises

51. Will a truck that is 2.4 m wide carrying a load that reaches 2.1 m feet above the ground clear the semielliptical arch on the one-way road that passes under the bridge shown in the figure?



52. A semielliptic archway has a height of 6 m and a width of 15 m. Can a truck 4 m high and 3 m wide drive under the archway without going into the other lane?
53. If an elliptical whispering room has a height of 6 m and a width of 30 m, where should two people stand if they would like to whisper back and forth and be heard?
54. A semielliptical arch over a tunnel for a one-way road through a mountain has a major axis of 15 m and a height at the center of 3 m.
    - Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.
    - Find an equation of the semielliptical arch over the tunnel.
    - You are driving a moving truck that has a width of 2.4 m and a height of 2,7 m. Will the moving truck clear the opening of the arch?
55. Halley's comet has an elliptical orbit, with the sun at one focus. The eccentricity of the orbit is approximately 0.967. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.)
    - Find an equation of the orbit. Place the center of the orbit at the origin, and place the major axis on the x −axis.
    - Use a graphing utility to graph the equation of the orbit.
    - Find the greatest (aphelion) and smallest (perihelion) distances from the sun's center to the comet's center.
56. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 50 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

APPENDIX 3: Homework

- Video: Ellipse
- Conic Section 3D Animation https://youtu.be/eTDaJ4ebK28
- What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo
- Intro to ellipses https://youtu.be/lvAYFUIEpFI (author: https://www.khanacademy.org/)
- Ellipse standard equation from graph https://youtu.be/_JrQF8Rkaio (author: https://www.khanacademy.org/)
- Ellipse graph from standard equation https://youtu.be/h5dIVNjVjXg (author: https://www.khanacademy.org/)
- Foci of an ellipse from equation https://youtu.be/QR2vxfwiHAU (author: https://www.khanacademy.org/)
- Eccentricity of an ellipse https://youtu.be/9hyQksn80ug

Quizzes: EllipseQuiz

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Lesson 2: The Hyperbola

Name of Unit	Workload	Handbook
The hyperbola	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.11 Curves**

DETAILED DESCRIPTION

In this lesson, we study the curve with two parts known as the hyperbola. We will use a geometric definition for a hyperbola to derive its equation. Students will complete a reading assignment, answer discussion questions, complete an activity and take a quiz. The class activity can also be used as an online individual project.

AIM: At the end of this lesson students will be able to construct a hyperbola and understand its graphical representation and write the equation of each when given appropriate information.

Learning Outcomes:

At the end of this lecture, each student should be able to

1. Graph a hyperbola centered at the origin.
2. Write equations of hyperbolas in standard form.
3. Graph hyperbolas not centered at the origin.
4. Solve applied problems involving hyperbolas.

Previous knowledge of mathematics:

- An ellipse

Relatedness with solving problems in the maritime field:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections.

Contents:

- THE HYPERBOLA
  - STANDARD FORM OF THE EQUATION OF A HYPERBOLA
  - STANDARD FORMS OF THE EQUATIONS OF A HYPERBOLA

- USING THE STANDARD FORM OF THE EQUATION OF A HYPERBOLA
- THE ASYMPTOTES OF A HYPERBOLA
- GRAPHING HYPERBOLAS CENTERED AT THE ORIGIN
- TRANSLATIONS OF HYPERBOLAS
- APPLICATIONS
- PRACTICE EXERCISES
    - APPLICATION EXERCISES

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation about hyperbola
- Videos: Hyperbola and
    - Conic Section 3D Animation - https://youtu.be/eTDaJ4ebK28
    - What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo&t=31s
    - Conic Sections Animated Gifs (mathwarehouse.com) https://www.mathwarehouse.com/animated-gifs/conic-sections.php
    - Easy Steps to Draw A Hyperbola using Focus Directrix Method https://youtu.be/dcaGNfplUbU
    - Intro to hyperbolas, https://www.youtube.com/watch?v=pzSyOTkAsY4 (author:https://www.khanacademy.org/)
    - Vertices and direction of a hyperbola https://youtu.be/oO3nWnJppqg (author:https://www.khanacademy.org/)
    - Vertices & direction of a hyperbola https://youtu.be/hnVFThmLW5Q (author:https://www.khanacademy.org/)
    - Graphing hyperbolas https://youtu.be/hl58vTCqVIY (author: https://www.khanacademy.org/)
    - https://www.youtube.com/watch?v=8nPMlW5NZSo&t=31s
- GeoGebra https://www.geogebra.org/
- Worksheets
- Quiz: Hyperbola
- Websites:
    - Applications of Hyperbolas (pleacher.com) https://www.pleacher.com/mp/mlessons/calculus/apphyper.html
    - https://www.geogebra.org/m/qZ8aGDzR
    - Tim Brzezinski, Conic Sections, https://www.geogebra.org/m/D55ER2yN
    - Constructing a Hyperbola https://www.geogebra.org/m/ZypVYe2n#material/mQdjNJq8
    - Hyperbola https://www.geogebra.org/m/uzmpazse

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Lesson: The Hyperbola

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter / Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use hyperbolas? Hyperbolas may be seen in many sundials. Every day, the sun revolves in a circle on the celestial sphere, and its rays striking the point on a sundial traces out a cone of light. The intersection of this cone with the horizontal plane of the ground forms a conic section. The angle between the ground plane and the sunlight cone depends on where you are and the axial tilt of Earth, which changes seasonally. At most populated latitudes and at most times of the year, this conic section is a hyperbola. Trilateration is he a method of pinpointing an exact location, using its distances to a given points. This can also be characterized as the difference in arrival times of synchronized signals between the desired point and known points. These types of problems arise in navigation, mainly nautical. A ship can locate its position using the arrival times of signals from GPS transmitters. Alternatively, a homing beacon can be located by comparing the arrival times of its signals at two separate receiving stations. This can be used to track people, cell phones, internet signals, and many other things. In the case in which a ship, or another object to be located, only knows the difference in distances between itself and two known points, the curve of possible locations is a hyperbola. One way of defining a hyperbola is as precisely this: the curve of points such that the absolute value of the difference

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					between the distances to two focal points remains constant.
10 min	Presentation Video: Hyperbola	Introduce the hyperbola concept, definition of the hyperbola	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Locate a hyperbola's vertices and foci.
10 min	Presentation	Standard forms of the equation of a hyperbola	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Can we determine the type of hyperbola from its equation? Sketch the graph of a hyperbola, The effect of the form of the equation on the orientation of the hyperbola.
30 min	Presentation Example 1,2,3,4	Using the standard form of the equation of a hyperbola	Frontal and questioning Group work Use GeoGebra, Graph. Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Finding vertices and foci from a hyperbola's equation. Determining the equation of a hyperbola by the given information. The asymptotes of a hyperbola. The asymptotes of a hyperbola entreat the origin. Graphing hyperbolas centered at the origin.
20 min	Presentation Example 5,6	Translations of hyperbolas	Frontal and questioning Group work Use GeoGebra, Graph Explains tasks and supports. Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Equation of a hyperbola under a horizontal and vertical translation. Graph of a hyperbola with center not at the origin. Converting the equation of hyperbolas to standard form
10 min	Presentation Application examples: Example 7	Application in engineering, maritime	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Hyperbolas are frequently used as models of situations that occur in the fields of optics and acoustics, because light and sound waves striking a hyperbolic surface at a certain angle (toward one focus) are reflected in a specific direction (toward the other focus). We can write equations for situations that involve hyperbolic shapes, as long as we have enough information to determine values for a and b in the given equations for the hyperbolas.
5 min	Summary		Giving homework	Complete the quizzes. View the video Complete the worksheets	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson https://maremathics.pfst.hr/wp-content/uploads/2022/04/IO2-5-Functions-11-2.pdf
Learning objectives	By the end of the lesson all students: • Graph hyperbolas centered at the origin • Write equations of hyperbolas in standard form • Graph hyperbolas not centered at the origin • Solve applied problems involving hyperbolas

A. The first section is a definition of the hyperbola, define the standard form of the equation of a hyperbola. Teacher presents and discusses with students' video: Hyperbola and shows students that they can use GeoGebra to plot graphs of a hyperbola. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
B. The second section is about horizontal and vertical translations of hyperbolas. Teacher presents and discusses with students' standard forms of equations of hyperbolas centered at (h, k) and shows students that they can use GeoGebra to plot graphs their graphs. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
C. The third section is a about some applications of hyperbolas. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
D. In the fourth section teacher introduces some practical applications. Teacher shows how to solve application exercises.

<u>Students Activity</u>

- Teacher asks students to exercise to be sure they understand how to do their task and how to use knowledge about the hyperbola to solve real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.
- Students get homework – quizzes. They have to solve their tasks on QUIZIZZ platform.

APPENDIX 1: Exercises

Find the vertices and locate the foci of each hyperbola with the given equation.

1. $\frac{x^2}{4} - \frac{y^2}{1} = 1;$
2. $\frac{y^2}{4} - \frac{x^2}{1} = 1;$
3. $\frac{x^2}{1} - \frac{y^2}{4} = 1;$
4. $\frac{y^2}{1} - \frac{x^2}{4} = 1;$

Find the standard form of the equation of each hyperbola satisfying the given conditions.

5. Foci: $(0,-3)$, $(0,3)$; vertices: $(0,-1)$, $(0,1)$;
6. Foci: $(0,-6)$, $(0,6)$; vertices: $(0,-2)$, $(0,2)$;
7. Foci: $(-4,0)$, $(4,0)$; vertices: $(-3,0)$, $(3,0)$;
8. Foci: $(-7,0)$, $(7,0)$; vertices: $(-5,0)$, $(5,0)$;
9. Endpoints of transverse axis: $(0,-6)$, $(0,6)$; asymptote: $y = 2x$;
10. Endpoints of transverse axis: $(-4,0)$, $(4,0)$; asymptote: $y = 2x$;
11. Center: $(4,-2)$; Focus: $(7,-2)$; vertex: $(6,-2)$;
12. Center: $(-2,1)$; Focus: $(-2,6)$; vertex: $(-2,4)$.

Use vertices and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

13. $\frac{x^2}{9} - \frac{y^2}{25} = 1;$
14. $\frac{x^2}{100} - \frac{y^2}{64} = 1;$
15. $\frac{y^2}{16} - \frac{x^2}{36} = 1;$
16. $\frac{x^2}{16} - \frac{y^2}{25} = 1;$
17. $\frac{x^2}{144} - \frac{y^2}{81} = 1;$
18. $\frac{y^2}{25} - \frac{x^2}{64} = 1;$
19. $4y^2 - x^2 = 1;$
20. $9x^2 - 25y^2 = 36;$
21. $9y^2 - 25x^2 = 225;$
22. $9y^2 - x^2 = 1;$
23. $4x^2 - 25y^2 = 100;$
24. $16y^2 - 9x^2 = 144;$
25. $y = \pm\sqrt{x^2 - 3};$
26. $y = \pm\sqrt{x^2 - 2}.$

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Find the standard form of the equation of each hyperbola.

27. $9x^2 - 4y^2 - 18x + 8y - 31 = 0$
28. $16x^2 - 4y^2 + 64x - 24y - 36 = 0$
29. $y^2 - x^2 - 4y + 2x - 6 = 0$
30. $4y^2 - 16x^2 - 24y + 96x - 172 = 0$
31. $9y^2 - x^2 + 18y - 4x - 4 = 0$

Use the center, vertices, and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

32. $\frac{(x+4)^2}{9} - \frac{(y+3)^2}{16} = 1;$
33. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{25} = 1;$
34. $\frac{(x+3)^2}{25} - \frac{y^2}{16} = 1;$
35. $\frac{(x+2)^2}{9} - \frac{y^2}{25} = 1;$
36. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1;$
37. $\frac{(y-2)^2}{36} - \frac{(x+1)^2}{49} = 1;$
38. $(x-3)^2 - 4(y+3)^2 = 4;$
39. $(x-1)^2 - (y-2)^2 = 3;$
40. $(x+3)^2 - 9(y-4)^2 = 9;$
41. $(y-2)^2 - (x+3)^2 = 5.$

Convert each equation to standard form by completing the square on x and y**.** Then graph the hyperbola. Locate the foci and find the equations of the asymptotes.

42. $x^2 - y^2 - 2x - 4y - 4 = 0;$
43. $4x^2 - y^2 + 32x + 6y + 39 = 0;$
44. $16x^2 - y^2 + 64x - 2y + 67 = 0;$
45. $-4x^2 + 9y^2 + 24x - 18y - 63 = 0;$
46. $4x^2 - 9y^2 - 16x + 54y - 101 = 0;$
47. $4x^2 - 9y^2 + 8x - 18y - 6 = 0;$
48. $4x^2 - 25y^2 - 32x + 164 = 0;$
49. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

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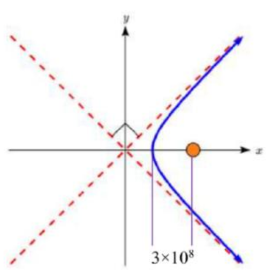
APPENDIX 2: Application exercises

50. Two microphones that are 1 mile apart record an explosion. Microphone M_1 received the sound 2 seconds before Microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.
51. Radio towers A and B, 200 kilometers apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 microseconds before the signal from A.
    a. Assuming that the radio signals travel 300 meters per microsecond, determine the equation of the hyperbola on which the ship is located.
    b. If the ship lies due north of tower B how far out at sea is it?
52. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250\,000$, where and are in yards. How far apart are the houses at their closest point?
53. Stations A and B are 100 kilometers apart and send a simultaneous radio signal to a ship. The signal from A arrives 0.0002 seconds before the signal from B. If the signal travels 300,000 kilometers per second, find an equation of the hyperbola on which the ship is positioned if the foci are located at A and B.
54. Anna and Julia are standing 3050 feet apart when they see a bolt of light strike the ground. Anna hears the thunder 0.5 seconds before Julia does. Sound travels at 1100 feet per second. Find an equation of the hyperbola on which the lighting strike is positioned if Anna and Julia are located at the foci.
55. A comet passes through the solar system following a hyperbolic trajectory with the sun as a focus. The closest it gets to the sun is 3×108 miles. The figure shows the trajectory of the comet, whose path of entry is at a right angle to its path of departure. Find an equation for the comet's trajectory. Round to two decimal places



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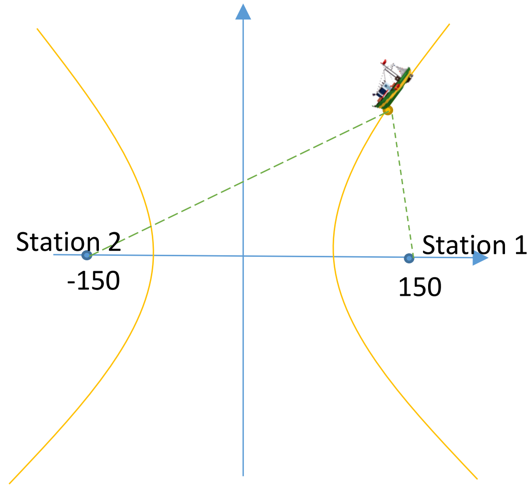
56. Write the standard form equation for the ship's location $P(x)$ in the diagram below. Assume that two stations, 300 miles apart, are positioned as pictured



APPENDIX 3: Homework

- Video1: hyperbola
- Intro to hyperbolas, https://www.youtube.com/watch?v=pzSyOTkAsY4 (author: https://www.khanacademy.org/)
- Vertices and direction of a hyperbola https://youtu.be/oO3nWnJppqg (author: https://www.khanacademy.org/)
- Vertices & direction of a hyperbola https://youtu.be/hnVFThmLW5Q (author: https://www.khanacademy.org/)
- Graphing hyperbolas https://youtu.be/hl58vTCqVIY (author: https://www.khanacademy.org/)
- What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo&t=31s

Quizzes: HyperbolaQuiz

Lesson 3: The Parabola

Name of Unit	Workload	Handbook
The parabola	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.11 Curves**

DETAILED DESCRIPTION

In the current lesson we will describe parabola and analyses the equation used to graph it. We will also discuss properties that make these curves so useful in so many different areas, from engineering to architecture to astronomy. We will use a geometric definition for a parabola to derive its equation. Students will complete a reading assignment, answer discussion questions, complete an activity and take a quiz. The class activity can also be used as an online individual project.

AIM: At the end of this lesson students will be able to construct a parabola and understand its graphical representation and write the equation of each when given appropriate information.

Learning Outcomes:

At the end of this lecture, each student should be able to

1. Graph a parabola centred at the origin.
2. Write equations of parabola in standard form.
3. Graph parabolas not centred at the origin.
4. Rewrite equations of parabola in standard form.
5. Solve applied problems involving parabolas.

Previous knowledge of mathematics:

A quadratic equations

Relatedness with solving problems in the maritime field:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections.

Contents:

- DEFINITION OF A PARABOLA
- STANDARD FORM OF THE EQUATION OF A PARABOLA
- STANDARD FORMS OF THE EQUATIONS OF A PARABOLA
- USING THE STANDARD FORM OF THE EQUATION OF A PARABOLA
- THE LATUS RECTUM AND GRAPHING PARABOLAS
- TRANSLATIONS OF PARABOLAS
- APPLICATIONS
- PRACTICE EXERCISES

Assessment strategies:

Evaluating students activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation about parabola
- Videos: Parabola and
  - Conic Section 3D Animation - https://youtu.be/eTDaJ4ebK28
  - What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo&t=31s
  - Conic Sections Animated Gifs (mathwarehouse.com) https://www.mathwarehouse.com/animated-gifs/conic-sections.php
  - Intro to parabolas, https://youtu.be/BGz3pkoGPag (author: https://www.khanacademy.org/)
  - Interpreting a parabola in context https://youtu.be/_Bk6XkV9O_0 (author: https://www.khanacademy.org/)
  - Intrepret a guadratic graph https://youtu.be/Tqxu53deWCo (author: https://www.khanacademy.org/)
  - Finding the vertex of a parabola https://youtu.be/IbI-l7mbKO4 (author: https://www.khanacademy.org/)
  - Graphing quadratics: standard form https://youtu.be/MQtsRYPx3v0 (author: https://www.khanacademy.org/)
  - Quadratic word problem: ball https://youtu.be/OZtqz_xw0SQ (author: https://www.khanacademy.org/)
  - Vertex and axis of symmetry of a parabola https://youtu.be/dfoXtodyiIA (author:https://www.khanacademy.org/)
  - Shifting parabolas https://youtu.be/ZmVOR6n_fzY (author:https://www.khanacademy.org/)
  - Finding The Focus and Directrix of a Parabola https://youtu.be/KYgmOTLbuqE
  - Equation for parabola from focus and directrix https://youtu.be/okXVhDMuGFg (author:https://www.khanacademy.org/)
- Worksheets
- Quiz: Parabola
- Websites:

- Applications of parabolas (pleacher.com)
  https://www.pleacher.com/mp/mlessons/calculus/appparab.html
- https://www.geogebra.org/m/qZ8aGDzR
- Tim Brzezinski, Conic Sections, https://www.geogebra.org/m/D55ER2yN
- Constructing a Parabola – GeoGebra
  https://www.geogebra.org/m/ZypVYe2n#material/Ey3kNJ3A
- Parabola (Locus)https://www.geogebra.org/m/nghvza2z
- Parabola https://www.geogebra.org/m/HFbfbs7z
- Parabolic car headlamp https://www.geogebra.org/m/BytSyufh

Lesson: The Parabola

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use parabolas? One well-known example is the parabolic reflector—a mirror or similar reflective device that concentrates light or other forms of electromagnetic radiation to a common focal point. Conversely, a parabolic reflector can collimate light from a point source at the focus into a parallel beam. This principle was applied to telescopes in the 17th century. Today, paraboloid reflectors are common throughout much of the world in microwave and satellite dish receiving and transmitting antennas. Paraboloids are also observed in the surface of a liquid confined to a container that is rotated around a central axis. In this case, liquid moves away from the center, and it “climbs” the walls of the container, forming a parabolic surface. This is the principle behind the liquid mirror telescope. Aircraft used to create a weightless state for purposes of experimentation, such as NASA’s “Vomit Comet,” follow a vertically parabolic trajectory for brief periods. This allows them to trace the course of an object in free fall. This can produce the same effect as zero gravity and lets the passengers on the aircraft experience the feeling of being in space.

10 min	Presentation Video: Parabola	Introduce the parabola concept, definition of the parabola	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Locate a parabola's directrix and focus.
10 min	Presentation	Standard forms of the equation of a parabola	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Can we determine the type of parabola from its equation? Sketch the graph of a parabola, Effect of the form of the equation on the orientation of the parabola.
30 min	Presentation Example 1, 2, 3	Using the standard form of the equation of a parabola	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Finding the focus and directrix of a parabola. Determining the equation of a parabola by the given information. The latus rectum of a parabola. Graphing parabolas centered at the origin.
20 min	Presentation Example 4, 5	Translations of parabolas	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Equation of a parabola under a horizontal and vertical translation. Graph of a parabola with center not at the origin. Converting the equation of a parabolas to standard form.
10 min	Presentation Application examples: Example 6	Application in engineering, maritime	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Parabolas have many applications. Cables hung between structures to form suspension bridges form parabolas. Arches constructed of steel and concrete, whose main purpose is strength, are usually parabolic in shape.
5 min	Summary		Giving homework	Complete the quizzes View the video uploaded to OneDrive Complete the worksheets	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard Lesson URL
Learning objectives	By the end of the lesson all students: • Graph parabolas centered at the origin • Write equations of parabolas in standard form • Graph parabolas not centered at the origin • Rewrite equations of parabolas in standard form • Solve applied problems involving parabolas

A. The first section is a definition of the parabola, define the standard form of the equation of a parabola. Teacher presents and discusses with students' video: Parabola and shows students that they can use GeoGebra to plot graphs of a parabola. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises

B. The second section is about horizontal and vertical translations of parabolas. Teacher presents and discusses with students' standard forms of equations of parabolas centered at (h, k) and shows students that they can use GeoGebra to plot graphs their graphs. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises

C. The third section is a about some applications of parabolas. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises

D. In the fourth section teacher introduces some practical applications. Teacher shows how to solve application exercises.

Students Activity

- Teacher asks students to exercise to be sure they understand how to do their task and how to use knowledge about the parabola to solve real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.
- Students get homework – quizzes. They have to solve their tasks on QUIZIZZ platform.

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APPENDIX 1: Exercises

Find the focus and directrix of each parabola with the given equation. $y^2 = 4x$;

1. $x^2 = 4y$;
2. $x^2 = -4y$;
3. $y^2 = -4x$.

Find the focus and directrix of the parabola with the given equation. Then graph the parabola.

4. $y^2 = 16x$;
5. $y^2 = -8x$;
6. $y^2 = 16x$;
7. $x^2 = -16y$;
8. $y^2 - 6x = 0$;
9. $8x^2 + 4y = 0$;
10. $y^2 = 4x$;
11. $y^2 = -16x$;
12. $x^2 = 8y$;
13. $x^2 = -20y$;
14. $x^2 - 6y = 0$;
15. $8y^2 + 4x = 0$.

Find the standard form of the equation of each parabola satisfying the given conditions.

16. Focus: $(7, 0)$; Directrix: $x = -7$;
17. Focus: $(9, 0)$; Directrix: $x = -9$;
18. Focus: $(-5, 0)$; Directrix: $x = 5$;
19. Focus: $(-10, 0)$; Directrix: $x = 10$;
20. Focus: $(0, 15)$; Directrix: $y = -15$;
21. Focus: $(0, 20)$; Directrix: $y = -20$;
22. Focus: $(0, -25)$; Directrix: $y = 25$;
23. Focus: $(0, -15)$; Directrix: $y = 15$;
24. Vertex: $(2, -3)$; Focus: $(2, -5)$;
25. Vertex: $(5, -2)$; Focus: $(7, -2)$;
26. Focus: $(3, 2)$; Directrix: $x = -1$;
27. Focus: $(2, 4)$; Directrix: $x = -4$;
28. Focus: $(-3, 4)$; Directrix: $y = 2$;

29. Focus: $(7,-1)$; Directrix: $y=-9$.

Find the vertex, focus, and directrix of each parabola with the given equation.

30. $(y-1)^2=4(x-1)$;
31. $(y-1)^2=-4(x-1)$;
32. $(x+1)^2=4(y+1)$;
33. $(x+1)^2=-4(y+1)$;

Find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

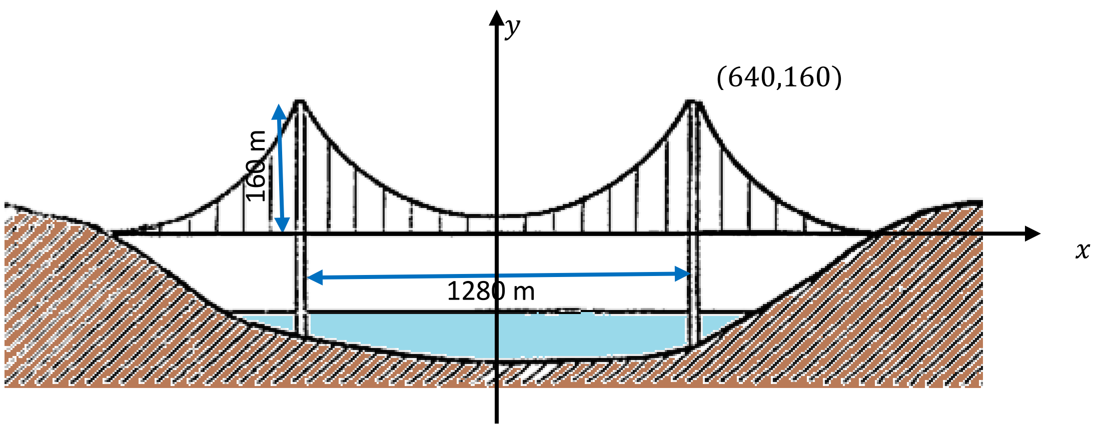
34. $(x-2)^2=8(y-1)$;
35. $(x+1)^2=-8(y+1)$;
36. $(y+3)^2=12(x+1)$;
37. $(y+1)^2=8x$;
38. $(x+2)^2=4(y+1)$;
39. $(x+2)^2=-8(y+2)$;
40. $(y+4)^2=12(x+2)$;
41. $(y-1)^2=-8x$.

Convert each equation to standard form by completing the square on x or y. Then find the vertex, focus, and directrix of the parabola. Finally, graph the parabola.

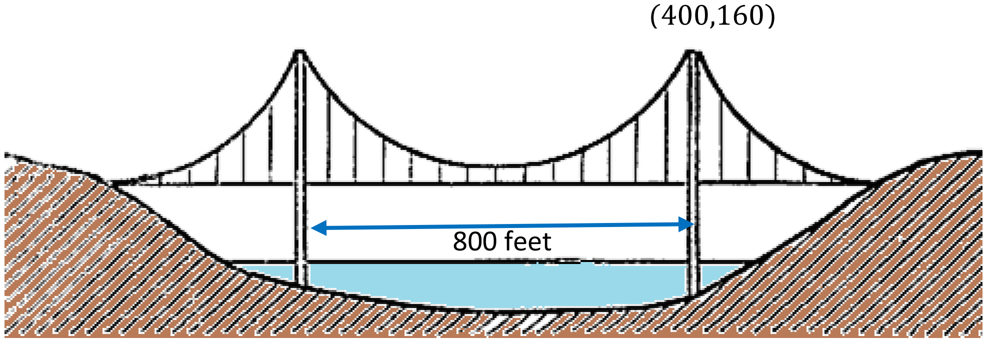
42. $x^2-2x-4y+9=0$;
43. $y^2-2y+12x-35=0$;
44. $x^2+6x-4y+1=0$;
45. $x^2+6x+8y+1=0$;
46. $y^2-2y-8x+1=0$;
47. $x^2+8x-4y+8=0$.

APPENDIX 2: Application exercises

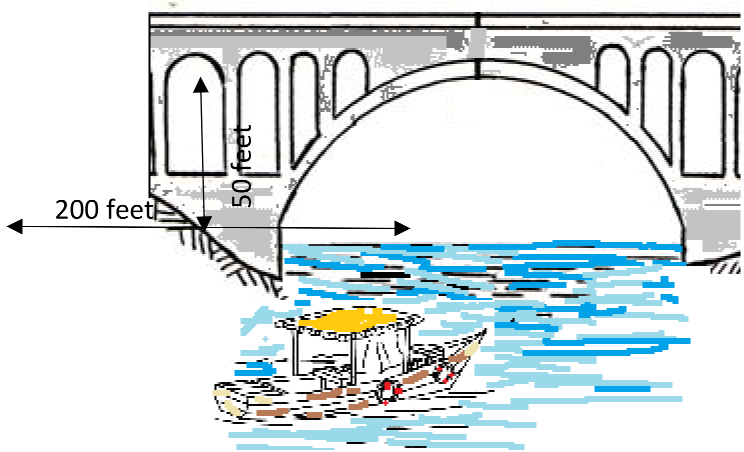
1. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 4 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
2. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 8 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
3. A satellite dish is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish has a diameter of 12 feet and a depth of 2 feet. How far from the base of the dish should the receiver be placed?
4. In Exercise 3, if the diameter of the dish is halved and the depth stays the same, how far from the base of the smaller dish should the receiver be placed?
5. The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 meters apart and rise 160 meters above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower? Round to the nearest meter.



6. The towers of a suspension bridge are 800 feet apart and rise 160 feet above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 feet from a tower?



7. The parabolic arch shown in the figure is 50 feet above the water at the center and 200 feet wide at the base. Will a boat that is 30 feet tall clear the arch 30 feet from the center?



8. A satellite dish in the shape of a parabolic surface has a diameter of 20 feet. If the receiver is to be placed 6 feet from the base, how deep should the dish be?
9. A domed ceiling is a parabolic surface. Ten meters down from the top of the dome, the ceiling is 15 meters wide. For the best lighting on the floor, a light source should be placed at the focus of the parabolic surface. How far from the top of the dome, to the nearest tenth of a meter, should the light source be placed?

APPENDIX 3: Homework

- Video: Parabola
- Intro to parabolas, https://youtu.be/BGz3pkoGPag (author: https://www.khanacademy.org/)
- Interpreting a parabola in context https://youtu.be/_Bk6XkV9O_0 (author: https://www.khanacademy.org/)
- Intrepret a guadratic graph https://youtu.be/Tqxu53deWCo (author: https://www.khanacademy.org/)
- Finding the vertex of a parabola https://youtu.be/IbI-l7mbKO4 (author: https://www.khanacademy.org/)
- Graphing quadratics: standard form https://youtu.be/MQtsRYPx3v0 (author: https://www.khanacademy.org/)
- Quadratic word problem: ball https://youtu.be/OZtqz_xw0SQ (author: https://www.khanacademy.org/)
- Vertex & axis of symmetry of a parabola https://youtu.be/dfoXtodyiIA (author:https://www.khanacademy.org/)
- Shifting parabolas https://youtu.be/ZmVOR6n_fzY (author:https://www.khanacademy.org/)
- What your teachers (probably) never told you about the parabola, hyperbola, and ellipse https://www.youtube.com/watch?v=8nPMlW5NZSo&t=31s

Quizzes: Parabola Quiz

Lesson 4: Rotation of axis

Name of Unit	Workload	Handbook
Rotation of axis	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.11 Curves**

DETAILED DESCRIPTION

As we have seen, conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a *cone*. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry. A parabola is formed by slicing the plane through the top or bottom of the double-cone, whereas a hyperbola is formed when the plane slices both the top and bottom of the cone. In previous lessons we have focused on the standard form equations for n conic sections. In this lesson, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order. Students will complete a reading assignment, answer discussion questions, complete an activity and take a quiz. The class activity can also be used as an online individual project.

AIM: At the end of this lesson students will be able to identify conics without completing the square and without rotating axes, use rotation of the axes formulas.

Learning Outcomes:

At the end of this lecture, each student should be able to

1. Identify conics without completing the square.
2. Use rotation of axes formulas.
3. Write equations of rotated conics in standard form.
4. Identify conics without rotating axes.

Previous knowledge of mathematics:

The ellipse, hyperbola and parabola

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Relatedness with solving problems in the maritime field:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections.

Contents:

- IDENTIFYING CONIC SECTIONS WITHOUT COMPLETING THE SQUARE
- ROTATION OF AXES
- USING ROTATIONS TO TRANSFORM EQUATIONS WITH XY- TERMS TO STANDARD EQUATIONS OF CONIC SECTIONS
- WRITING THE EQUATION OF A ROTATED CONIC IN STANDARD FORM
- IDENTIFYING CONIC SECTIONS WITHOUT ROTATING AXES

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define inverse trigonometric function
- Videos: Rotation of axes
  - Conic Section 3D Animation - https://youtu.be/eTDaJ4ebK28
  - Conic Sections -- Rotations https://youtu.be/wRun1LQYmuY
- Worksheets
- Quiz: ConicSections
- Websites:
  - Rotation of Axis https://slideplayer.com/slide/9310810/
  - Rotate a conic section https://www.geogebra.org/m/fMDeCr6g
  - Hyperbola rotation https://www.geogebra.org/m/kphmdkve
  - Parabols rotation https://www.geogebra.org/m/nqWEhpvJ
  - Ellipse rotation https://www.geogebra.org/m/WzheFdmv

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use rotation of the axes ?
15 min	Presentation Video: rotation of axes Example 1	Introduce the general form equation, which can be used for any conic	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Identifying conic sections without completing the square
20 min	Presentation Video: rotation of axes Example 2	Rotation of axes	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Rotation of axes forrmulas
30 min	Presentation Example 3,4	Using rotations to transform equations with $xy-$ terms to standard equations of conic sections	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Amount of Rotation Formula Writing the equation of a rotated conic in standard form Graphing the equation of a rotated conic
15 min	Presentation Example 5	Identifying conic sections without rotating axes	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	Identifying a conic section without rotating axes.
5 min	Summary		Giving homework	Complete the quizzes View the video uploaded to OneDrive Complete the worksheets	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Lesson: Rotation of Axes

RESOURCES	Whiteboard Lesson URL
Learning objectives	By the end of the lesson all students: Identify conics without completing the square. Use rotation of axes formulas. Write equations of rotated conics in standard form. Identify conics without rotating axes.

E. The first section introduce the general form equation, which can be used for any conic. Teacher presents and discusses with students video: Rotation of Axes and shows students that they can use GeoGebra for that needs. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
F. The second section is about using rotations to transform equations with $xy -$ terms to standard equations of conic sections Teacher presents and discusses with students how to use rotation equations. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
G. The third section is a about identifying conic sections without rotating axes. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises

Students Activity

- Teacher asks students to exercise to be sure they understand how to do their task and how to use knowledge about the rotation of axes to solve problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.
- Students get homework – quizzes. They have to solve their tasks on QUIZIZZ platform.

APPENDIX 1: Exercises

Identify each equation without completing the square

1. $y^2 - 4x + 2y + 21 = 0$;
2. $y^2 - 4x - 4y = 0$;
3. $4x^2 - 9y^2 - 8x - 36y - 68 = 0$;
4. $9x^2 + 25y^2 - 54x - 200y + 256 = 0$;
5. $4x^2 + 4y^2 + 12x + 4y + 1 = 0$;
6. $y^2 + 8x - 6y + 25 = 0$.

Write each equation in terms of a rotated $x'y'$ −system using θ the angle of rotation. Write the equation involving x' and y' in standard form

7. $xy = -1,\ \theta = 45°$;
8. $13x^2 - 10xy + 13y^2 - 72 = 0,\ \theta = 45°$;
9. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0,\ \theta = 30°$;
10. $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0,\ \theta = 60°$.

Write the appropriate rotation formulas so that in a rotated system the equation has no $x'y'$ −term

11. $x^2 + xy + y^2 - 10 = 0$;
12. $x^2 + 4xy + y^2 - 3 = 0$;
13. $3x^2 - 10xy + 3y^2 - 32 = 0$;
14. $5x^2 - 8xy + 5y^2 - 9 = 0$;
15. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$;
16. $x^2 + 4xy - 2y^2 - 1 = 0$;
17. $3xy - 4y^2 + 18 = 0$;
18. $34x^2 - 24xy + 41y^2 - 25 = 0$;
19. $6x^2 - 6xy + 14y^2 - 45 = 0$.

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Rewrite the equation in a rotated $x'y'-$ system without an $x'y'-$ term.

- Use the appropriate rotation formulas from Exercises 15–26.
- Express the equation involving x' and y' in the standard form of a conic section.
- Use the rotated system to graph the equation

20. $x^2 + xy + y^2 - 10 = 0;$
21. $x^2 + 4xy + y^2 - 3 = 0;$
22. $3x^2 - 10xy + 3y^2 - 32 = 0;$
23. $5x^2 - 8xy + 5y^2 - 9 = 0;$
24. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0;$
25. $x^2 + 4xy - 2y^2 - 1 = 0;$
26. $3xy - 4y^2 + 18 = 0;$
27. $34x^2 - 24xy + 41y^2 - 25 = 0;$
28. $6x^2 - 6xy + 14y^2 - 45 = 0.$

Identify each equation without applying a rotation of axes.

29. $5x^2 - 2xy + 5y^2 - 12 = 0;$
30. $10x^2 + 24xy + 17y^2 - 9 = 0;$
31. $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0;$
32. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0;$
33. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0;$
34. $4xy + 3y^2 + 4x + 6y - 1 = 0.$

In the next exercises

- If the graph of the equation is an ellipse, find the coordinates of the vertices on the minor axis.
- If the graph of the equation is a hyperbola, find the equations of the asymptotes.
- If the graph of the equation is a parabola, find the coordinates of the vertex.

Express answers relative to an $x'y'-$ system in which the given equation has no $x'y'-$ term. Assume that the $x'y'-$ system has the same origin as the $xy-$ system.

35. $5x^2 - 6xy + 5y^2 - 8 = 0;$
36. $2x^2 - 4xy + 5y^2 - 36 = 0;$
37. $x^2 - 4xy + 4y^2 + 5\sqrt{5}y - 10 = 0;$
38. $x^2 + 4xy - 2y^2 - 6 = 0.$

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APPENDIX 2: Homework

- Videos: Rotation of axes
- Conic Section 3D Animation - https://youtu.be/eTDaJ4ebK28
- Conic Sections -- Rotations https://youtu.be/wRun1LQYmuY
- ROTATION OF AXES HOMEWORK worksheet

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Lesson 5: The Parametric equations of conic sections

Name of Unit	Workload	Handbook
The Parametric equations of conic sections	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.11 Curves**

DETAILED DESCRIPTION

In the current lesson we are going to define x and y in terms of a third variable, t. There are so many things that change over time and are thus connected. For example, your height is a function of your age (time). How far and how long a soccer ball travels when kicked is also a function of time. And, the sun's position in the sky throughout the course of the day will determine if you need sunglasses.

Parametric equations are useful for drawing curves, as the equation can be integrated and differentiated term-wise. Equations can be converted between parametric equations and a single equation.

Students will complete a reading assignment, answer discussion questions, complete an activity and take a quiz. The class activity can also be used as an online individual project.

AIM: At the end of this lesson students will be able to use point plotting to graph plane curves described by parametric equations, eliminate the parameter, find parametric equations for functions, understand the advantages of parametric representations.

Learning Outcomes:

At the end of this lecture, each student should be able to

- Use point plotting to graph plane curves described by parametric equations;
- Eliminate the parameter;
- Find parametric equations for functions;
- Understand the advantages of parametric representations.

Previous knowledge of mathematics:

Conic sections

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Relatedness with solving problems in the maritime field:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections.

Contents:

- PARAMETRIC EQUATIONS
- PLANE CURVES AND PARAMETRIC EQUATIONS
- GRAPHING PLANE CURVES
- FINDING PARAMETRIC EQUATIONS
- ADVANTAGES OF PARAMETRIC EQUATIONS OVER RECTANGULAR EQUATIONS

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation about the parametric equations of the curves
- Videos: Parametric equation
- Worksheets
- Websites:
    - Parametric equations https://www.slideshare.net/usersshoulddie/lesson-15-polar-curves?next_slideshow=12210540
    - Calculus II - Parametric Equations and Curves (lamar.edu) https://tutorial.math.lamar.edu/classes/calcii/parametriceqn.aspx

Lesson 5: The Parametric equation of conic sections

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use parametric equations of the curves?
20 min	Presentation Video: Parametric equations Example 1, 2, 3	Introduce the parametric equations concept	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Plane curves and parametric equations. Graphing plane curves. Finding and graphing the rectangular equation of a curve defined parametrically. Eliminating the parameter.
15 min	Presentation Example 4	Finding parametric equations	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Can you start with any choice for the parametric equation for x? The answer is no. Advantages of parametric equations over rectangular equations
5 min	Summary		Giving homework	Complete the quizzes View the video uploaded to OneDrive Complete the worksheets	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** **Lesson: https://maremathics.pfst.hr/wp-content/uploads/2022/04/IO2-5-Functions-11-5.pdf**
Learning objectives	By the end of the lesson all students: • Use point plotting to graph plane curves described by parametric equations; • Eliminate the parameter; • Find parametric equations for functions; • Understand the advantages of parametric representations.

A. The first section is introduction of the parametric equations concept. Teacher presents and discusses with students' video: parametric equations, how to eliminate parameter and find and graph the rectangular equation of a curve defined parametrically. Teacher shows students that they can use GeoGebra to plot graphs. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises.

A. The second section is about how to find the parametric equation of the curves. Teacher presents and discusses with students' theme. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises.

B. In the third section teacher introduces some practical applications. Teacher shows how to solve exercises. Teacher asks a student to solve the exercises.

Students Activity

- Teacher asks students to exercise to be sure they understand how to do their task and how to use knowledge about the parametric equations to solve problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.

APPENDIX 1: Exercises

Parametric equations and a value for the parameter t are given. Find the coordinates of the point on the plane curve described by the parametric equations corresponding to the given value of t

1. $x = 3 - 5t,\ y = 4 + 2t,\ t = 1;$
2. $x = 7 - 4t,\ y = 5 + 6t,\ t = 1;$
3. $x = t^2 + 1,\ y = 5 - t^3,\ t = 2;$
4. $x = 2 + 3\cos t,\ y = 4 + 2\sin t,\ t = \pi;$
5. $x = 60t\cos 30°,\ y = 5 + 60t\cos 30° - 16t^2,\ t = 2;$
6. $x = 80t\cos 45°,\ y = 6 + 80t\cos 45° - 16t^2,\ t = 2.$

Use point plotting to graph the plane curve described by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t

7. $x = t + 2,\ y = t^2, -2 \le t \le 2;$
8. $x = t - 1,\ y = t^2, -2 \le t \le 2;$
9. $x = t - 2,\ y = 2t + 1, -2 \le t \le 3;$
10. $x = t - 3,\ y = 2t + 2, -2 \le t \le 2;$
11. $x = t + 1,\ y = \sqrt{t}, t \ge 0;$
12. $x = t^2 + 2,\ y = t^3 - 1, -\infty < t < \infty;$
13. $x = 2t,\ y = |t - 1|, -\infty < t < \infty;$
14. $x = |t + 1|,\ y = t - 2, -\infty < t < \infty.$

Eliminate the parameter t. Then use the rectangular equation to sketch the plane curve represented by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t (If an interval for t is not specified, assume that $-\infty < t < \infty$)

15. $x = t,\ y = 2t;$
16. $x = t,\ y = -2t;$
17. $x = 2t - 4,\ y = 4t^2;$
18. $x = t - 2,\ y = t^2;$
19. $x = \sqrt{t},\ y = t - 1;$
20. $x = \sqrt{t},\ y = t + 1;$
21. $x = 2\,sin\,t,\ y = 2\,cos\,t,\ 0 \le t < 2\pi;$

22. $x = 2\,sin\,t\,,\; y = 2\,cos\,t\,,\; 0 \le t < 2\pi$;
23. $x = 1 + 3\,cos\,t\,,\; y = 2 + 3\,sin\,t\,,\; 0 \le t < 2\pi$;
24. $x = -1 + 2\,cos\,t\,,\; y = 1 + 2\,sin\,t\,,\; 0 \le t < 2\pi$;
25. $x = 2\,cos\,t\,,\; y = 3\,sin\,t\,,\; 0 \le t < 2\pi$;
26. $x = 2^t,\; y = 2^{-t}, t \ge 0$;
27. $x = e^t,\; y = e^{-t}, t \ge 0$;
28. $x = \sqrt{t} + 2,\; y = \sqrt{t} - 2$.

Eliminate the parameter. Write the resulting equation in standard form.

29. A circle: $\mathrm{x = h + r\cos t\,,\; y = k + r\sin t\,}$;
30. An ellipse: $\mathrm{x = h + a\cos t\,,\; y = k + b\sin t\,}$;
31. A hyperbola: $\mathrm{x = h + a\sec t\,,\; y = k + b\tan t\,}$;

Find a set of parametric equations for the conic section or the line

32. Circle: Center: $(3, 5)$, Radius: 6;
33. Circle: Center: $(4, 6)$, Radius: 9
34. Ellipse: Center: $(-2, 3)$, Vertices: 5 units to the left and right of the center; Endpoints of Minor Axis: 2 units above and below the center;
35. Ellipse: Center: $(4, -1)$, Vertices: 5 units above and below the center; Endpoints of Minor Axis: 3 units to the left and right of the center;
36. Hyperbola: Vertices: $(4, 0)$ and $(-4, 0)$, Foci: $(6, 0)$ and $(-6, 0)$;
37. Hyperbola: Vertices: $(0, 4)$ and $(0, -4)$ Foci: $(0, 5)$ and $(0, -5)$;
38. Line: Passes through $(-2, 4)$ and $(1, 7)$;
39. Line: Passes through $(3, -1)$and $(9, 12)$.

Find two different sets of parametric equations for each rectangular equation

40. $y = 4x - 3$;
41. $y = x^2 + 4$;
42. $y = 2x - 5$;
43. $y = x^2 - 3$.

The parametric equations of four plane curves are given. Graph each plane curve and determine how they differ from each other.

44. a) $x = t,\ y = t^2 - 4$; b) $x = t^2,\ y = t^4 - 4$; c) $x = \cos t\,,\ y = \cos^2 t - 4$; d) $x = e^t,\ y = e^{2t} - 4$;

a) $x = t,\ y = \sqrt{4 - t^2},\ -2 \le t \le 2$; b) $x = \sqrt{4 - t^2},\ y = t\ , -2 \le t \le 2$; c) $x = 2\sin t\,,\ y = 2\cos t\,,\ 0 \le t < 2\pi$; d) $x = 2\cos t\,,\ y = 2\sin t\,,\ 0 \le t$

APPENDIX 2: Homework

- Video: Parametric equation
- Worksheet