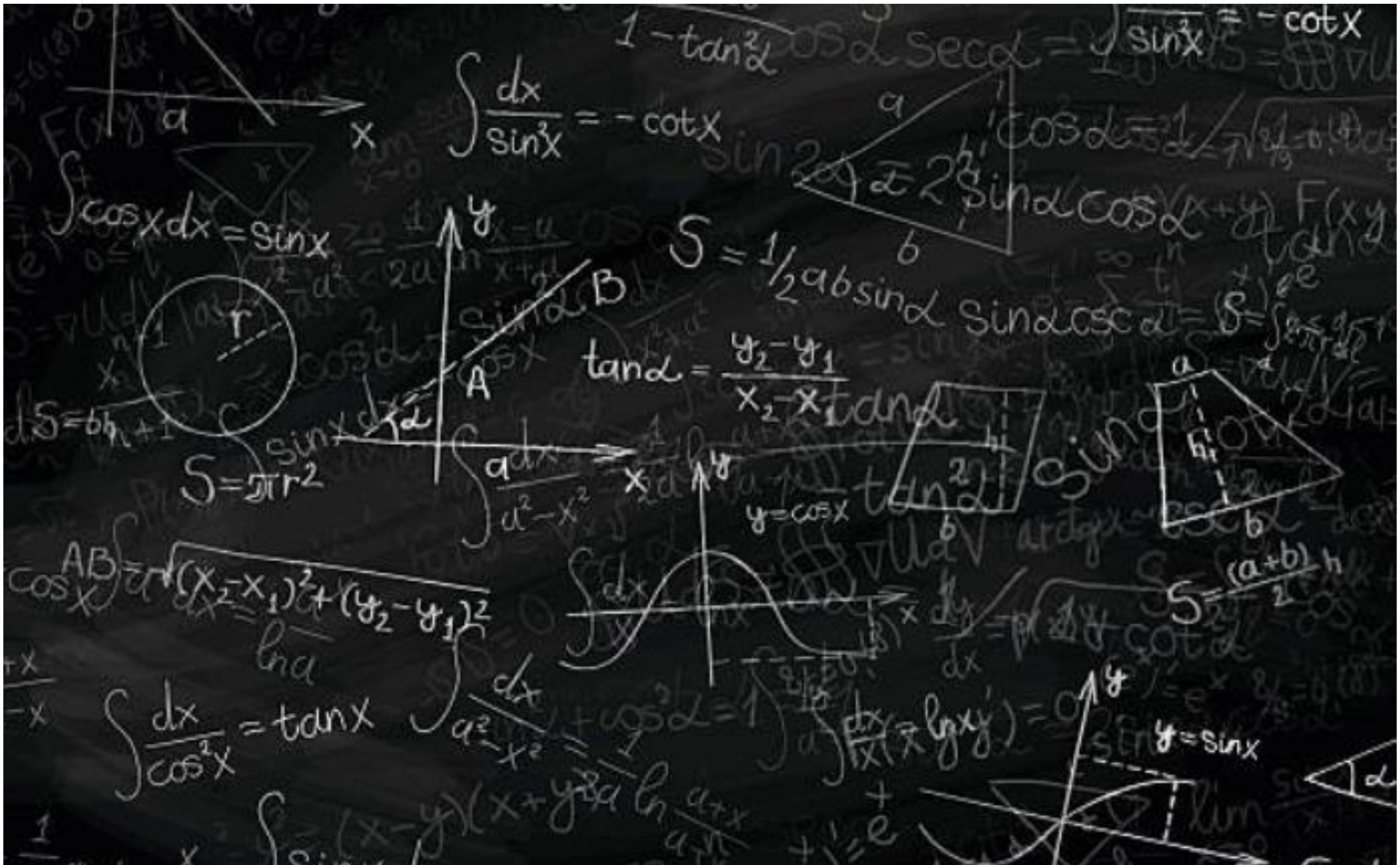
Innovative Approach in Mathematical Education for Maritime Students



Teacher's manual

Functions

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MareMathics

Innovative Approach in Mathematical Education for Maritime Students

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https://maremathics.pfst.hr/

Manual for teachers

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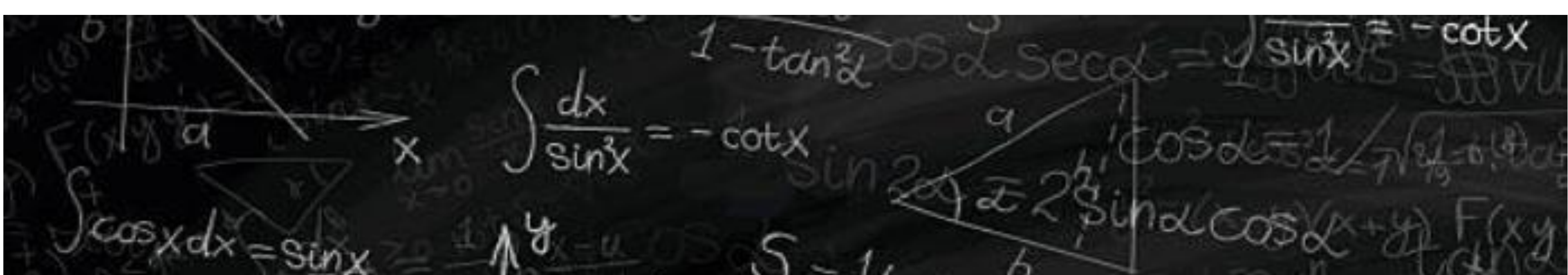

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The Manual is the outcome of the collaborative work of all the Partners for the development of the MareMathics Project.

Partners in the project:

University of Split Faculty of Maritime Studies
CROATIA

Tallinn University of Technology, Estonian Maritime Academy
ESTONIA

Latvian Maritime Academy
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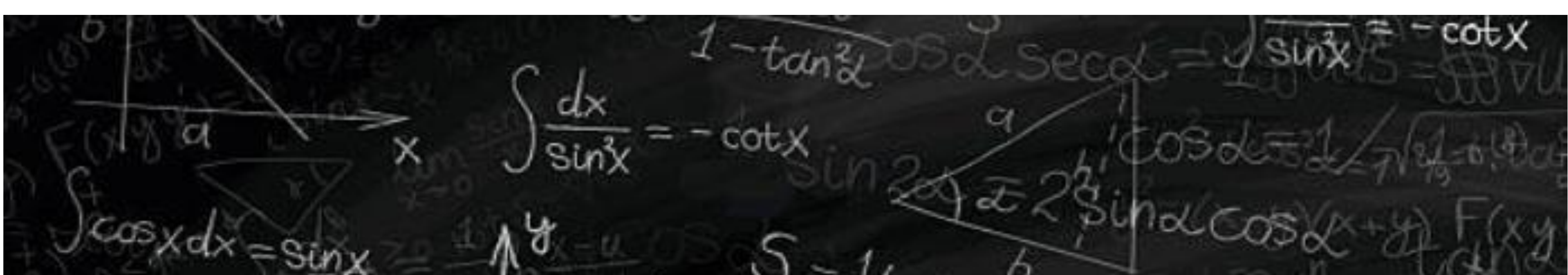

CONTENTS

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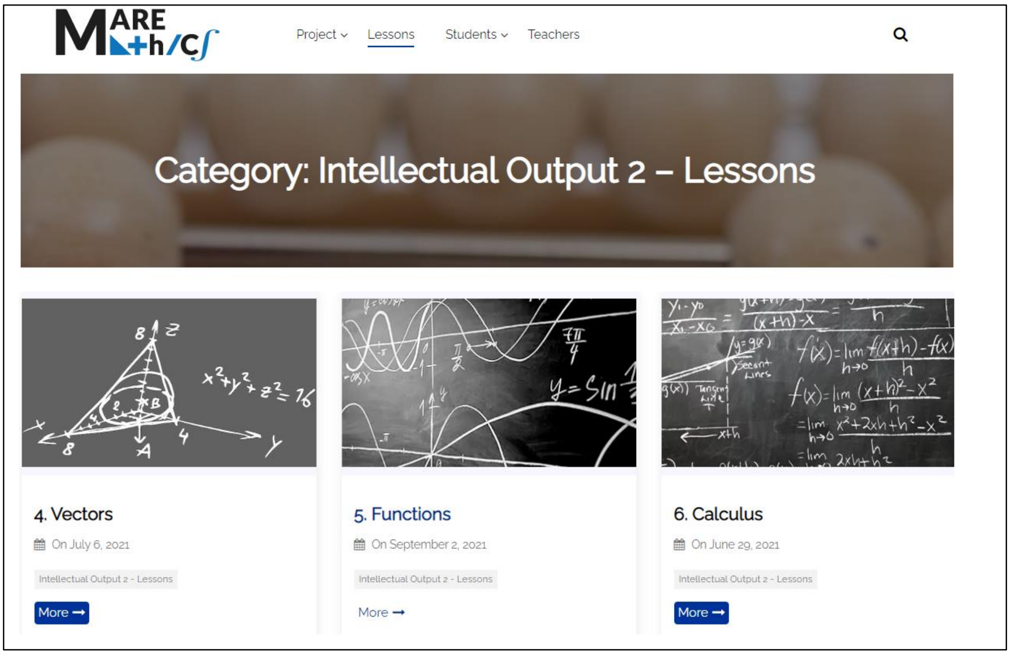
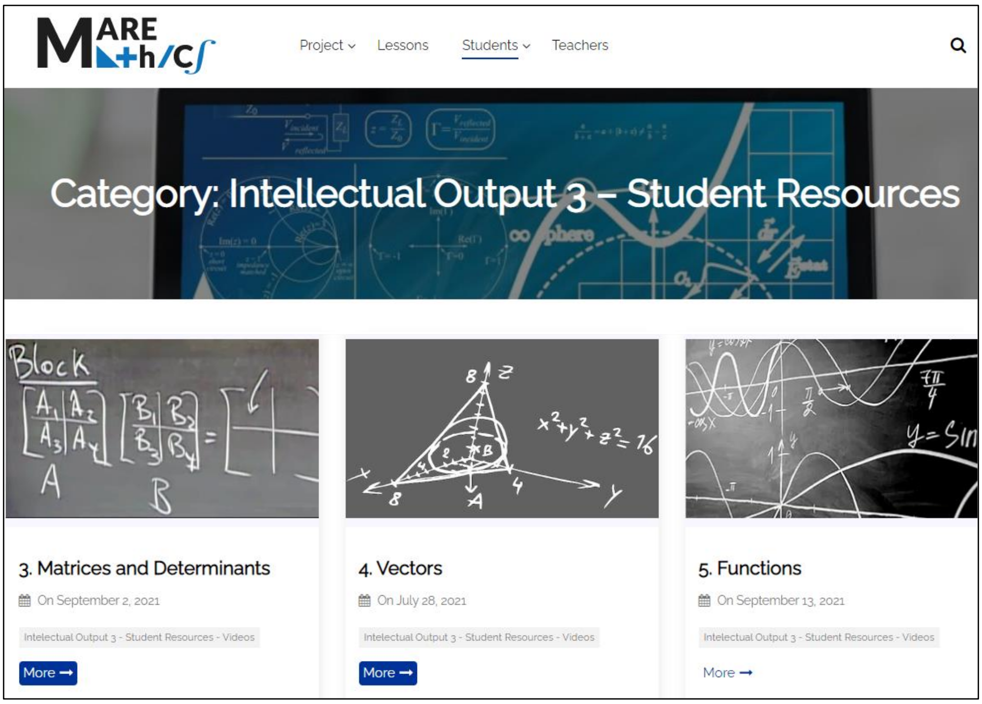
FUNCTION – teaching and learning plan

The resources are picked from project MareMathics and available on the The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

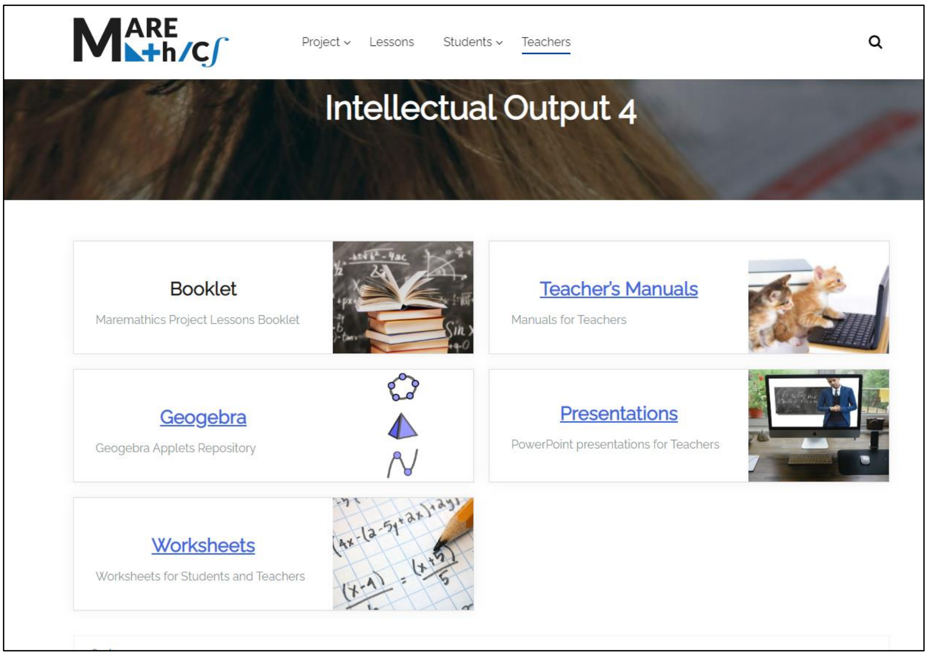
The resources are picked from project ***MareMathics*** and available on the https://marematics.pfst.hr/.



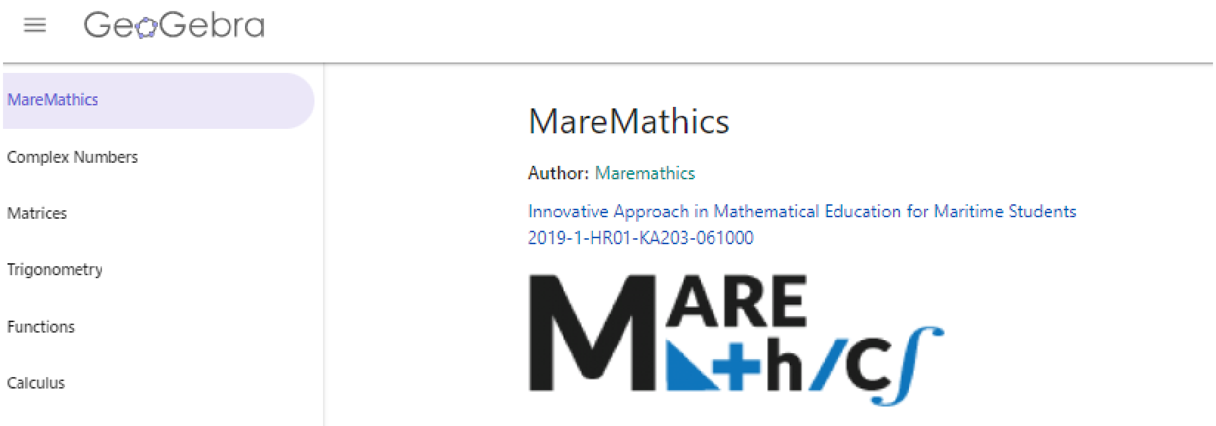
Functions: https://marematics.pfst.hr/?p=3526



Videos: https://marematics.pfst.hr/?p=3526



https://maremathics.pfst.hr/?p=3770



https://www.geogebra.org/u/marematics

Lesson 1. Functions: basics

Name of Unit	Workload	Handbook
Functions: basics	**Lecture: 90 min**	**Unit 5.1 Functions: basics**

DETAILED DESCRIPTION

The unit *Function basics* starts with the definition of a function. The properties of functions are introduced through real life examples and examples related to maritime problems. Since some maritime problems like determining the hydrostatic pressure or global positioning using longitude and latitude can be solved using functions, this knowledge is important to maritime students.

AIM: To learn basic concepts in functions, being able to recognize if a function is an injection, surjection, and bijection. Solve maritime problems using functions.

Learning Outcomes:

1. Understanding the definition of a function and recognising whether the mapping is a function.
2. Determining whether the function is an injection, surjection, and bijection.

Keywords of this Unit: function, domain, codomain, injection, surjection, bijection, longitude, latitude

Prior Knowledge: algebraic expressions, algebraic identities, linear equations, and inequalities.

Relationship to real maritime problems:

Functions are widely used in solving many engineering problems. Practical application of functions is in navigation theory, calculating the set and drift, the velocity of a vehicle, the traveling time, and distance.

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Lesson: Functions: basics

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
10 min	Presentation + two videos	The definition of a function and examples	Introduction Moderator Using videos	Active listening + discussion	What kind of relations can be called functions?
15 min	Examples 1, 2, and 3	Examples of functions	Frontal and questioning	Active listening and contributing to questions	What kind of relations can be called functions?
5 min	Video	Vertical line test	Frontal and questioning Using video and Geogebra	Active listening and contributing to questions Practical work using Geogebra	What kind of relations can be called functions?
15 min	Presentation and example	The definition of an injection, surjection, and bijection	Frontal and questioning	Active listening and contributing to questions	What functions are injections, surjections, and bijections?
10 min	Example 4	Function: Determining the latitude and longitude of a place of Earth	Moderator Using mobile phone	Practical work using mobile application Maps Contributing to questions	Is the mapping that accompanies each point on the Earth's surface an ordered pair of number coordinates a function, injection, surjection, and bijection?
5 min	Example 5	Function: Boarding on a ship	Moderator and questioning	Contributing to questions	Can boarding on a ship can be a presentation of a function?
10 min	Example 6	Function: Defining position and velocity od ship	Moderator and questioning Using Geogebra	Practical work using Geogebra and contributing to questions Complete the worksheet	
15 min	Example 7 and video	Defining function for hydrostatic pressure	Moderator and questioning Using online application and video	Participating in solving real life problem Complete the worksheet	
5 min	Final quiz in MS Forms	Functions	Frontal	Individual work	

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Assessment strategies:

Evaluating students' activity during lesson and quiz.

MareMathics Teacher Toolkit and Digital Recources:

- Powerpoint presentation (link: https://marematics.pfst.hr/index.php/2022/04/11/presentations/
- Video link https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#relation-function-1
- Video link https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#relation-function-2
- Video link https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#vertical-test
- Video link: https://www.youtube.com/watch?v=ierAsmNk4pI
- MareMathics Geogebra link: https://www.geogebra.org/m/ttu4reyn
- MareMathics Geogebra link: https://www.geogebra.org/m/shtqu5kq
- MareMathics Geogebra link: https://www.geogebra.org/m/ehagjzfm
- Quiz:
    - MS Form link: https://forms.gle/yrxiqpHQVe6vHnms9
    - MareMathics Geogebra link: https://www.geogebra.org/m/f49nggyb

Useful Online application (link: https://phet.colorado.edu/sims/html/under-pressure/latest/under-pressure_en.html)

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Activity 1 – definition and video with examples

The students are given worksheets with the following statements and are asked to try to fill in the blanks.

WORKSHEET
Function f is a relation in which _________ from the set of inputs X is associated to ________ object from the set of outputs Y. Each function must have three elements defined:
1.________ – a set of inputs, i.e., a set of all arguments of the function
2. Mapping rule f – the way this data is transformed - functional equation
3. Codomain Y - ____________________

- There can be a discussion.
- The teacher plays the video functions 01
- After watching video students can fill in the blanks.

SOLVED WORKSHEET
Function f is a relation in which every object from the set of inputs X is associated to exactly one object from the set of outputs Y. Each function must have three elements defined:
1.Domain X – a set of inputs, i.e. a set of all arguments of the function
2. Mapping rule f – the way this data is transformed - functional equation
3. Codomain Y – the set of outputs

The teacher asks the question for each of six relations in video 01: Is this relation a function? After students' voting, the teacher plays the following video functions 02 with explanations

Activity 2 – Examples – real life example

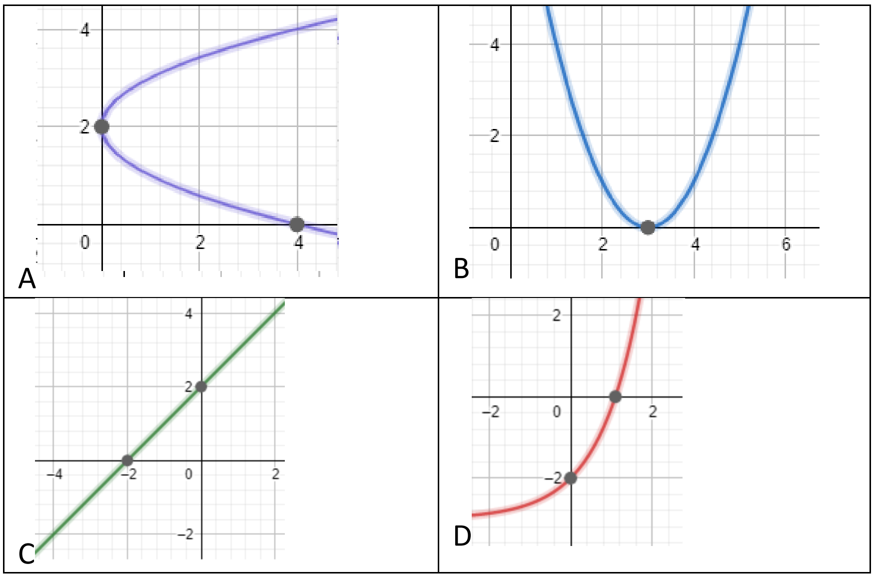
Teacher asks the question:

More than 31 000 students graduated from high school in Croatia in 2021. Some of them enroled in college. Is the mapping in which each student is asociated with a college a function?

SOLUTION: Not necessarily, because some students didn't enrole in college, and some students enroled in two colleges.

Then, the following example is shown to the students (using a projector or as a worksheet):

Example 2: Which of the following is NOT a graph of a function?



SOLUTION: A, because some x-values, i.e., x=2 are mapped to multiple y-values, so this is not a function. Students can discuss the domains and codomains of the functions.

Example 3. Are the following functions EQUAL?

$f\colon \mathbb{R} \to \mathbb{R},\quad f(x) = x^2 - 1,\qquad\qquad g\colon \langle -5, \infty\rangle \to \langle -1, \infty\rangle,\ \ g(x) = x^2 - 1$

SOLUTION: NO, because the domain and the codomain are not equal.

The teacher and students discuss the solutions, the teacher solves the tasks in front of the class and clearly explains the reasoning.

Activity 3 – vertical line test - video

The teacher introduces the **vertical line test** for determining if a relation is a function by playing a video [Video vertical line test]

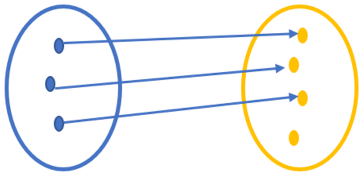
Student can actively participate using [Geogebra vertical line test]

A relation is a function if every vertical line in the domain **intersects the mapping exactly once**.

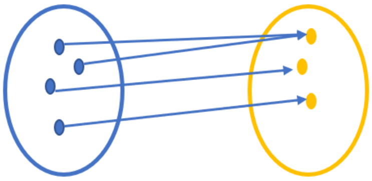
Activity 4 – injection, surjection, bijection – definition and examples

The teacher starts the next part of the lesson by explaining that functions can have certain (desirable) properties and names them:

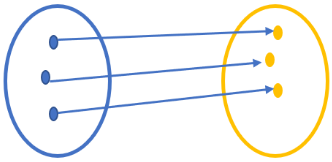
1. The function is called an **injection** if it is a one-to-one function. It is a function that maps distinct elements of the domain to distinct elements of the codomain.



2. A function is called a **surjection** if every element of the codomain is mapped by at least one element of the domain.



3. A function is called a **bijection** if it is both an injection and a surjection.



<u>Real life example 1b is presented:</u>

More than 20 000 students enroled in college in Croatia in 2021. Is the mapping in which each student who enroled in exactly one college is asociated with a college:

a) function

b) injection

c) surjection

d) bijection

Solution: Mapping is a) function and probably b) surjection. There can be a discussion on constant change in the world. Is there any college without enroled students? Is it possible that some today's colleges would become unnecessary in the future?

Mapping is not an injection nor bijection.

Activity 5 – real life examples

The teacher shows the following example:

EXAMPLE 1 teaching material .

For example, the city of Split is associated with coordinates (43.508695°N, 16.440303°E) (43° 30’ 31’’ N and 16° 26’ 25’’ E).

Is the mapping that accompanies each point on the Earth's surface an ordered pair of number coordinates:

a) function

b) injection

c) surjection

d) bijection

SOLUTION:

The mapping that accompanies each point on the Earth's surface an ordered pair of number coordinates is a function, injection, surjection, and bijection.

a) The mapping is a function because each point on the surface has corresponding coordinates.

b) The function is an injection because two different places on the Earth’s surface will surely have different coordinates. There cannot be two different cities with the same latitude and longitude.

c) The function is a surjection because all possible values of latitude and longitude are hit.

d) Since the function is an injection and surjection, it is a bijection.

This is important because it means we can go “backwards”. Knowing the coordinates of one place, for instance (43.508695°N, 16.440303°E), we can distinctively say which point on the surface it is (Split).

EXAMPLE 2.

Port and 300 passengers. There can be a discussion about mapping.

The GPS system receives messages about the coordinates of the Nautilus ship every hour during the four-hour voyage.

Example 4

Hydrostatic pressure application hidrostatic preassure

Students are actively participating in class. They are using their mobile phones in this activity.

Using the link above, we will examine some properties of the hydrostatic pressure formula.

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Directions for the setup of the experiment.

0) Open the link

1) In the lower left corner, pick the first model (one fosset and one pool).

2) In the upper right corner tick the grid option (as to measure the water depth)

3) Pour water into the pool to the brim

4) Drag and drop the barometer to measure the pressure above the pool and in the pool

First try the air pressure above the pool:

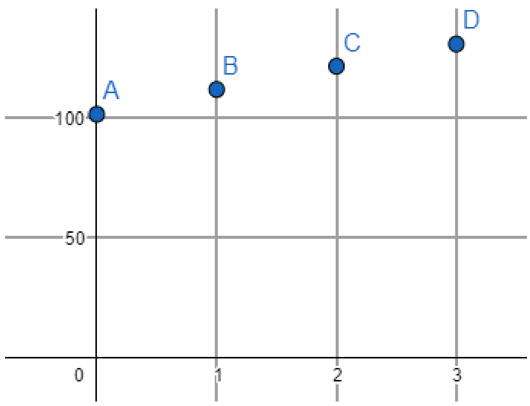
a) When the depth increases, the pressure __________.

b) What is the atmospheric pressure (the pressure at 0 m, in pool level)?

____

c) Fill out the table:

Water depth (X)	Water pressure (Y)
0	
1	
2	
3	



d) What can you notice about the points? Is there a pattern? Explain your reasoning.

____

e) The hydrostatic pressure formula is:

f) Given the saltwater density is $\rho = 1\,025\ kg/m^3$, the atmospheric pressure is $p_{atm} = 101\,325\ Pa$ and $g = 9.81 m/s^2$, determine the hydrostatic pressure at a depth of 40 m?

g) The world record in deep sea diving (without injury) is ___. With conditions described above ($\rho = 1025\ kg/m^3, p_{atm} = 101325\ Pa,\ g = 9.81 m/s^2$) determine the pressure at the given depth.

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h) What is the percent increase in water pressure at depth 214 m when compared with atmospheric pressure?

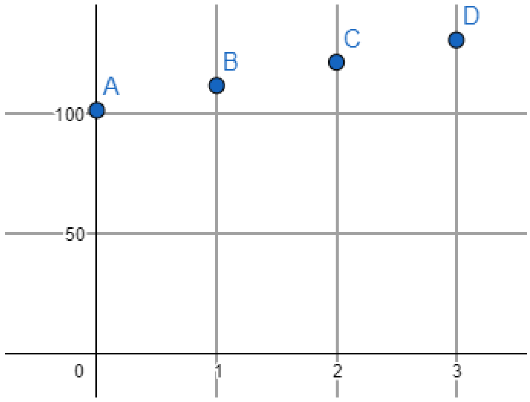
SOLUTION:

a) When the depth increases, the pressure **increases**.

b) What is the atmospheric pressure is 101.325 kPa, so 101 325 Pa.

c) Fill out the table:

Water depth (X)	Water pressure (Y)
0	101.326
1	111.126
2	120.926
3	130.726



d) When the points from table in part 3) are graphed in a coordinate plane, a pattern can be seen. All the points seem to lie on the same line.

This means that the pressure at various depths can be determined from the graph, by extending the line.

e) The hydrostatic pressure formula is:

$$p_{hyd}(h) = p_{atm} + \rho g h$$

where ρ is the liquid density, g is the Earths gravitational constant (9.81 m/s^2) , and h is the depth.

f) To calculate the pressure, plug in h = 40 m into the equation:

$$p_{hyd}(40) = 101\,325 + 1025 \cdot 9{,}81 \cdot 40 = 503\,535 \ \ Pa$$

g) Video world record deep sea diving shows the setting of world record in deep sea diving. The To calculate the pressure, plug in h = 214 m into the equation:

$$p_{hyd}(214) = 101\,325 + 1025 \cdot 9{,}81 \cdot 214 = 2\,253\,148\,Pa$$

h) To find the percent increase, calculate the quotient $\frac{p_{hyd}(214)}{p_{atm}} = 22.23$

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So, the pressure at depth 214 m is 22.23 times greater than the atmospheric pressure. This is a percent increase of 2123%!

APPENDIX 1:

Quiz with 6 questions.

Lesson 2. Analysing the graph of a function

Name of Unit	Workload	Handbook
Analysing the graph of a function	**Lecture: 30 min**	**Unit 5.2**

DETAILED DESCRIPTION

In this unit few examples of graphs are given. Students will learn how to recognize if the function is increasing or decreasing, and how to find if the values of a function are positive or negative. The increase or the decrease of the sea level can be associated with this lecture.

AIM: To learn concept of sign of a function and increasing ang decreasing function and to apply the knowledge in maritime problems.

Learning Outcomes:

1. Find the intervals where the function has positive or negative values
2. Find the intervals of monotonicity of a function.

Keywords of this Unit: function, increasing, decreasing, monotonicity, positive, negative, sign of a function

Prior Knowledge: algebraic expressions, algebraic identities, linear equations, and inequalities.

Relationship to real maritime problems:

Practical application of basic analysing the graph of functions can be found in a wave theory.

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Lesson 2. Analysing the graph of a function

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
10 min	Example: Activity 1	The definition of a sign of a function with an example	Introductio n Moderator Frontal and questioning Geogebra	Active listening + discussion	When do we say that the function if positive or negative on intervals?
5 min	Example: Activity 2	The definition of an increasing or decreasing function	Frontal and questioning Geogebra	Active listening and contributing to questions	What kind of relations can be called functions?
5 min	Example: Activity 3	Example: sign of a function and monotonicity	Frontal and questioning	Contributing to questions	
5 min	Example: Activity 4 – application in maritime	Example: sign of a function and monotonicity in real life problems	Frontal and questioning Using Geogebra	Active listening and contributing to questions	
5 min	Appendix: Final quiz in MS Forms	Functions: sing of a function, increasing and decreasing	Frontal	Individual work	

Assessment strategies:

Evaluating students' activity during lesson and quiz.

MareMathics Teacher Toolkit and Digital Recources:

- Lesson: https://maremathics.pfst.hr/wp-content/uploads/2022/06/Functions-2.pdf
- MareMathics Geogebra link: https://www.geogebra.org/m/fdfpqxww
- MareMathics Geogebra link: https://www.geogebra.org/m/jtxkcrhf
- MareMathics Geogebra link: https://www.geogebra.org/m/bzkej7p4
- Quiz link: https://docs.google.com/forms/d/e/1FAIpQLSeZE9HkGNk3FHn0VsxgVjEDIYUabF90XBXbQnZ43finYwrZNg/viewform?usp=sf_link

Useful Websites

https://www.youtube.com/watch?v=nrbHENlGq_0

https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:intervals-where-a-function-is-positive-negative-increasing-or-decreasing/v/increasing-decreasing-positive-and-negative-intervals https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:functions/x2f8bb11595b61c86:intervals-where-a-function-is-positive-negative-increasing-or-decreasing/v/when-a-function-is-positive-or-negative

https://www.youtube.com/watch?v=VSzUM7nw5s0

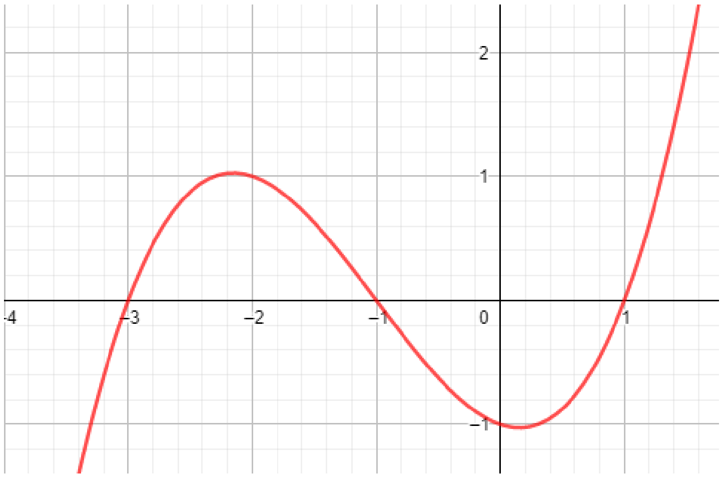
https://www.mathsisfun.com/sets/functions-increasing.html

https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:poly-graphs/x2ec2f6f830c9fb89:poly-intervals/a/positive-and-negative-intervals-of-polynomials

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

ACTIVITY 1

The teacher shows the following function:



The function can also be accessed via the following link:

https://www.geogebra.org/m/fdfpqxww

The students must determine:

a) All numbers for which the function values are positive
b) All numbers for which the function values are negative
c) All numbers for which the function values are zero

SOLUTION

a) The function is positive for all numbers between -3 and -1, and from 1 to infinity
b) The function is negative for all numbers less than -3 and from -1 to 1.
c) The function is equal to zero for -3, -1, and 1.

The goal is to steer the conversation towards phrasings like “all numbers between a and b “. The emphasis should be on the fact that those are all **real** numbers between points a and b. The teacher then explains that this is called an **interval**, and that the interval can be **open** or **closed** or **semi-open/semi-closed**.

The teacher then explains the different types of parenthesis (< , >, [,] , { , }) to denote an open/closed interval and a set.

The teacher then shows the correct solution to the problem above:

The function values are positive for all numbers in the intervals $< -3, -1 > \cup < 1, +\infty >$.

The function values are positive for all numbers in the intervals $< -\infty, -3 > \cup < -1,\ 1 >$.

The function values are zero for the following numbers: { -3,-1,1}

If the function value is positive for all numbers in an interval, we say that the **function is positive** on that interval.

For instance, the function above is positive on the interval $< -3, -1 >$.

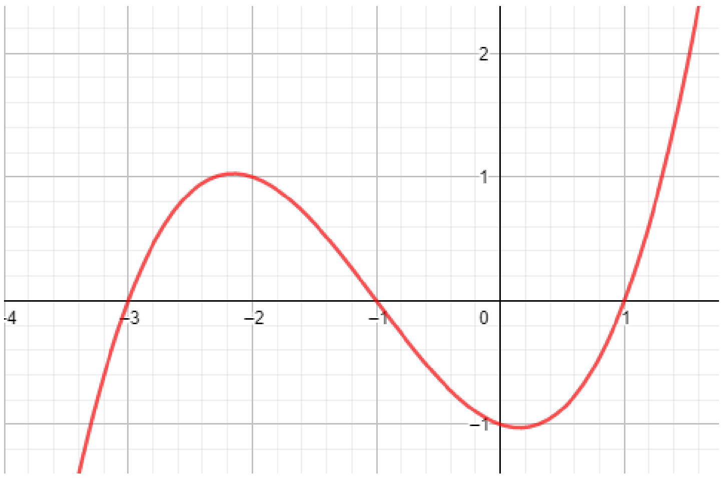
Lastly, the teacher emphasizes the definition with the graph above, showing that the function is positive if the graph of the function is **above the x-axis**, and the function is negative if the graph of the function is **below the x-axis**.

If the function values are positive, we say that the function has a **positive sign**, if the function values are negative the sign of the function on that interval is also **negative**.

For all values where the function value is 0, the function has a **sign of 0** (because zero technically does not have a sign).

ACTIVITY 2.

The teacher shows the following function:



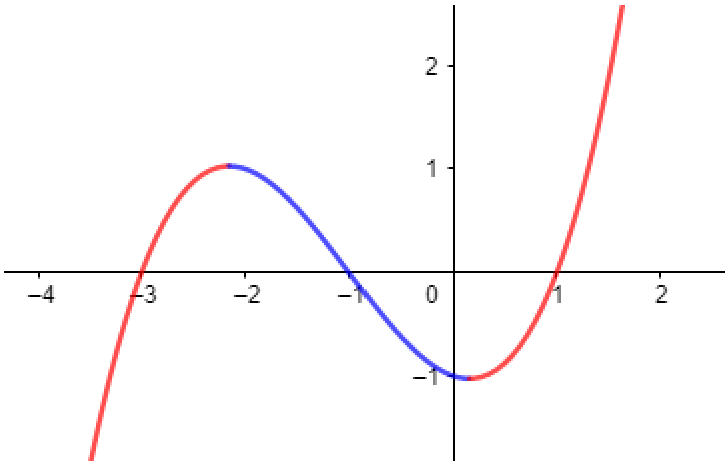
The function can also be accessed via the following link:

https://www.geogebra.org/m/jtxkcrhf

The students must determine:

a) All intervals for which the function is **increasing**
b) All intervals for which the function is **decreasing**

SOLUTION:



For all values colored red the function is increasing.

For all values colored blue the function is decreasing.

The teacher writes the definitions of increasing and decreasing functions on the blackboard:

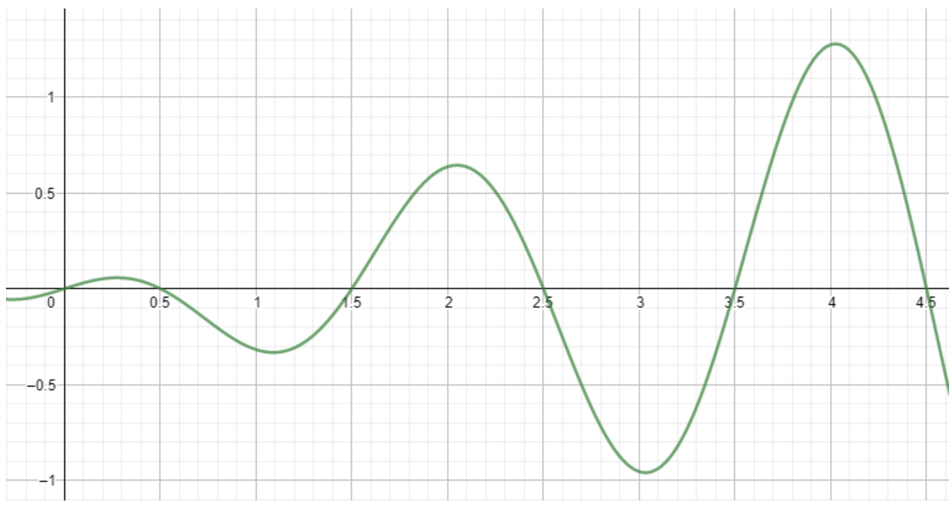
If for every $x_1, x_2 \in [a, b]$ when $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ we say that function is **increasing** on $[\mathrm{a}, \mathrm{b}]$.

If for every $x_1, x_2 \in [a, b]$ when $x_1 < x_2$ then $f(x_1) \geq f(x_2)$ we say that function is **decreasing** on $[\mathrm{a}, \mathrm{b}]$.

A function can be increasing on an interval and decreasing on another interval. If the function is increasing/decreasing on the entire domain, we say that the function is **increasing/decreasing**.

The increasing or decreasing behavior is called **monotonicity**.

ACTIVITY 3.



Given the function graph above from $[0,4.5]$, determine the following:

All intervals where the function is positive (all x values for which the function value is positive)

b) All zero points

c) All intervals where the function is decreasing

SOLUTION

a) $x \in \langle 0,0.5\rangle \cup \langle 1.5,2.5\rangle \cup \langle 3.5,4.5\rangle$

b) Zero points are $(0,0), (0.5,0), (1.5,0), (2.5,0), (3.5,0), (4.5,0)$

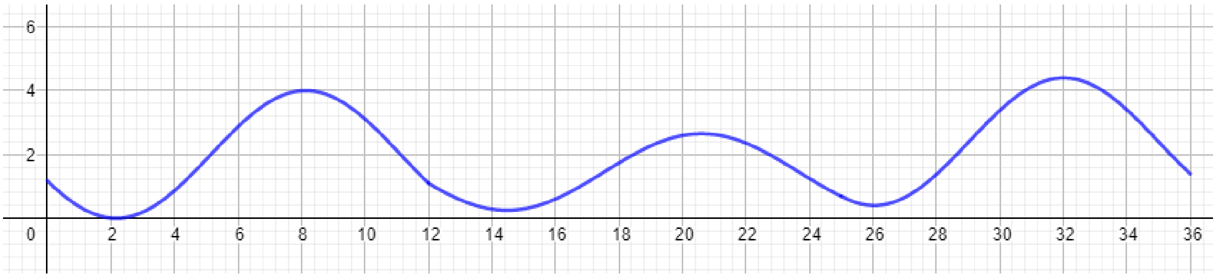
c) $x \in \langle 0.5,1.5\rangle \cup \langle 2.5,3.5\rangle$

ADDITIONAL RESOURCES

ACTIVITY 4.

The graph depicts the sea level in the Bay of Fundy in a period of 36 hours.

https://www.geogebra.org/m/bzkej7p4



Determine:
a) The tides
b) The maximum sea level
c) The sea level at t=10 hours, t=18 hours, t=32 hours.
d) At what times was the sea level equal to 2 meters
e) All intervals when the sea level was increasing

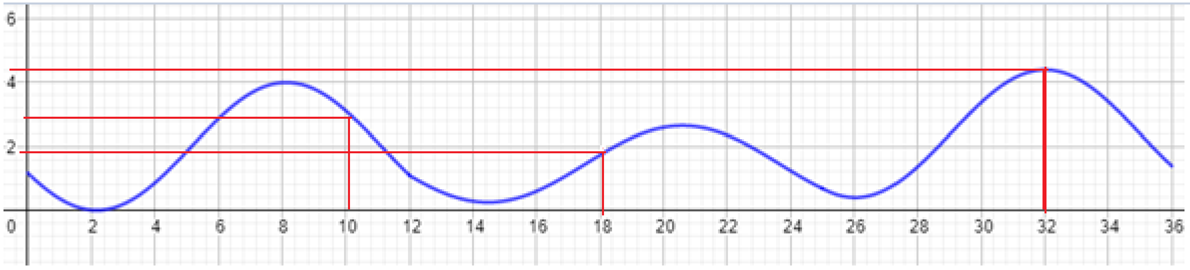
SOLUTION:

a) The increase of the sea level indicates flood, so flood occurs in the following approximate intervals $[2,8], [14,20.5], [26,32]$.

The decrease of the sea level indicates ebb, so ebb occurs in the following approximate intervals $[0,2], [8,14], [20.5,26], [32,36]$.

b) The maximum sea level is the highest point on the graph, which occurs at time $t = 32$. The corresponding sea level is 4.2 m.

c) To find the sea level at given points, read the y-coordinate of points on the graph at $t = 10, t = 18, t = 32$, like this:



The corresponding points are approximately $(10,2.8), (18,2), (32,4.4)$.

d) To find the times corresponding to the sea level of 2 meters, draw the line $y = 2$ and read the x-coordinates of all intersection points. The solutions are approximately: $t = 5, t = 11, t = 18, t = 23, t = 29, t = 35$.

e) The increasing sea level intervals are equivalent to flood, solved in part a).

APPENDIX 1:

Quiz with 6 questions. [quiz functions]

Lesson 3. Odd and even functions

Name of Unit	Workload	Handbook
Functions	**Lecture + exercise: 60 min**	**Unit 5.3 Odd and even functions**

DETAILED DESCRIPTION

In this lesson we are learning difference between odd and even functions.

Learning Outcomes:

1) Students know how to distinguish odd and even function
2) Students know how to decide is function odd, even or neither by seeing the graph

Key words of this Unit:

Odd function, even function, symmetry

Previous knowledge of mathematics:

Knows how to draw graphics and decide about symmetry.

Teacher Toolkit and Digital Resources:

- Lesson
- Video
- Presentation
- Quiz

Lesson: Odd and even functions

LESSON FLOW						
	Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points of discussion**
I.	5 min	Introduction	Pre teaching	Motivation	Active listening and discussion.	What do you remember about odd and even functions?
II.	5 min	Presentation	Video	Frontal	Active listening and contributing to questions	
III.	10 min	Quiz	quiz	Frontal . Moderator	Working on a quiz	
IV.	10 min	Presentation			Active listening and contributing to questions.	
V.	15 min	Exercise	Worksheet	Explain task and supports	Complete the worksheet	
VI.	5 min	Summary	Post teaching	Frontal		

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• **Whiteboard** • **Lesson** https://maremathics.pfst.hr/wp-content/uploads/2022/06/Functions-3.pdf • **Video** https://maremathics.pfst.hr/?p=3526#oddevenfun **https://www.dropbox.com/s/u049d75l7ap36ws/odd_n_even.mp4?raw=1** **Presentation** https://livettu-my.sharepoint.com/:p:/g/personal/julia_tammela_ttu_ee/EfoVIDGxjnhDr2qhymF56rAB4UEJvAWVL_DtDGhqWcBd7A?e=K6TQbq • **Worksheet Odd and even function** https://marematics.pfst.hr/?p=4022 • **Quiz https://quizizz.com/admin/quiz/621603ee8fd453001e35315a**

I. Introduction

Teacher introduce topic of this lesson. There is a discussion what do students remember about odd and even functions.

Students discuss whether they studied this topic in school or not. Name some odd functions.

II. Presentation

Teacher shows students video about odd and even function to remind what it is.

Students watch video.

III. Quiz

Teacher gives students link to a quiz and explains assignment. Later gives some feedback after quiz.

Students work on a quiz and are remembering previously learned material.

IV. Presentation

Teacher explains given topic and answer some questions. Gives some examples how to solve tasks on odd and even functions. Odd and even functions are used in physics and math (series).

Students write down given information and examples and ask questions if necessary.

Presentation is given as a link.

V. Exercise

Teacher gives student worksheet with tasks to solve and explains the tasks. In process gets and gives students feedback.

Some students solve exercises on a whiteboard while others work in there notebooks.

Link to worksheet is given in the end of a document.

VI. Summary

At the end of a lesson teacher gives homework and concludes what new was learned today.

Students write down homework, ask question about it and discuss whether this topic was new to them or not.

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Appendix worksheet

Exercise 1

Is this function odd, even or neither?

1) $y = x^2 + 5x$; 2) $y = -x^2 + 2$; 3) $y = x^2 + 2x - 3$;

4) $y = 5x - 2$; 5) $y = x^4 + 2x^2$; 6) $y = x^2 - 5x + 3$;

7) $y = x + \frac{1}{x}$; 8) $y = \frac{2x^2}{3x^2+x}$; 9) $y = \frac{3}{x^2-5}$;

10) $y = x^4 + x^2 - 3x$; 11) $y = x^3 + x^3$; 12) $y = 3x^4 + x^2 + 4$

Exercise 2

Is this function odd, even or neither?

$a)$ $y = 5x$; $e)$ $y = 5x^2$; $i)$ $y = 5x - 1$; $m)$ $y = \frac{3}{x}$;

$b)$ $y = 5x^2 + 1$; $f)$ $y = 6x$; $j)$ $y = 6x^2 + 4$; $n)$ $y = 6x + 1$;

c) $y = 2x^3$; $g)$ $y = 5x^2 + 1$; $k)$ $y = x^2 - 2x + 3$; $o)$ $y = 3x^4$;

d) $y = 7x^2 + x$; $h)$ $y = 3x^4 + x^2 + 4$; $l)$ $y = x^3 - 5x - 3$; $p)$ $y = x + \frac{1}{x}$.

Lesson 4. Linear functions

Name of Unit	Workload	Handbook
Linear functions	**Lecture: 45 min** **Exercises: 45 min**	**Unit 5.4.1 Linear functions**

DETAILED DESCRIPTION

Linear functions apply to real world problems that involve a constant rate. If you know a real-world problem is linear, such as the distance you travel when you go for a jog, you can graph the function and make some assumptions with only two points. The slope of a function is the same as the rate of change for the dependent variable y. For instance, if you're graphing distance vs. time, then the slope is how fast your distance is changing with time, or in other words, your velocity. Linear equations often include a rate of change. For example, the rate at which distance changes over time is called velocity. If two points in time and the total distance traveled is known the rate of change, also known as slope, can be determined. From this information, a linear equation can be written and then predictions can be made from the equation of the line.

If the unit or quantity in respect to which something is changing is not specified, usually the rate is per unit of time. The most common type of rate is "per unit of time", such as speed, heart rate and flux. Ratios that have a non-time denominator include exchange rates, literacy rates, and electric field (in volts/meter).

In describing the units of a rate, the word "per" is used to separate the units of the two measurements used to calculate the rate (for example a heart rate is expressed "beats per minute").

Linear mathematical models describe real world applications with lines. linear algebra, is a fundamental part of science. Scientists – as well as people working in many other fields or just going about their regular routines – make use of linear equations every day. Among other things, linear equations can help us describe the relationship between two quantities or phenomena (i.e. femur length and overall height), calculate rates (such as how quickly an object is moving), or convert from one unit of measurement to another (for example, inches to centimeters).

AIM: To acquire skills in solving equations and inequalities and also to understand concepts standing behind those calculations. Linear functions are also important for some real-life applications in the maritime field.

Learning outcomes

At the end of this lecture, students should be able to

1. Know the definition of linear function

2. Plot a graph of linear functions and read properties of the function from the graph.
3. Know how to apply linear functions in real life problems and solve practical problems.

Key words of this Unit:

- rate of change: ratio between two related quantities that are changing.
- linear equation, slope, point of intersection.

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Previous knowledge of mathematics: Content from basic algebraic operations.

Relatedness with solving problems in the maritime field: Linear equations can be used to describe many relationships and processes in the physical world, and that is why they play a meaningful role in science. Frequently, linear equations are used to calculate rates, such as how quickly a projectile is moving or a chemical reaction is proceeding. They can also be used to convert from one unit of measurement to another, such as meters to miles or degrees Celsius to degrees Fahrenheit $(F = 1.8C + 32)$ or to calculate the total monthly income for a salesperson with a base salary of \$1,500 plus a commission of $\$400/unit$ sold: $I = 400T + 1{,}500$, where T represents the total number of units sold. In some cases, scientists *discover* linear relationships during the course of research. For example, an environmental scientists analyzing data they have collected about the concentration of a certain pollutant in a lake or a sea or an ocean may notice that the pollutant degrades at a constant rate. Using those data, they may develop a linear equation that describes the concentration of the pollutant over time. The equation can then be used to calculate how much of the pollutant will be present in five years or how long it will take for the pollutant to degrade entirely. Specific real world problems that can be solved using a linear function include: predicting equipment rental cost given a rate and rental period, calculating profit for a small business, taxes due on a particular amount of income, in structural engineering design.

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define linear functions
- GeoGebra https://www.geogebra.org/ https://www.geogebra.org/graphing?lang=en
- https://www.padowan.dk/
- Graph

Version 4.4

Graph - plotting of mathematical functions

- Work Sheets
- Websites:

— https://linearequations.org/linear-equations-examples/
— https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#lin-model
— https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-5.pdf
— https://www.reference.com/world-view/real-world-uses-linear-functions-7f42bdfccbfac550
— https://www.padowan.dk/
— https://courses.lumenlearning.com/boundless-algebra/chapter/applications-of-linear-functions/
— https://www.visionlearning.com/en/library/Math-in-Science/62/Linear-Equations-in-Science/194
— https://www.mathway.com/examples/calc/linear-equations

Lesson : Linear functions

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can find linear functions ?
30 min	Presentation	Description, definition, Properties of linear functions.	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Are students able to recognize the slope and y-intercept , point-slope form of the equation of a line, vertical and horizontal lines. Are they able to graph linear functions
30 min	Presentation Exercises 1-7	Working examples of the equations of the lines	Frontal Explains task and supports	Active listening and contributing to questions Complete the worksheets	Are students able to solve exercises with linear models.
25 min	Time for open learning for students – solving exercises and real life problems	Break-even analysis, linear models	Frontal Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
5 min	Summary		Giving homework		

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• Whiteboard • Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#lin-model • https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-5.pdf
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve linear equations and inequalities, to sketch their graphs • ***all*** students should be able to apply linear function in real life problems.

A. The first section is a description of linear function, defining a slope, y-intercept, properties of a linear function, its graphs.

A linear function has the following form

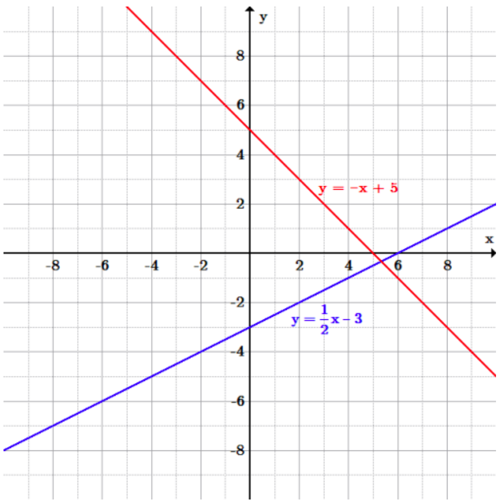
$$\boldsymbol{y = ax + b}, \text{ (slope } \boldsymbol{a}, \ \boldsymbol{y} - \text{intercept } \boldsymbol{b}).$$

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y.
b is the constant term or the y *−intercept*. It is the value of the dependent variable when $x = 0$.
a is the coefficient of the independent variable. It is also known as the *slope* and gives the *rate of change* of the dependent variable. The slope of a line is a number that describes both the *direction* and the *steepness* of the line. Slope is often denoted by the letter a . Recall the slop-intercept form of a line, $y = ax + b$. Putting the equation of a line into this form gives you the slope (a) of a line, and its y-intercept (b).
Slope is calculated by finding the ratio of the "vertical change" to the "horizontal change" between any two distinct points on a line. This ratio is represented by a quotient ("rise over run"), and gives the same number for any two distinct points on the same line. It is represented by $a = \frac{rise}{run}$. Slope describes the direction and steepness of a line, and can be calculated given two points on the line:

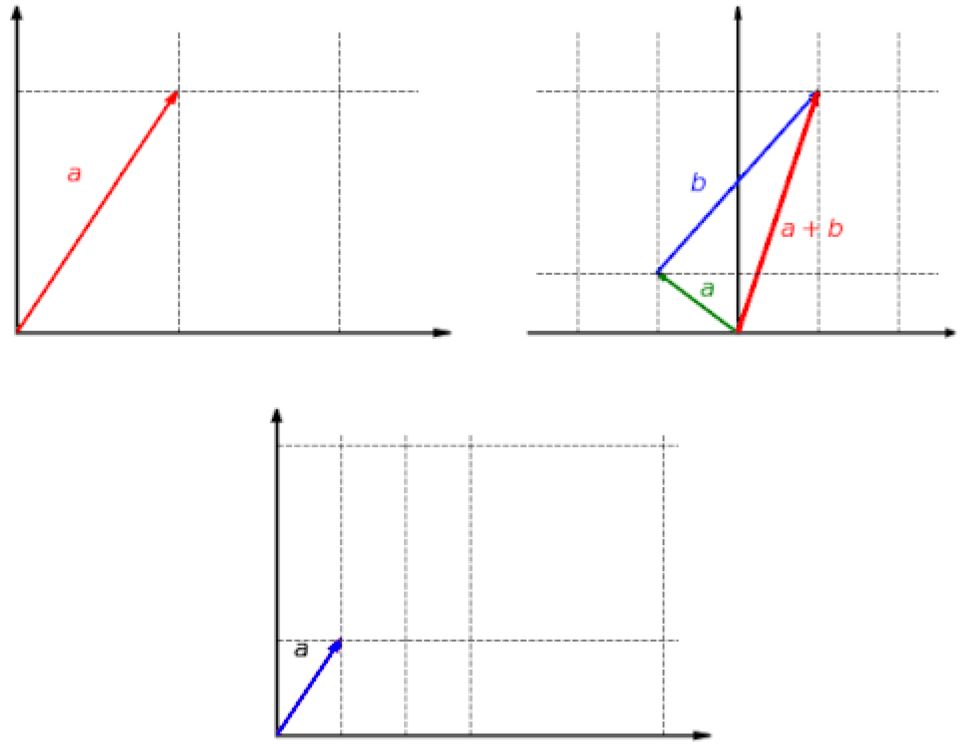
$$\boldsymbol{a} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

B. Teacher shows students that they can use Graph or GeoGebra to plot graphs of linear functions and on the basis of the graph recognize properties of those functions.

https://study.com/learn/lesson/positive-slope-graph.html



C. A teacher explains and remains vertical line, horizontal lines, graphing the linear functions:



[source: https://www.calculushowto.com/types-of-functions/linear-function/]

Vertical lines have an undefined slope, and cannot be represented in the form $y = ax + b$, but instead as an equation of the form $x = c$ for a constant c, because the vertical line intersects a value on the x-axis, c. For example, the graph of the equation $x = 3$ includes the same input value of 4 for all points on the line, but would have different output values, such as $(3, -3), (3, 0), (3, 1), (3, 6), (3, 2), (3, -1)$, etc. Vertical lines are **NOT functions**, however, since each input is related to more than one output. If a line is vertical the slope is undefined.

Horizontal lines have a slope of zero and they are represented by the form, $y = b$, where b is the y-intercept. A graph of the equation $y = 3$ includes the same output value of 3 for all input values on the line, such as $(-1, 3)$, $(0, 3)$, $(2, 3)$, $(3, 3)$, $(6,3)$ etc. Horizontal lines **ARE functions** because the relation (set of points) has the characteristic that each input is related to exactly one output.

Graphing a linear function

To graph a linear function, students have to:

1. Find 2 points which satisfy the equation;

2. Plot them;

3. Connect the points with a straight line.

Point-slope form of the equation of a line

The equation of the line through $(x_0,\ y_0)$ with slope a is

$$\boldsymbol{y - y_0 = a(x - x_0)}.$$

The equation of the line through two points: $(x_1,\ y_1)$, $(x_2,\ y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

D. Teacher shows how to draw graphs and students use GeoGebra or Graph to plot graphs

- Teacher presents a video: https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#lin-model

E. Teacher introduces some practical applications of exponential functions and explains students the possibility of meeting exponential functions in real life problems. Teacher gives students links to some useful articles or websites.

1. The total cost in euros of manufacturing x units of a certain commodity is

   $f(x) = 30x + 1000.$

   a) Compute the cost of manufacturing 10 units.

   b) Compute the cost of manufacturing 10^{th} unit.

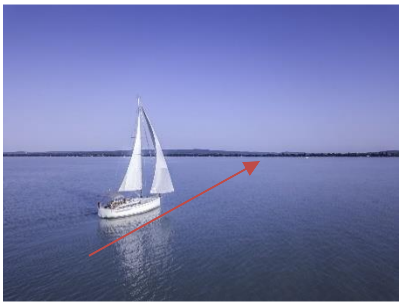
2. Find the slope and y −intercept of the line $3x + 2y = 6$ and draw the graph.
3. Find the equation of the line through $(2,5)$ and $(1,-2)$.
4. Since the beginning of the month, a local reservoir has been losing water at constant rate.

   On the 12^{th} of the month, the reservoir held 400 million liters of water and on the 22^{th}, it held 250 million liters.

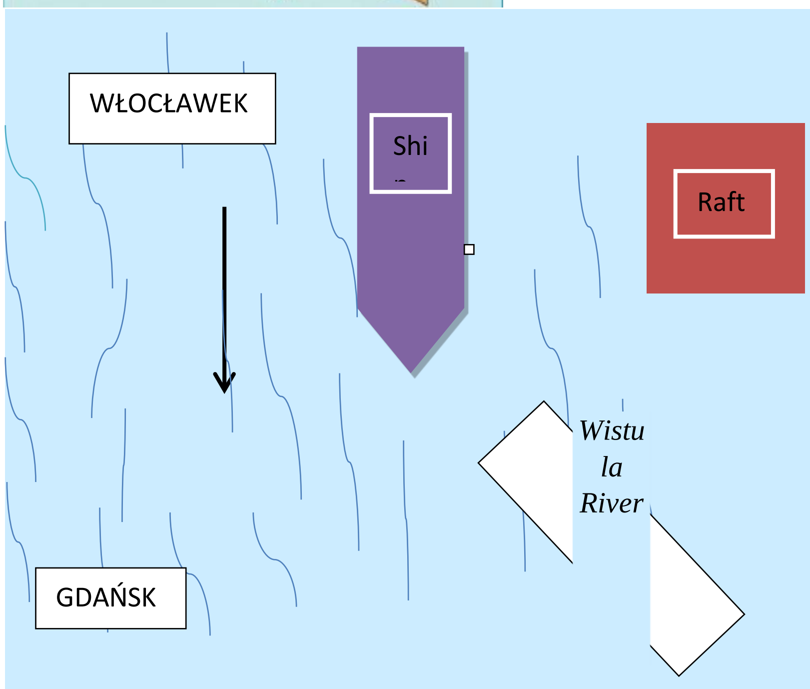
   a) Express the amount of water in the reservoir as a function of time.
   b) How much water was in the reservoir on the eighth of the month?

5. Find the point of intersection of the lines: $y = -x + 5$ and $y = 4x + 3$.

6. Membership in a private fitness club costs 800€ per year and entitles the member to use the courts for a fee of 3€ per hour. At a competing club, membership costs 650€ per year and the charge for the use of the gym is 5€ per hour. If only financial considerations are to be taken into account, how should a client choose which club to join?



7. A river ship with a drive does not stop along its way along the Vistula River from the city of Włocławek in Poland to the city of Gdańsk also in Poland for three days, while from Gdańsk to Włocławek for four days. How many days will the raft sail (without a drive) from Włocławek to Gdańsk?



Teacher presents and discusses with students' parts of following videos:

https://study.com/learn/lesson/what-is-a-linear-equation.html

https://study.com/academy/lesson/linear-equations-intercepts-standard-form-and-graphing.html https://study.com/academy/lesson/abstract-algebraic-examples-and-going-from-a-graph-to-a-rule.html

https://study.com/academy/lesson/graphing-undefined-slope-zero-slope-and-more.html

https://www.calculushowto.com/types-of-functions/linear-function/

Students Activity

- Teacher asks students to find the slope and y −intercept to be sure they understand the meaning of the slope and y-intercept how to do their task and how to use linear functions to real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions: (test on-line: https://docs.google.com/forms/d/1LXFC4dNREOXciQCxWayB0AgKVl6JqYh2QdCMbN7i2kU/edit?usp=sharing)
- Students get sample chapter exam – 6 problems. They have to solve their tasks and show the solutions to teacher next lesson or send the solution to teacher by mail or via EduPlatform (an educational platform functioning in PNA).

APPENDIX 1: Exercises

Exercises

1. If $f(x) = 3x - 2$, compute $f(2), f(0), f(-1)$.
2. If $f(x) = 2x + 6$, find the value of x for which $a)\ f(x) = 6$, $\ b)\ f(x) = 4$.
3. The cost of renting a car is $f(x) = 14 + 0.12x$ €, where x is a number of kilometres driven.

   a) What is the cost of renting a car for 60 km trip?

   b) How much is charged for each additional kilometer?

   c) If the total rental cost was 23,60 €, how far was the car driven?
4. Find the slope and y intercept of each of the following lines:

   a) $2y = 6x + 4$, b) $5x - 4y = 20$, c) $2y + 3x = 0$.
5. Find the equation of the line which passes through the origin $(0, 0)$ and has slope -2.
6. Find the equation of the line through the points $(2, 4)$ and $(1, -3)$.
7. A doctor owns 1,500 € worth of medical books which for tax purposes, are assumed to depreciate at a constant rate over the 10-year period so that at the end of the 10-year period, their value will have been reduced to zero.

   a) Express the value of the books as a function of time.

   b) By how much does the value decrease each year?
8. Find the point of intersection (if any) of the lines:

   a) $y = x + 4$ and $y = -2x + 1$,

   b) $y = 4x + 9$ and $y = 4x - 6$.
9. First plumber charges 16 € plus 6 € per half hour. A second charges 21 € plus 4 € per half hour. If only financial considerations are to be taken into account, how should you decide which plumber to call?
10. A ship sails from Włocławek to Gdańsk two days, while from Gdańsk to Włocławek it sails six days. How many days does the river flow from Włocławek to Gdańsk?
11. a) As a dry air moves upward, it expands and cools. If the ground temperature is $20°C$ and the temperature at the height of 1 km is $10°C$, express the temperature T (in $°C$) as a function of the height h (in km) assuming the function is linear.

    b) draw the graph of the function in part a). What does the slope represent?

    c) What is the temperature at a height of 2.5 km?

ANSWERS

Exercises.

1. $f(2) = 4, f(0) = -2,\ f(-1) = -5$
2. $x = 0,\ x = -1$.

3. a) 21.20 €, b) 12 cents, c) 80 kilometers.

4. a) $a = 3,\ b = 2$ b) $a = 1.25,\ b = -5$ c) $a = -\frac{3}{2},\ \ b = 0$.

5. $y = -2x$.

6. $y = 7x - 10$.

7. a) $y = -150x + 1500$ b) 150 €

8. a) $(-1, 3)$ b) none.

9. Call the first plumber if work will take less than 75 minutes and call the second if work will take more than 75 minutes.

10. 6 days.

11. a) $T(h) = -10h + 20$

b) The rate of change of temperature with respect to height.

c) -5℃.

APPENDIX 2: Sample chapter exam

1. It is estimated that t years from now, the population of a certain community will be

$f(t) = 600t + 12000$.

a) What will the population be 8 years from now?

b) What is the current population?

c) By how much does the population increase each year?

d) When will the population be 15000?

2. Find the slope and y intercept of the given lines: $5x - 4y = 20$, $\frac{x}{3} + \frac{y}{4} = 4$.

3. Write an equation for the line with a given properties: a) through $(1, 4)$ and $(5, 0)$;

b) through $(-1, 3)$ with slope -5.

4. Under the provisions of a proposed property tax bill a homeowner will pay 100 € plus 8% of the assessed value of the house. Under the provisions of a competing bill, the homeowner will pay 1900 € plus 2% of the assessed value of the house. If only financial considerations are taken into account, how should a homeowner decide which bill to support?

Lesson 5. Quadratic functions

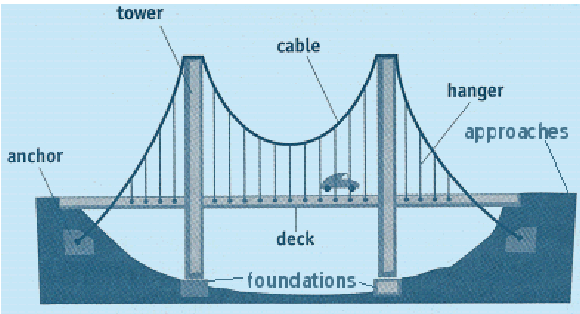
Name of Unit	Workload	Handbook
Quadratic functions	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.4.2.**

DETAILED DESCRIPTION

Quadratic functions hold a unique position in the school and study programs. They are functions whose values can be easily calculated from input values, so they are a slight advance on linear functions and provide a significant move away from attachment to straight lines.

Engineers of all sorts use these equations. They are necessary for the design of any piece of equipment that is curved, such as auto bodies. Automotive engineers also use them to design brake systems. For similar reasons, aerospace engineers work with them on a regular basis.

Quadratic equations are actually used in everyday life, as when calculating areas, determining a product's profit or formulating the speed of an object. Quadratic equations refer to equations with at least one squared variable, with the most standard form being $ax^2 + bx + c = 0$. For example, when working with area, if both dimensions are written in terms of the same variable, we use a quadratic equation. Because the quantity of a product sold often depends on the price, we sometimes use a quadratic equation to represent revenue as a product of the price and the quantity sold. Quadratic equations are also used when gravity is involved, such as the path of a ball or the shape of cables in a suspension bridge.



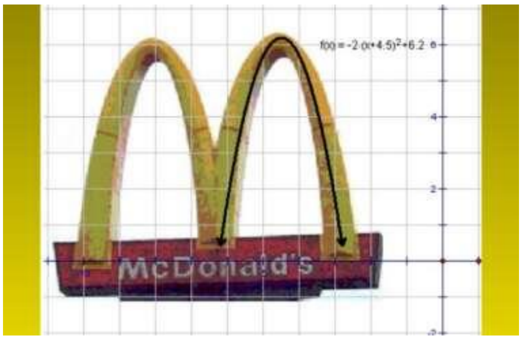
[source: https://mathimages.swarthmore.edu/index.php/Parabolic_Bridges]

AIM: To answer the following questions:

- [Why are quadratic functions important?]
- [How engineers use quadratic equations?]

- [What is the use of quadratic equations in real life?]
- [When do you use a quadratic equation for area?]
- [When do you use a quadratic equation for revenue?]
- In this module students will learn how to graph and analyze quadratic functions and discover many other useful applications in which they play a role.

To acquire skills in solving equations and inequalities and also to understand concepts standing behind those calculations. Quadratic functions are also important for some real-life applications in the maritime field.



Learning outcomes

At the end of this lecture, students should be able to
1. Understand quadratic functions
2. Recognize, sketch and produce graphs of quadratic functions of one variable
3. Use quadratic graphs to estimate values of y for given values of x
4. Solve quadratic equations and inequalities
5. Solve real life problems which are connected with quadratic functions

Key words of this Unit:

- rate of change: ratio between two related quantities that are changing.
- linear equation, slope, point of intersection.

Previous knowledge of mathematics: Content from basic algebraic operations.

Relatedness with solving problems in real life problems, in science: We can point some careers that use quadratic functions, they are: military and law enforcement, quadratic equations are often used to describe the motion of objects that fly through the air, engineers of all sorts use these equations, we can meet them in management and clerical work, agriculture. Problems involving gravity and projectile motion are typically dependent upon a second-order variable, usually time or initial velocity depending on the relationship. Coulomb's Law, which relates electrostatic force, charge amount and distance between two charged particles, has a second-order dependence on the separation of the particles. Solving for either charge results in a quadratic function. Quadratic equations appear often in physics.

The basic kinematic equations for the position x of a particle as a function of time t, with an initial velocity v_0 (a constant) and constant acceleration a can be written as,

$$x(t) = x_0 + v_0 t + 12at^2$$

This is a quadratic function in t. The function therefore gives the position as a quadratic function of time t. If we are dealing with a free-falling object under Earth's gravitational field, we might write this function in the form,

$$h(t) = h_0 + v_0 t + 12gt^2$$

to express the "height" h of the object at a given time t falling with a constant acceleration $g = -9.8m/s^2$. Here h_0 the initial height (a constant). The units for acceleration are meters-per-square second m/s^2. The negative acceleration is a convention to signify that the direction of the acceleration is downward.
Most all equations involving gravity include a second-order relationship. If we consider the equation relating gravitational force F between two objects to the masses of each object (m_1 and m_2) and the distance between them r

$$F = G\frac{m_1 m_2}{r^2}$$

The shape of this function is not a parabola, but becomes such a shape if rearranged to solve for $m_1\ or\ m_2$, as

$$m_2 = \frac{Fr^2}{Gm_1}.$$

The equation relating electrostatic force F between two particles, the particles' respective charges q_1 and q_2, and the distance between them r is very similar to the aforementioned formula for gravitational force:

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}.$$

This is known as Coulomb's Law. Solving for either charge results in a quadratic equation where the charge is dependent on r^2.

And something to smile:

A. The curved shape of a banana closely resembles a parabola. Hence, it is one of the best examples of parabolic objects used in everyday life.

B. A rainbow is a U-shaped seven coloured curve that appears in the sky after or during the rain. Hence, it is yet another example of a parabolic shape in nature.

C. While doing yoga you must have practiced the wheel pose or the chakra-asana. While practicing the wheel pose, the shape of the human body appears to be an inverted U alphabet. Hence, it is one of the best examples of parabola in real life.

D. The jump of a dolphin is known as proposing. Several researches show that dolphins use their jump as a method of communication. If you look at a dolphin proposing, you will observe that they trace a parabolic path while performing the jump. Hence, it is yet another example of parabolic shape in real life.

Useful websites:

https://studiousguy.com/parabola-examples/

https://alison.com/course/introduction-to-quadratic-equations-and-inequalities-for-general-studies

https://www.varsitytutors.com/hotmath/hotmath_help/topics/quadratic-function

List of Training Manuals and Correspondence Courses

Autorzy United States. Naval Education and Training Support Command

Contents:

QUADRATIC FUNCTIONS

- Graph and general form of a quadratic function
- Domain and range of a quadratic function
- Finding the x– and y-intercepts of a quadratic function
- Vieta's formulas
- Solving quadratic equations
- Solving quadratic inequalities
- Exercises
- Sample chapter exam

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define linear functions
- GeoGebra https://www.geogebra.org/
- https://www.geogebra.org/graphing?lang=en
- https://www.padowan.dk/

Graph

Version 4.4

Graph - plotting of mathematical functions

- Work Sheets
- Websites:

— https://shortinformer.com/why-is-quadratic-equation-important/
— https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-6.pdf
— https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/#quadratic
— https://www.padowan.dk/

Lesson : Quadratic functions

LESSON FLOW

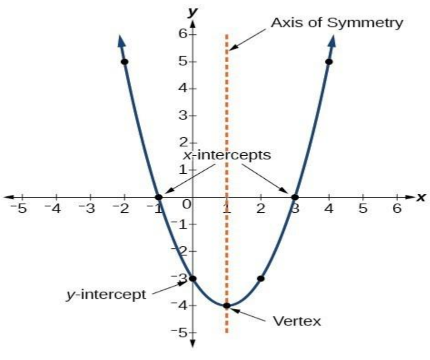
Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
35 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can find quadratic functions ?
45 min	Presentation	Description, definition, General form of quadratic function, standard form of quadratic function, discriminant, x-intercepts, Properties of a quadratic function. Quadratic equations and inequalities, Vieta's formulas	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Are students able to recognize the form of a quadratic function, to sketch the graph, to compute the x-intercepts? Do they know the definition and the meaning of discriminant?
45 min	Presentation Exercises 1-6	Working examples of the quadratic equations and quadratic inequalities	Frontal Explains task and supports	Active listening and contributing to questions Complete the worksheets	Are students able to solve quadratic equations? Are students able to solve quadratic inequalities?
45 min	Time for open learning for students – solving exercises and real life problems		Frontal Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
10 min	Summary		Giving homework		

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

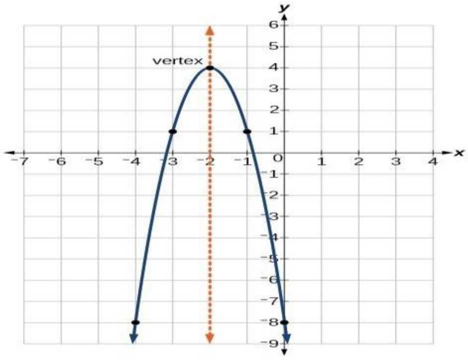
	• **Whiteboard** • Lesson https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-6.pdf • Video https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#quadratic
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve quadratic equations and inequalities, to sketch the graphs of quadratic functions • ***all*** students should be able to apply quadratic functions in some real life problems.

A. The first section is a description of quadratic function, definition of its range and domain, presentation different forms of quadratic functions: general form, standard form, quadratic formula, discriminant, Vieta's formulas, graphs of quadratic functions.

The graph of a quadratic function is a U-shaped curve called a parabola. Important feature of the graph is that it has an extreme point, called the vertex. If the parabola opens upward, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. The graph is also symmetric with a vertical line drawn through the vertex, called the axis of symmetry.



The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$. The **standard form** of a quadratic function is represented by $f(x) = a(x-p)^2 + q$, where $(p,\ q)$ is the vertex.



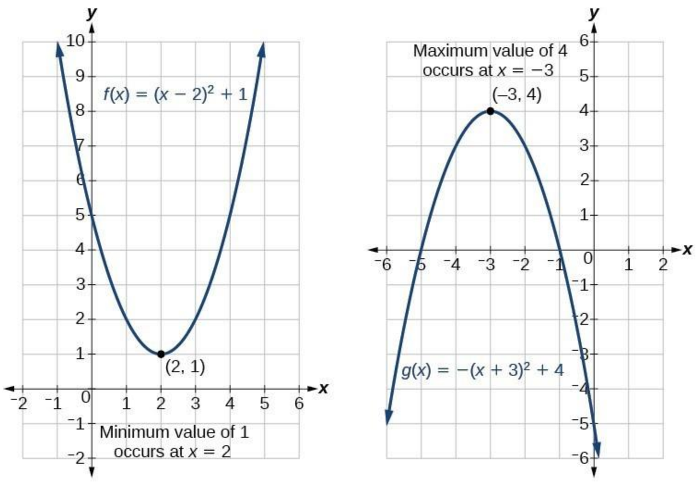
Discriminant of a square trinomial $ax^2 + bx + c$:

$$\Delta = b^2 - 4ac$$

Quadratic formula:

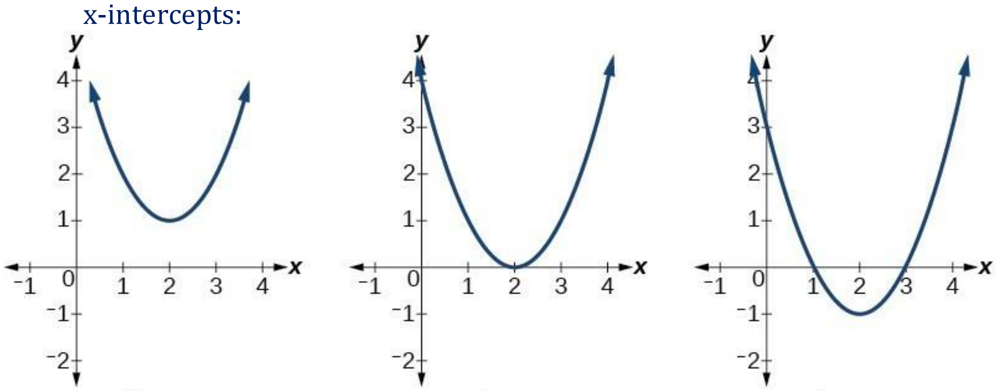
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is a formula for solving quadratic equations in terms of the coefficients.



(a) (b)

x-intercepts:



No x-intercept One x-intercept Two x-intercepts

Vieta's formulas give a simple relation between the roots of a polynomial and its coefficients. In the case of the quadratic equation, they take the following form:

$$x_1 + x_2 = -\frac{b}{a}$$
$$x_1 \cdot x_2 = \frac{c}{a}.$$

B. Teacher shows and remains students how to solve quadratic equations and inequalities of various levels of difficulty:

- $x^2 - 16 = 0$
- $x^2 + 6x = 0$
- $2x^2 - 8x + 6 = 0$
- $x^2 + 6x + 9 = 0$
- $4x^2 + 2x + 1 = 0$
- $x^2 - 5x + 6 \geq 0$

C. Teacher presents and explains applying a quadratic function in physics:

→ Suppose that you head out on a river boat cruise that takes 4 hours to go 20 km upstream and then turn around and go 20 km back downstream. When you get back, you notice that the speedometer of the boat wasn't working during the cruise, so you want to calculate the boat's speed. The river has a current of 3 kilometers per hour.

→ v_b-speed of the boat, $s = v \cdot t$, $t = \frac{s}{v}$,

we can create an equation: $\frac{20}{v_b - 3} + \frac{20}{v_b + 3} = 4$.

D. Teacher shows how to draw graphs and students use GeoGebra or Graph to plot graphs

- Teacher presents a video: https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#quadratic

E. Teacher gives some exercises illustrating the above mentioned. knowledge. Students in groups try to solve them. Teacher introduces some hints and control the solutions, explains any doubts.

Exercises:

1. Solve the quadratic equations and inequalities:
   a. $9x^2 + 42x + 49 = 0$.
   b. $12x^2 - 25x + 12 = 0$.
   c. $x^2 + 4x + 1 = 0$.
   d. $x^4 - 3x^2 + 2 = 0$ Hint: Substitute $x^2 = t,\ t > 0$.
   e. $9 - x^2 \leq 0$.
   f. $x^2 - 8x > 0$.
   g. $-x^2 + 4x - 3 < 0$.
   h. $-x^2 + x - \sqrt{2007} < 0$.
   i. $x^2 + \sqrt{1999}x + 500 < 0$.
   j. $x(x + 1) \leq 0$.

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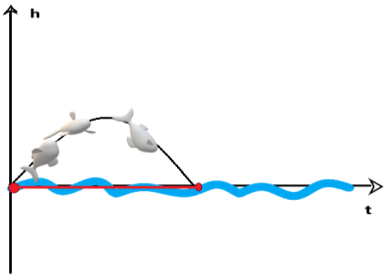
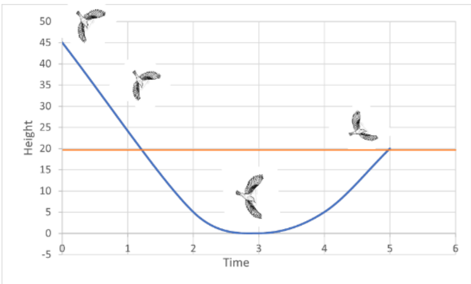
2. A dolphin jumps out of the sea with an initial velocity of 20 feet per second (assume its starting height is 0 feet). Use the vertical motion model, $h = -16t^2 + vt + s$, where v is initial velocity in feet/second and s is the dolphin starting height in feet, to calculate the amount of time the dolphin is in the air before it hits the water again (see the Figure below). Round you answer to the nearest tenth if necessary.



3. An osprey, a fish-eating bird of prey, dives toward the water to a salmon.

The height h, in meters of the osprey above the water t seconds after it begins its dive can be approximated by the function $h(t) = 5t^2 - 30t + 45$. Determine the time it takes for the osprey to reach a return height of 20 m. See figure below.



4. A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, is given the equation $h = -4.9t^2 + 52t + 376$. At what time does the rocket hit the sea? (Round answer to 2 decimal places).

Answers

1.

a. $x = -\frac{7}{3}$

b. $x_1 = \frac{3}{4}, \quad x_2 = \frac{4}{3}$

c. $x_1 = -2 - \sqrt{3}, \quad x_2 = -2 + \sqrt{3}$

d. $x_1 = 1, \quad x_2 = -1\,, \quad x_3 = \sqrt{2}, \quad x_4 = -\sqrt{2}$

e. $x \in (-\infty, -3] \cup [3, \infty)$

f. $x \in (-\infty, 0) \cup (8, \infty)$

g. $x \in (-\infty, 1) \cup (3, \infty)$

h. $x \in \mathbb{R}$

i. No solutions

j. $x \in [-1, 0]$.

2. The dolphin is in the air for $\frac{5}{4} = 1.25 \approx 1.3$ seconds.

3. It takes 5 seconds for the osprey to reach a return height of 20 m.

4. $t = 15{,}55$ seconds.

F. Teacher presents and discusses with students parts of following videos:

→ https://www.schooltube.com/media/Properties-of-Quadratic-Functions/1_mymyqyhz
→ https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:quadratic-functions-equations/x2f8bb11595b61c86:standard-form-quadratic/v/application-problem-with-quadratic-formula
→ https://www.khanacademy.org/math/algebra-home/alg-quadratics/alg-quadratic-inequalities/v/quadratic-inequalities-visual-explanation
→ https://www.youtube.com/watch?v=V2SSgCZQNRU

Students Activity

- Teacher asks students to find the slope and y −intercept to be sure they understand the meaning of the slope and y-intercept how to do their task and how to use linear functions to real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions: (test on-line:

https://docs.google.com/forms/d/1PxrO00eM_g2JDhjnFa0BjaOiMuyedE4OEFSf5qidPM4/edit?usp=sharing)

- Students get sample chapter exam – 6 problems. They have to solve their tasks and show the solutions to teacher next lesson or send the solution to teacher by mail or via EduPlatform (an educational platform functioning in PNA).

APPENDIX 1: Sample chapter exam

SAMPLE CHAPTER EXAM

1. Sketch the following polynomials:

a. $f(x) = (x-3)(x+6)$
b. $f(x) = x^2 + 4x + 1.$

2. Find the $y-$ and $x-$intercepts of a parabola $f(x) = 2x^2 - 3x - 2.$

3. Find the domain and range of $f(x) = 3x^2 + 9x - 1.$

4. Find the vertex of a quadratic function $f(x) = x^2 - x - 2.$ Rewrite the quadratic in standard form (vertex form).

5. Solve the equations:

a. $x^2 - 2x - 15 = 0.$
b. $x^2 + 4x + 15 = 0.$

6. Solve the inequalities:

a. $x^2 - x < 2$
b. $x^2 + 1 \geq 2x^2 - x.$

7. The seaman launch the flare from a crow's nest of a height of 5 meters. The height (h, in meters) of the flare t seconds after taking off is given by the formula:

$$h = -3t^2 + 14t + 5.$$

a. How long will it take for the flare to hit the sea?

b. Find the time when the flare is 5 meters form hitting the sea. (See short film: http://xurl.pl/aqul)

Lesson 6. Exponential functions

Name of Unit	Workload	Handbook
Exponential functions	Lecture: 90 min Exercises: 90 min	Unit 5.4.3.

DETAILED DESCRIPTION

The unit Exponential functions begins with the definition and properties of the exponential function and then presents the graphs of exponential function with various bases. From the graphs, student is able to identify the properties of the function.

Knowledge of exponential functions is important in developing students' understanding of solving exponential equations and inequalities. The fact that $e^{x+y} = e^x \cdot e^y$ together with its continuity and e^0 =1 make the exponential function unique. Then, it is the unique solution of $f' = f$, $f(0) = 1$ which is the basis for very many things about differential equations and dynamical systems. $e^{iz} = \cos(z) + i\sin(z)$ is fundamental for trigonometry. Exponential function is central for Fourier analysis, $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ is the density of the most important distribution in probability theory. Exponential decay occurs naturally when a quantity is decaying at a rate which is proportional to how much is left. Exponential growth occurs when a quantity is increasing at a rate proportional to the current amount. A positive time constant corresponds to exponential decay, while a negative time constant corresponds to exponential growth. In signal processing, we almost always deal exclusively with exponential decay (positive time constants). Exponential functions are indispensable in science since they can be used to determine growth rate, decay rate, time passed or the amount of something at a given time. This module describes methods of solving exponential equations, inequalities and shows how they are graphed. Sample problems, including a look at the compound of interest, exponential decrease illustrate how exponential functions are used in the real world.

AIM: To acquire skills in solving equations and inequalities and also to understand concepts standing behind those calculations. Exponential functions are also important for some real-life applications in the maritime field.

Learning outcomes

At the end of this lecture, students should be able to

1. Know the definition of exponential function
2. Plot a graph of exponential functions with different bases and read properties of the function from the graph.
3. Know the basic rules for exponentials.
4. Solve exponential equations.

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5. Solve exponential inequalities
6. Solve practical problems like compound interest, depreciation formula, exponential decay.

Key words of this Unit:

Base, Exponent, Exponential equation, exponential inequality, exponential decay, compound of interest.

Previous knowledge of mathematics: Content from basic algebraic operations, functions, graphs, quadratic equations, quadratic inequalities.

Relatedness with solving problems in the maritime field: Exponential functions are important to understand exponential population growth, exponential disease transmission (i.e. COVID 19) [https://www.cebm.net/covid-19/exponential-growth-what-it-is-why-it-matters-and-how-to-spot-it/], exponentially worse wildfires caused by climate changes. Exponential equations are applied in a wide variety of situations in science, from modeling the spread of a viral disease in a population to estimating the atmospheric pressure at a given altitude to chain reactions in nuclear fission. All of these processes involve a geometric progression: One person with a virus can infect ten others, for example, and each of those ten people can infect ten more. In all of these cases, real-world data can be modeled using exponential equations, and these equations can provide predictions of future behavior. Solving exponential equations is a valuable tool for finding variables such as growth rate, decay rate, the amount of time that has passed, or an amount of something at a given time.

Contents:

Assessment strategies:

Evaluating student's activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define exponential functions
- GeoGebra https://www.geogebra.org/ https://www.geogebra.org/graphing?lang=en
- https://www.padowan.dk/

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Graph

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Graph - plotting of mathematical functions

- Work Sheets
- Websites:

— https://ccrma.stanford.edu
— https://www.geogebra.org/t/exponential-function
— https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/
— https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-explogfns-2009-1.pdf https://www.padowan.dk/
— https://www.visionlearning.com/en/library/Math-in-Science/62/Exponential-Equations-in-Science-I/206
— https://users.math.msu.edu/users/liuqinbo/Chapter4-103.pdf
— https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:exponential-growth-decay

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Lesson : Exponential functions

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
25 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use exponential function ?
40 min	Presentation	Definition, Rules of exponents, properties	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Are students able to recognize the difference between exponential functions with different bases
55 min	Presentation Exercise 1-3	EXPONENTIAL EQUATION Exponential inequalities	Frontal Explains task and supports	Active listening and contributing to questions Complete the worksheets	Are students able to solve equations and inequalities with bases a>1 and 0 <a<1.
55 min	Presentation	Compound interest, Depreciation formula exponential decrease	Frontal Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
15 min	Summary		Giving homework		

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/ https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-7.pdf
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve exponential equations • ***all*** students should be able to solve exponential inequality.

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A. The first section is a definition of exponential function, its properties, graphs. Then remaining rules od exponentials.

1. A function of the form

$$y = a^x$$

is called **exponential**, x is any number, $a > 0$, $a \neq 1$,

a is called **the base**, while x is called **the exponent**

The domain of the exponential function is $\mathbb{R}$, while the range is $(0, \infty)$.

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MareMathics Summer School – Split 2021, 20– 24 September

The exponential function we define in four stages:

1. If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}};$$

2. If $x = 0$, then $a^0 = 1$;
3. If $x = -n$, where n is a positive integer, then $a^{-n} = \frac{1}{a^n}$;
4. If x is a rational number, $x = \frac{p}{q}$, where p, q are inegers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

B. Teacher shows students that they can use Graph or GeoGebra to plot graphs of exponential functions with different bases and on the basis of the graph recognize properties of those functions.

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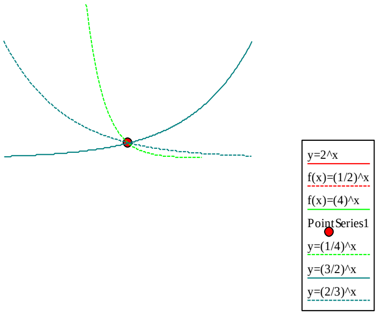

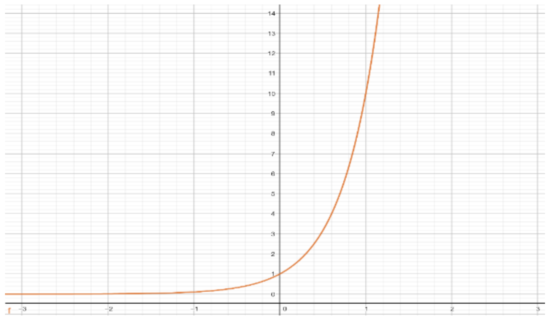

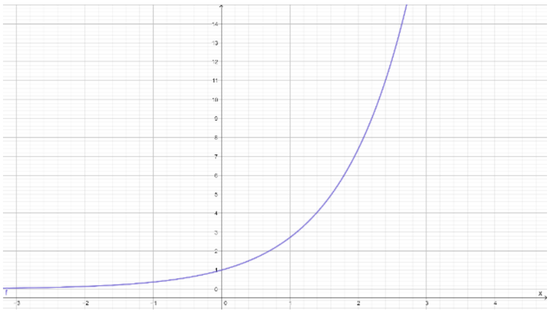

https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/#exp-graphs

C. Teacher reminds the rules of exponentials:

Rule or special case	**Formula** $a > 0, \quad x \in \mathbb{R}, y \in \mathbb{R}$	**Example**
Product	$a^x a^y = a^{x+y}$	$2^3 2^4 = 2^7$
Quotient	$\frac{a^x}{a^y} = a^{x-y}$	$\frac{2^4}{2^3} = 2^{4-3} = 2$
Power of power	$(a^x)^y = a^{x \cdot y}$	$(2^3)^2 = 2^{2\cdot 3} = 2^6$=64
Power of a product	$(a \cdot b)^x = a^x b^x$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144$
Power of one	$a^1 = a$	$2^1 = 2$
Power of zero	$a^0 = 1$	$2^0 = 1$

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Power of negative one	$a^{-1} = \frac{1}{a}$	$2^{-1} = \frac{1}{2}$
Change sign of exponents	$a^{-x} = \frac{1}{a^x}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
Fractional exponents	$a^{\frac{x}{y}} = \sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$	$4^{\frac{3}{2}} = \sqrt{4^3} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

D. Teacher shows how to solve exponential equation and inequalities. Recalls and asks students about the rules of exponentials, how to solve quadratic equations, plot graphs of quadratic functions (students can use GeoGebra or Graph to plot graphs)

1. Solve the following equations and inequalities.

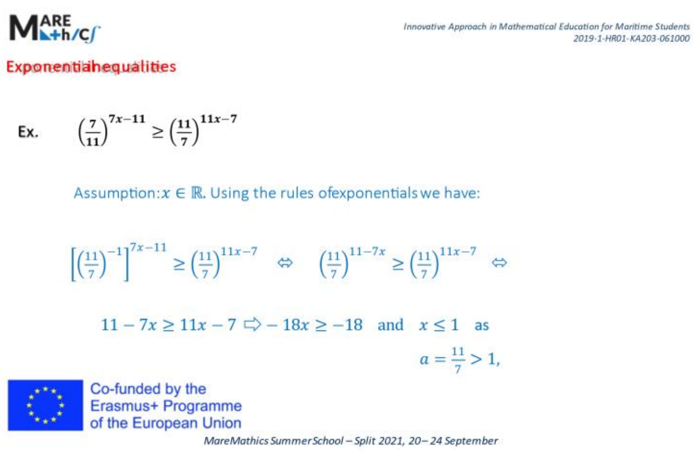
a. $2^x + 2^{x+1} + 2^{x+2} = 6^x + 6^{x+1}$

Solution

Assumption: $x \in \mathbb{R}$. Using the rules of exponentials we have:

$$2^x + 2 \cdot 2^x + 2^2 \cdot 2^x = 6^x + 6 \cdot 6^x \;\Leftrightarrow\; 2^x(1+2+4) = 6^x(1+6) \;\Leftrightarrow$$
$$2^x = 6^x \;\Leftrightarrow\; 1 = \frac{6^x}{2^x} \;\Leftrightarrow\; 1 = \left(\frac{6}{2}\right)^x \;\Leftrightarrow\; 1 = 3^x \;\Leftrightarrow\; x = 0.$$

Teacher presents a video: https://marematHics.pfst.hr/index.php/2021/09/13/5-functions/#exp-equation



Teacher asks a student to solve the exercise, it requires to reminds the rules of solving also quadratic inequalities:

c. $2^{2x} \leq 3 \cdot 2^{x+\sqrt{x}} + 4 \cdot 2^{2\sqrt{x}}.$

Solution

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Assumption: $x \geq 0$. Let us divide both sides of the inequality by positive $2^{2\sqrt{x}}$ and obtain

$$2^{2x-2\sqrt{x}} \leq 3 \cdot 2^{x+\sqrt{x}-2\sqrt{x}} + 4 \quad \Leftrightarrow \quad 2^{2(x-\sqrt{x})} \leq 3 \cdot 2^{x-\sqrt{x}} + 4 \;.$$

Now we substitute $2^{x-\sqrt{x}} = t,\ t > 0$. Then we get

$$t^2 \leq 3t + 4 \quad \Leftrightarrow \quad t^2 - 3t - 4 \leq 0 \quad \Leftrightarrow \quad (t-4)(t+1) \leq 0\,.$$

As we see in Fig.3.4. $-1 \leq t \leq 4 \;\Leftrightarrow\; -1 \leq 2^{x-\sqrt{x}} \leq 4$.

Hence $2^{x-\sqrt{x}} \leq 4 \;\Leftrightarrow\; 2^{x-\sqrt{x}} \leq 2^2 \;\Leftrightarrow\; x - \sqrt{x} \leq 2$.

Now we use substitution $\sqrt{x} = u,\ u \geq 0$ and obtain

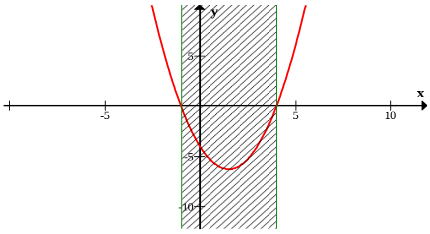

Fig.3.4. The illustration of $x^2 - 3x - 4 \leq 0$.

$$u^2 - u \leq 2 \quad \Leftrightarrow \quad u^2 - u - 2 \leq 0 \quad \Leftrightarrow \quad (u+1)(u-2) \leq 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -1 \leq u \leq 2 \quad \text{(see Fig.3.5.)}$$

$$\Leftrightarrow \quad -1 \leq \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \begin{cases} x \geq 0 \\ x \leq 4 \end{cases} \quad \Leftrightarrow \quad x \in [0, 4].$$

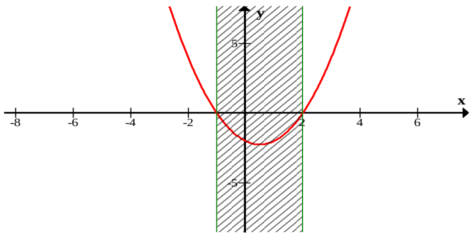

Fig.3.5. The illustration of $x^2 - x - 2 \leq 0$.

E. Teacher introduces some practical applications of exponential functions and explains students the possibility of meeting exponential functions in real life problems. Teacher gives students links to some useful articles or websites.

1. Compound interest

Most people who have a savings account with a bank or other financial institution leave their deposits for a period of time expecting to accrue money as time passes. If the deposits are made in an account carrying **simple interest** (flat rate of interest) the interest received is calculated on the original deposit for the duration of the account.

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This would mean that if one invested 1 000 € at a **flat interest rate** of 3.5% then in the first year he or she would have accrued:

total earned= principal + 3.5% of principal over 1 year:

$$= 1000 + \frac{3,5}{100} \cdot 1000 \cdot 1 = 1000(1 + 0.035) = 1035 \text{ €}.$$

The compound interest formula is as follows

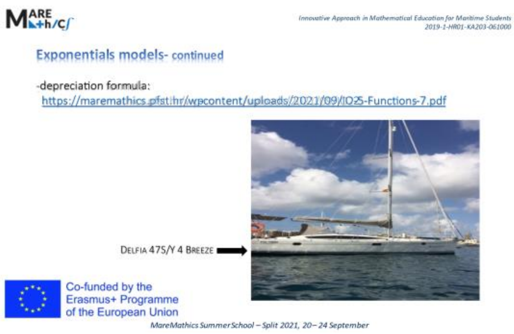
$$A = P\left(1 + \frac{r}{100}\right)^n,$$

where A is the total amount returned, P is the principal (initial amount)?
r is the rate as a percentage returned in each investment period and n is the number of investment periods.

2. Depreciation formula:

$$D = P(1 - \frac{r}{100})^n,$$

where D is final value of the asset, P is the initial value of the asset, r is the rate of depreciation per period and n is the number of depreciation periods.
Suppose in 2010 a man purchased a yacht Delfia 47S/Y 4 Breeze valued at \$275 000. We know that yacht depreciate at 11.2% each year. What would the value of the yacht be after a period of time? Examine the table below for calculations for 5 years.



3. Exponential decrease can be modeled as:

$$N(t) = N_0 e^{-\lambda t}$$

where N is the quantity, N_0 is the initial quantity, λ is the decay constant (specific for each element), and t is time.
Oftentimes, half-life is used to describe the amount of time required for half of a sample to decay. It can be defined mathematically as:

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

where $t_{1/2}$ is half-life.
Half-life can be inserted into the exponential decay model as such:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}.$$

Remark

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- **half-life**: The time it takes for a substance (drug, radioactive nuclide, or other) to lose half of its pharmacological, physiological, biological, or radiological activity.
  Imagine we have 100 kg of a substance with a half-life of 5 years. Then in 5 years half the amount (50 kg) remains. In another 5 years there will be 25 kg remaining. In another 5 years, or 15 years from the beginning, there will be 12.5. The amount by which the substance decreases, is itself slowly decreasing.
- **isotope**: Any of two or more forms of an element where the atoms have the same number of protons, but a different number of neutrons. As a consequence, atoms for the same isotope will have the same atomic number but a different mass number (atomic weight).

Teacher presents and discusses with students' parts of following videos:

https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/#exp-decay

https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/#exp-grow

https://maremathics.pfst.hr/index.php/2021/09/13/5-functions/#exp-models

Students Activity

- Teacher asks students to solve exponential equations and inequalities to be sure they understand how to do their task and how to use exponentials to real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions: (test on-line: https://docs.google.com/forms/d/1sj4gmZlYDqDtxY4R924YWqre4HcZYhKyKjkKr7nhkV0/edit?usp=sharing)
- Students get sample chapter exam – 6 problems. They have to solve their tasks and show the solutions to teacher next lesson or send the solution to teacher by mail or via EduPlatform (an educational platform functioning in PNA).

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APPENDIX 1: Exercises

1. Solve the equations

a) $3^{x+2} - 3^{x-1} = \frac{26}{9}$. $\qquad x = -1$

b) $3 \cdot 5^x - 2 \cdot 5^{x-1} = 5^{x+1} - \frac{12}{5}$. $\qquad x = 0$

c) $\frac{3}{10} \cdot \left(\frac{3}{2}\right)^{x-2} = \frac{6}{5}\left(\frac{3}{2}\right)^{x-3} - \frac{1}{2}$. $\qquad x = 2$

2. Solve the inequalities

a) $\left(\frac{8}{9}\right)^{8x^2-9} \geq \left(\frac{9}{8}\right)^{9x^2-8}$. $\qquad x \in [-1,\ 1]$

b) $\left(\frac{1}{2}\right)^{2x^2+x-1} > \left(\frac{1}{4}\right)^{\frac{1}{2}x^2+x-\frac{1}{8}}$ $\qquad x \in \left(-\frac{1}{2}, \frac{3}{2}\right)$

c) $2^{x+3} - 5^x < 7 \cdot 2^{x-2} - 3 \cdot 5^{x-1}$ $\qquad x \in (3, \infty)$

d) $7^{-x} - 3 \cdot 7^{x+1} > 4$ $\qquad x \in (-\infty, -1]$.

3. Suppose in 2020 a man purchased a motor yacht Chris Craft Launch 25GT valued at $234 750. We know that yacht depreciate at 11.2% each year. What would the value of the yacht be in 2026? $ 115102.14

4. The decay of radium is modelled by the function $R = R_0 e^{-0{,}077t}$, where R is the amount remaining (g), t is time (weeks) and R_0 is the original amount. Generate a table of values to find the half-life of 10 g of radium.(Remember that half-life means time to reach half of the original amount).

Answer:

Original function	
Weeks t	**Radium $f(t)$ (g)**
0	10,00
1	9,26
2	8,57
3	7,94
4	7,35
5	6,80
6	6,30
7	5,83
8	5,40
9	**5,00**

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APPENDIX 2: Sample chapter exam

1. Solve the equation: $2 \cdot 4^{\sqrt{x}} = \sqrt[4]{2} \cdot 8^{x-1}$.

2. Solve the inequality: $5^x - 20 > 5^{3-x}$.

3. Find all the values of x for which $f(x) > 0$, if $f(x) = \left(\frac{3}{5}\right)^{x^2-x-6} - 1$.

4. There are given functions: $f(x) = 4^{x+1} - 7 \cdot 3^x$ and $g(x) = 3^{x+2} - 5 \cdot 4^x$. Solve the inequality $f(x) \leq g(x)$.

5. Find the domain and a range of a function $f(x) = e - e^x$.

6. The equation $P = 20 \cdot 10^{0,1n}$ can be used to convert any number of decibels (n) to the corresponding number of micropascals (P) used to measure loudness. Show that a 60 decibel sound is 10 times as loud as a 50 decibel sound, and 100 times as loud as a 40 decibel sound.

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Lesson 7. Logarithmic functions

Name of Unit	Workload	Handbook
Logarithmic functions	Lecture: 90 min Exercises: 90 min	Unit 5.4.4. Functions

DETAILED DESCRIPTION

Like many types of functions, the exponential function has an inverse. This inverse is called the logarithmic function, and it is the focus of this chapter. The first section explains the meaning of the logarithmic function $f\,(x) = k \cdot \log_a(x - p) + q$. It describes how to evaluate logarithms and how to graph logarithmic functions. This section also addresses the domain and range of a logarithmic function, which are inverses of those of its corresponding exponential function.

The second section presents two special logarithmic functions-the common logarithmic function and the natural logarithmic function. The common logarithm is $\log x$, and it corresponds to the "log" button on most calculators. The natural logarithm is $\log_e x = \ln x$ and it corresponds to the "ln" button on most calculators. The natural log has a particular use in economics-it is used to perform calculations involving compound interest. This section addresses these calculations.

The lesson e deals with the properties of logarithms. The properties discussed in this section are helpful in evaluating logarithmic expressions by hand or using a calculator. They are also useful in simplifying and solving equations containing logarithms or exponents, which is the focus of the final section.

Logarithmic functions are important largely because of their relationship to exponential functions. Logarithms can be used to solve exponential equations and to explore the properties of exponential functions. They will also become extremely valuable in calculus, where they will be used to calculate the slope of certain functions and the area bounded by certain curves. In addition, they have practical applications specially in economics.

AIM: To acquire skills in solving equations and inequalities and also to understand concepts standing behind those calculations. Logarithmic functions are also important for some real-life applications we can meet also in the maritime field.

Learning outcomes

At the end of this lecture, students should be able to:

1. Know the definition of logarithm and logarithmic function;
2. Plot a graph of logarithmic functions with different bases and read properties of the function from the graph;
3. Know the basic rules for logarithms;
4. Solve logarithmic equations;
5. Solve logarithmic inequalities.
6. Solve some practical problems like measuring loudness, stellar magnitude etc.

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Key words of this Unit:

Base, Logarithm, common logarithm, natural logarithm, logarithmic equation, logarithmic inequality.

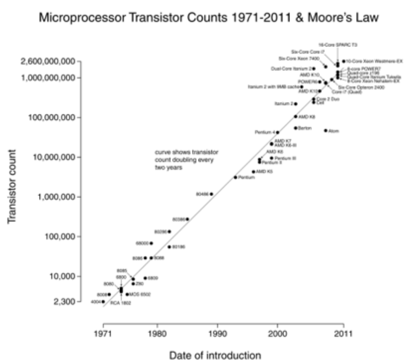
Previous knowledge of mathematics: Content from basic algebraic operations, functions, graphs, quadratic equations, quadratic inequalities, exponents, exponential equations, exponential inequalities.

Relatedness with solving problems in the maritime field: *Why are logarithms useful?*
Humans use logarithms in many ways in everyday life, from the music one hears on the radio to keeping the water in a swimming pool clean. They are important in measuring the magnitude of earthquakes, radioactive decay and population growth. In the financial world they help in the calculation of interest rates

- **Logarithms put numbers on a human-friendly scale.**

Large numbers break our brains. Millions and trillions are "really big" even though a million seconds is 12 days and a trillion seconds is 30,000 years. The trick to overcoming "huge number blindness" is to write numbers in terms of "inputs" (i.e. their power base 10). We're describing numbers in terms of their digits, i.e. how many powers of 10 they have (are they in the tens, hundreds, thousands, ten-thousands, etc.). Adding a digit means "multiplying by 10", i.e.: 1[1 digit]$\cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5 = 100{,}000$. Logarithms count the number of multiplications *added on*, so starting with 1 (a single digit) we add 5 more digits (10^5) and $100{,}000$ get a 6-figure result. Talking about "6" instead of "One hundred thousand" is the essence of logarithms. It gives a rough sense of scale without jumping into details

- Logarithms are how we figure out how fast we're growing.
- **Measurement Scale: Richter, Decibel.** The idea is to put events which can vary drastically (earthquakes) on a single scale with a small range (typically 1 to 10). Decibels are similar, though it can be negative. Sounds can go from intensely quiet (pin drop) to extremely loud (airplane) and our brains can process it all. In reality, the sound of an airplane's engine is millions (billions, trillions) of times more powerful than a pin drop, and it's inconvenient to have a scale that goes from 1 to a gazillion. Logs keep everything on a reasonable scale.
- Logarithmic graphs. We often see items plotted on a "log scale". This means one side is counting "number of digits" or "number of multiplications", not the value itself. Again, this helps show wildly varying events on a single scale (going from 1 to 10, not 1 to billions). Moore's law is a great example: we double the number of transistors every 18 months (image courtesy Wikipedia).



- Logarithms find the root cause for an effect (see growth, find interest rate).

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➡ They help count multiplications or digits, with the bonus of partial counts.

Meanwhile some other problems do directly use logarithms like radioactive dating. It is just a tool for figuring things out. That being said, logarithmic scales are used a lot, and to actually use them in a useful way, we have to involve exponents and logarithms. The invention of logarithms as far as dealing with complex calculation is concerned is second only in importance to the invention of the decimal and place value system (including the zero concept) pioneered by Indian and Arab mathematicians. As Lord Moulton said, *“The invention of logarithms came to the world as a bolt from the blue“.* (https://royalsocietypublishing.org/doi/10.1098/rsnr.2013.0056)

Summarising: If the music at a party is above the number of decibels set by noise regulation of the local authority, the police have the authority to issue a citation to the responsible party. Decibels, derived from “bels,” are units useful in measuring many types of wave functions of from acoustics, electronics and telecommunications. The bel is a base 10 logarithmic function. If the pH of the swimming pool is too high, algae is more likely to grow in the water. Pool owners adjust the pH in their pools to keep the water clear and ensure the comfort of swimmers. This logarithmic function measures the concentration of hydrogen ions in solution. Scientists use the Richter scale to compare the seriousness of earthquakes. The increase from a 4.0 earthquake to a 5.0 quake is tenfold. Bankers use natural logarithms to calculate the time required for a sum of money deposited at an interest rate to reach the desired balance.

Read more at https://www.amansmathsblogs.com/real-life-scenario-logarithms/
https://prezi.com/hlhkzzwu2l0g/logarithms-in-real-life-applications/

Contents:

8. LOGARITHMIC FUNCTIONS

Assessment strategies:

Evaluating students activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define logarithmic functions
- GeoGebra https://www.geogebra.org/ https://www.geogebra.org/graphing?lang=en
- https://www.padowan.dk/

Graph

Version 4.4

Graph - plotting of mathematical functions

- Work Sheets
- Websites:

- https://www.youtube.com/watch?v=1dUSNdZspQc
- https://www.youtube.com/watch?v=jBKnzcboJ84
- Demystifying the Natural Logarithm (ln)
- A Visual Guide to Simple, Compound and Continuous Interest Rates
- Using Logarithms in the Real World
- https://www.amansmathsblogs.com/real-life-scenario-logarithms/
- https://janav.wordpress.com/2013/09/29/logarithms-in-real-life/

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Lesson : Logarithmic functions

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
30 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can find logarithmic function ?
30 min	Presentation	Definition, graphs Properties of logarithms Common logarithm, natural logarithms	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Are students able to recognize the difference between logarithmic functions with different bases. Are they able to plot logarithmic graphs?
45 min	Presentation Examples	Logarithmic equations and inequalities	Frontal Explains task and supports	Active listening and contributing to questions Complete the worksheets	Are students able to solve equations and inequalities with bases $a>1$ and $0<a<1$.
30 min	Presentation	Measuring loudness, stellar magnitude	Frontal Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
40 min	Active students work	Solving equations and inequalities	Encouraging students to solve examples	Students solve their task, present their solutions, compare solutions with the proper answers	
5 min	Summary		Giving homework		

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#logarithms • Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#logarithms • Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#log-ex2 • Lesson https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-8.pdf
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve logarithmic equations • ***all*** students should be able to solve logarithmic inequality.

A. The first section is a definition of logarithmic function, its properties, graphs. Then remaining properties of logarithms.

LOGARITHMIC FUNCTIONS

Logarithmic functions are the inverses of exponential functions and any exponential function can be expressed in logarithmic form.

- ❑ Similarly, all logarithmic functions can be rewritten in exponential form.

A logarithmic function is a function of the form

$$y = log_a x, \quad x > 0, \quad a > 0, \quad a \neq 1,$$

$$y = log_a x \quad \text{is equivalent to } x = a^y.$$

There are no restrictions on y.

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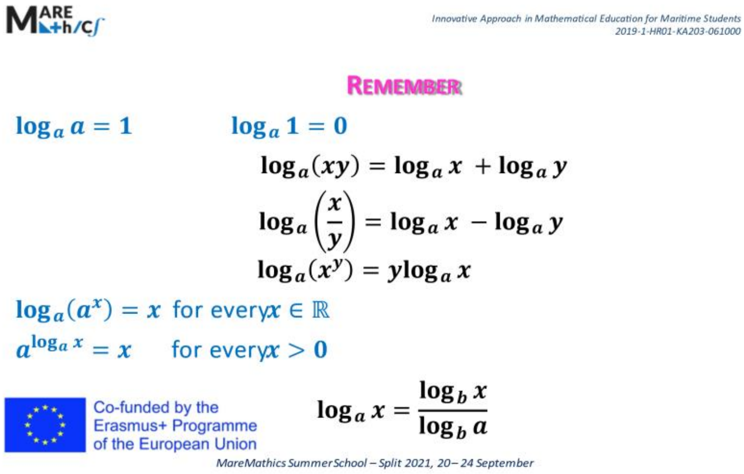
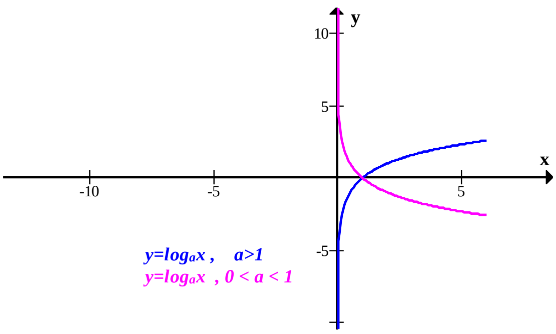
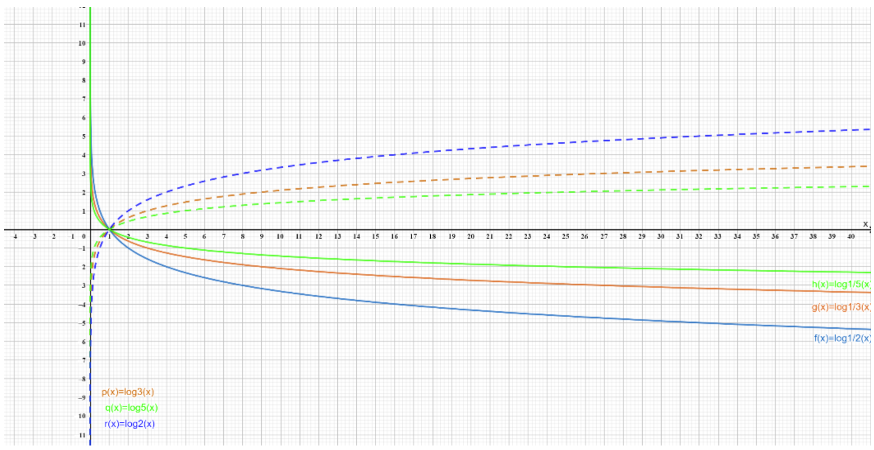


B. Teacher shows students that they can use Graph or GeoGebra to plot graphs of exponential functions with different bases and on the basis of the graph recognize properties of those functions.





Graphs of logarithmic function when the base $a > 1$ and $0 < a < 1$.

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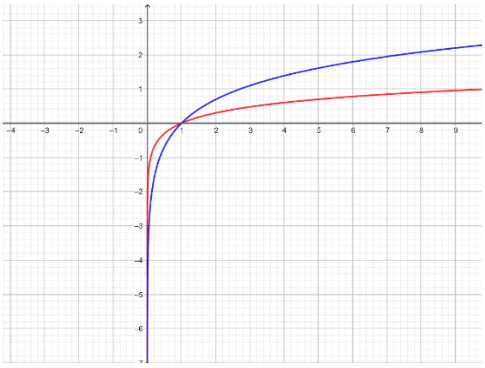

Graphs of $y = \log x$, $y = \ln x$.

C. Teacher reminds the properties of logarithms:

Theorem: If $a > 1$, the function $f(x) = \log_a x$ is one-to-one, continuous, increasing function with domain $(0, \infty)$ and range R. If $x, y > 0$, then

1. $\mathbf{\log_a(xy) = \log_a x + \log_a y}$
2. $\mathbf{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$
3. $\mathbf{\log_a(x^y) = y\log_a x}$

For any positive number a, $a \neq 1$, we have **4.** $\mathbf{\log_a x = \frac{\ln x}{\ln a}}$.

Generally, if we need to change the base a for another, let say b , we can do it as follows **5.** $\mathbf{\log_a x = \frac{\log_b x}{\log_b a}}$.

D. Teacher shows how to solve exponential equation and inequalities. Recalls and asks students about the rules of exponentials, how to solve quadratic equations, plot graphs of quadratic functions (students can use GeoGebra or Graph to plot graphs)

Examples.

1. a. Evaluate $\log_4 2 + \log_4 32$.
   b. Evaluate $\log_2 80 - \log_2 5$.
   c. Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Teacher presents parts of videos:

Solving problems with logarithms

- https://www.youtube.com/watch?v=CZeTfXCnaRk
- https://www.youtube.com/watch?v=IEa5vHNWQ_
- Logarithmic models https://marematthics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-8.pdf

E. Teacher asks a student to solve the exercise, it requires to reminds the rules of solving also quadratic inequalities:

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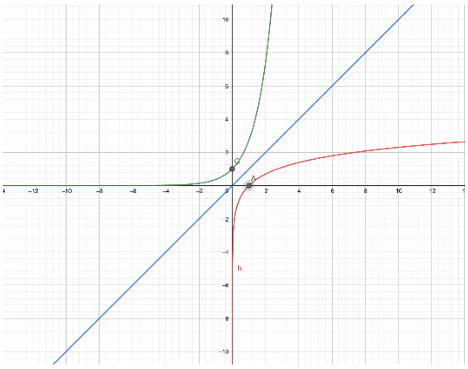
Exercises.

1. **Determine which of the numbers are greater: $\log_3 222$ or $\log_2 33$.**

Solution:

As the function $f(x) = \log_a x,\, a > 1$ is increasing, we have

$$\log_3 222 < \log_3 243 = \log_3 3^5 = 5 = \log_2 2^5 = \log_2 32 < \log_2 33.$$



Graphs of $y = \ln x,\, y = e^x$ are symmetric with regard to $y = x$.

2. **Solve the equations:**

a. $\log(3-x)(x-5) = \log(x-3) + \log(5-x)$.
b. $\log_3(4\cdot 3^{x-1} - 1) = 2x - 1$.
c. $\log(5x^2 + 2x - 1) - \log(x+2) = 1$.

Examples of solutions

a. As $(3-x)(x-5) = (x-3)(5-x)$, so we can write

$$\log(x-3)(5-x) = \log(x-3) + \log(5-x).$$

So the equation is satisfied if and only if

$$(x-3>0 \text{ and } 5-x>0) \Leftrightarrow (x>3 \text{ and } x<5) \Leftrightarrow 3<x<5.$$

b. Assumption: $4\cdot 3^{x-1} - 1 > 0,\ 3^{x-1} > \frac{1}{4}$ and we take log base 3 of both sides of the inequality and we get

$$x - 1 > \log_3 \tfrac{1}{4},$$
$$x - 1 > -\log_3 4,$$
$$x > 1 - \log_3 4,$$
$$x > \log_3 3 - \log_3 4,$$
$$x > \log_3 \tfrac{3}{4}.$$

Now:

$$\log_3(4\cdot 3^{x-1} - 1) = \log_3(3^{2x-1})$$

$$4\cdot 3^{x-1} - 1 = 3^{2x-1} \quad |\cdot 3$$

$$4\cdot 3^x - 3 = 3^{2x} \Leftrightarrow (3^x)^2 - 4\cdot 3^x + 3 = 0.$$

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Let $3^x = t,\ t > 0$, then

$$t^2 - 4t + 3 = 0 \ \Leftrightarrow\ (t-1)(t-3) = 0 \ \Leftrightarrow\ t = 1 \text{ or } t = 3.$$

Hence $3^x = 1,\ 3^x = 3 \ \Leftrightarrow\ x = 0,\ x = 1.$

Both $x = 0,\ x = 1$ satisfy the condition $x > \log_3 \frac{3}{4}$.

c. Assumption 1: $5x^2 + 2x - 1 > 0.$

Assumption 2: $x + 2 > 0.$

The equation can be written as follows:

$$\log(5x^2 + 2x - 1) = \log(x+2) + 1 \ \Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log(x+2) + \log(10) \ \Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log[10(x+2)] \ \Leftrightarrow$$

$5x^2 + 2x - 1 = 10(x+2)$ for all x satisfying Assumptions: 1 and 2.

Notice that if x_0 satisfies the equation and the Assumption 2 then x_0 satisfies also Assumption 1. Therefore after solving the equation it is enough to verify if its solutions satisfy Assumption 2 – it is much more easier.

Now we solve the quadratic equation:

$$5x^2 - 8x - 21 = 0:$$

$$\Delta = 64 + 420 = 484,\ \sqrt{\Delta} = 22,\ x_1 = \frac{8-22}{10} = -\frac{7}{5},\ x_2 = \frac{8+22}{10} = 3.$$

It is easy to check that both solutions satisfy Assumption 2, thereby

Assumption 1 which means that

$$x_1 = -\frac{7}{5},\ x_2 = 3$$

are solutions of the equation

$$\log(5x^2 + 2x - 1) - \log(x+2) = 1.$$

Now we substitute $2^{x-\sqrt{x}} = t,\ t > 0$. Then we get

$$t^2 \le 3t + 4 \ \Leftrightarrow\ t^2 - 3t - 4 \le 0 \ \Leftrightarrow\ (t-4)(t+1) \le 0\,.$$

As we see in Fig.3.4. $-1 \le t \le 4 \ \Leftrightarrow\ -1 \le 2^{x-\sqrt{x}} \le 4.$

Hence $2^{x-\sqrt{x}} \le 4 \ \Leftrightarrow\ 2^{x-\sqrt{x}} \le 2^2 \ \Leftrightarrow\ x - \sqrt{x} \le 2.$

Now we use substitution $\sqrt{x} = u,\ u \ge 0$ and obtain

$$u^2 - u \le 2 \ \Leftrightarrow\ u^2 - u - 2 \le 0 \ \Leftrightarrow\ (u+1)(u-2) \le 0 \ \Leftrightarrow\ -1 \le u \le 2$$

$$\Leftrightarrow\ -1 \le \sqrt{x} \le 2 \ \Leftrightarrow\ \sqrt{x} \le 2 \ \Leftrightarrow\ \begin{cases} x \ge 0 \\ x \le 4 \end{cases} \ \Leftrightarrow\ x \in [0, 4].$$

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Remark

If we need to solve the logarithmic inequalities we will use the following facts:

- If $a > 1$, $g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \Leftrightarrow f(x) \geq g(x)$.
- If $0 < a < 1$, $f(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \Leftrightarrow f(x) \leq g(x)$.

3. Solve the inequalities

a. $\log(x-4) + \log x \leq \log 21$.
b. $\log(2^x + x - 13) > x - x\log 5$.
c. $3^{(\log_3 x)^2} + x^{\log_3 x} \leq 162$.
d. $\log_{(x-2)} \frac{x-1}{x-3} \geq 1$.

E. Teacher introduces some practical applications of exponential functions and explains students the possibility of meeting exponential functions in real life problems. Teacher gives students links to some useful articles or websites.

4. **Measuring loudness**

Sounds can vary in intensity from the lowest level of hearing (a ticking watch 7 meters away) to the pain threshold (the roar of a jumbo jet). Sound is detected by the ear as changes in air pressure measured in micropascals (μP). The ticking watch is about 20 (μP), conversational speech about 20 000 (μP), a jet engine close up about 200 000 000 (μP), an enormous range of values. A scale was required to compress the range of 20 to 200 000 000 into a more manageable and useful form from 0 to 140. The decibel scale was invented for this purpose. If P is the level of sound intensity to be measured and P_0 is a reference level, then

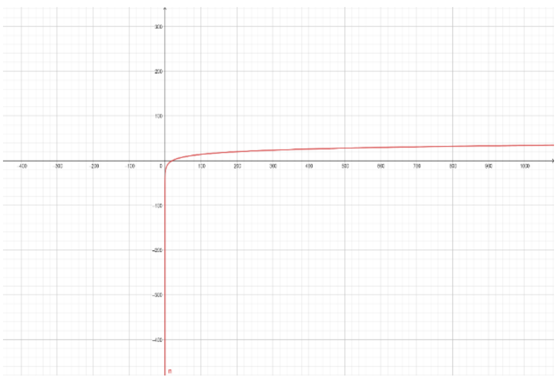
$$n = 20\log\left(\frac{P}{P_0}\right),$$

where n is the decibel scale level.
If we assume 20 (μP) to be the threshold level, then the equation would be:

$$n = 20\log\left(\frac{P}{20}\right)$$

and the graph of the relationship would resemble the one below (Fig.4.7)

Decibel scale for loudness of sound



Sound level (μP)

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As the sound is measured in a logarithmic scale using a unit called a decibel then we can use the following formula:

$$d = 10\log\left(\frac{P}{P_0}\right)$$

where P is the power or intensity of the sound and P_0 is the weakest sound that the human ear can hear.

One hot water pump has a noise rating of 50 decibels. One device in engine room, however, has a noise rating of 62 decibels. The device's in engine room noise is how many times more intense than the hot water pump noise?

Solution

We can't easily compare the two noises using the formula, but we can compare them to P_0. Start by finding the intensity of noise for the hot water pump. Use h for the intensity of the hot water pump's noise:

$$50 = 10\log\left(\frac{h}{P_0}\right),$$

$$5 = \log\left(\frac{h}{P_0}\right),$$
$$10^5 = \frac{h}{P_0}$$
$$h = 10^5 P_0,$$

then repeat the same process to find the intensity of the noise for the device in engine room

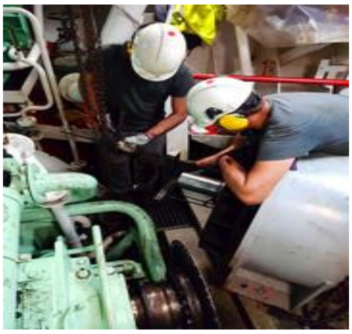
$$62 = 10\log\left(\frac{d}{P_0}\right),$$
$$6{,}2 = \log\left(\frac{d}{P_0}\right),$$
$$10^{6{,}2} = \frac{d}{P_0},$$
$$d = 10^{6{,}2} P_0.$$

To compare d to h, we divide $\frac{d}{h} = \frac{10^{6.2}P_0}{10^5 P_0} = 10^{1.2}$

Answer: The device's in engine room noise is $10^{1.2}$ (or about 15.85) times as intense as the hot water pump.

Remark

The conditions of people on board the ships are particularly difficult, because even after working in crew cabins are at risk of being in high-level areas vibration and noise. During the cruise, the crew cannot escape to the forest or park in areas of peace and quiet. The crew (especially the mechanics) is exposed to danger related to hearing loss. In addition, excessive levels of noise and vibration cause others ailments such as cardiovascular and nervous system diseases. Ailments syndrome health related to noise, infrasonic noise and vibrations is called vibroacoustic disease. For example at room of marine power plant a permissible noise level is 90 decibels ($90\ dB$), at control room, navigation cabin - $65dB$.
https://www.youtube.com/watch?v=IFB4_Snp6wA



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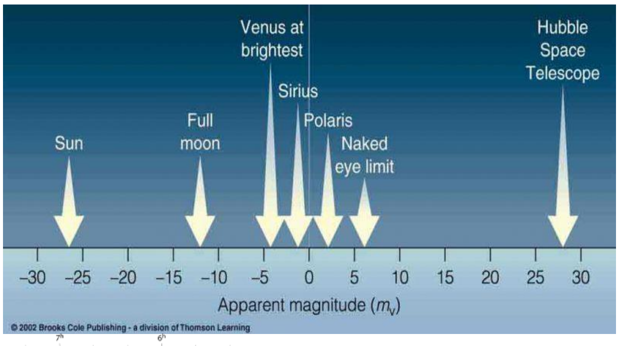
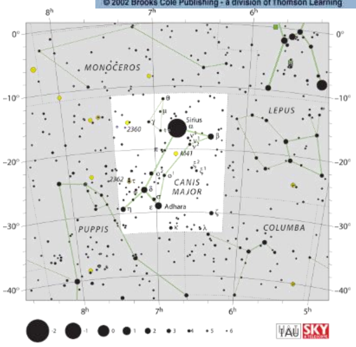
5. The stellar magnitude of a star is negative logarithmic scale, and the quantity measured is
the brightness of the star. If $SM = -\log B$, where SM is stellar magnitude and B is brightness, answer the following questions:
a) What is the stellar magnitude of star A which has a brightness of 0.7943?
b) Star B has a magnitude of 2.1, what is its brightness?
c) Compare the brightness of the two stars.





Solution

a) If the star has a brightness of 0.7943 from the formula $SM = -logB$ it will have a
stellar magnitude of $-\log(0.7943)$ or 0.1. This can be confirmed from the graph.

b) If the magnitude is 2.1 then
$$2{,}1 = -\log B$$
$$-2{,}1 = \log B$$
$$B = 10^{-2{,}1}$$
$$B \approx 0{,}0079.$$

The brightness is 0.007943.

c) Comparing the star A with the star B : A is about 100 times as bright as B.

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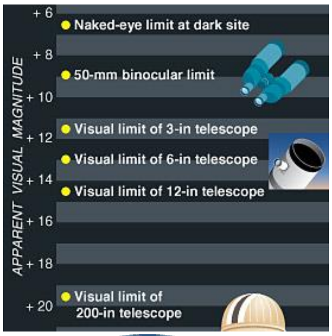

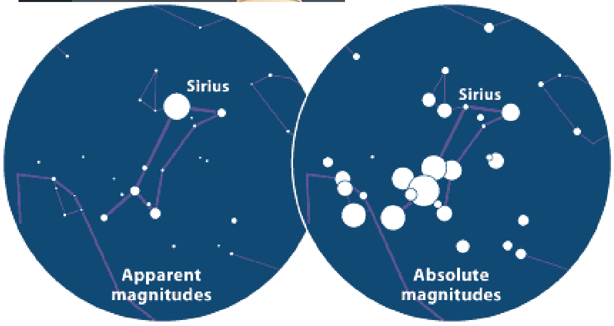

https://en.wikipedia.org/wiki/Decibel
https://logarithmstutorial.wordpress.com/decibel-scale/
https://en.wikipedia.org/wiki/Richter_magnitude_scale

Students Activity

- Teacher asks students to solve logarithmic equations and inequalities to be sure they understand how to do their task and how to use logarithms to real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions: tests on-line:

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- https://docs.google.com/forms/d/16CKlD_jUA9Xjr4YZk5cZQ1SG5i_-2EIfk3x3aReD0B8/edit?usp=sharing

- https://docs.google.com/forms/d/1jaTwl62GMzfebjcLJE-VVDV2f6zi7KucvKpuRdPwIL8/edit?usp=sharing

- Students get sample chapter exam – 8 problems. They have to solve their tasks and show the solutions to teacher next lessons or send the solution to teacher by mail or via EduPlatform (an educational platform functioning in PNA).

APPENDIX 1: Exercises

1. **Evaluate:**
   a. $\log_{\sqrt{2}} 16$
   b. $\log_2 \frac{1}{8}$
   c. $\log_4 0.5$
   d. $\log_{\sqrt{2}} 0.25$
   e. $\log_{\frac{2}{3}} 2.25$
   f. $\log_{\frac{1}{9}} 3\sqrt[3]{3}$
   g. $16^{\log_2 3}$.

2. **Evaluate $\log_{35} 28$ if we know that $\log_{14} 2 = a$, $\log_{14} 5 = b$.**

3. **Solve the equations:**

   a. $\ln(5x - e) = 1$
   b. $\log_{1.5}\left(2x - \sqrt[3]{1.5}\right) = \frac{1}{3}$
   c. $\log_x 3\sqrt{3} = \frac{1}{2}$
   d. $\log_{\frac{3}{4}}\left(1 - \frac{x-2}{2x-5}\right) = -1$
   e. $\ln(\log_2 x) = 0.$
4. **Solve the equations:**
   a. $\log_3\left(x + \sqrt{3}\right) = -\log_3\left(x - \sqrt{3}\right)$
   b. $\log_3(5x + 1) - \log_3(x - 1) = 2$
   c. $\log_4 x + \log_4(12 - 2x) = 2$
   d. $\log(5 - x) + 2\log\sqrt{x - 3} = 0$
   e. $\frac{1}{2}\log(2x + 7) + \log\sqrt{7x + 5} = 1 + \log\frac{9}{2}$
   f. $\log_3 x + \log_5 x = \frac{\log 15}{\log 3}$
   g. $(\log_3 x)^2 = \frac{1}{2}\log_3 x$.
5. **Solve inequalities**
   a. $\log(x - 3) - \log(27 - x) \leq -\log 5 - 1$
   b. $\log_{\frac{1}{2}}(\log_5 x) \geq 0$
   c. $\log_{\frac{1}{3}}(|x| - 1) > -2$

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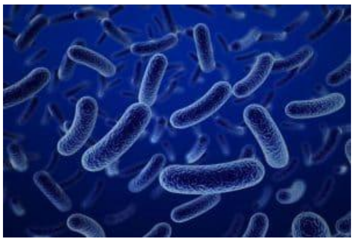
d. $3^{\log_{\frac{1}{5}}(x^2-4x-4)} < 1$
e. $\log_{x^2}(x+6) \geq 1$
f. $\log_{\frac{1}{2}}\frac{2x+1}{3x+2} > 3$

6. A particular dangerous bacteria culture that threatens the marine fauna of the Maldives doubles every 20 minutes and follows the exponential function $N(t) = 200 \cdot 2^{3t}$, where $N(t)$ is the number of bacteria in the culture after t hours. Estimate how long it will be before the number of bacteria in the culture reaches $1\,000\,000$.

7. Rearrange the following formula to make x the subject: $y = 1{,}4e^{-0{,}6x} - 3$.

Answers

1.
a. 8
b. -3
c. $-\frac{1}{2}$
d. -4
e. -2
f. $-\frac{2}{3}$
g. 81

2. $\frac{a+1}{b-a+1}$

3.
a. $x = \frac{2}{5}e$
b. $x = \sqrt[3]{1.5}$
c. $x = 27$
d. $x = \frac{11}{5}$
e. $x = 2$

4.
a. $x = 2$
b. $x = \frac{5}{2}$
c. $x_1 = 2,\ x_2 = 4$
d. $x = 4$
e. $x = 10$
f. $x = 5$
g. $x_1 = 1,\ x_2 = \sqrt{3}$.

5.
a. $x \in \left(3, \frac{59}{17}\right]$
b. $x \in (1, 5]$
c. $x \in (-10, -1) \cup (1, 10)$
d. $x \in (-\infty, -1) \cup (5, \infty)$
e. $x \in [-2,\ -1) \cup (1,\ 3]$
f. $x \in \left(-\frac{1}{2}, -\frac{6}{13}\right)$.

6. It will be 4.11 hours before the number of bacteria in the culture reaches $1\,000\,000$.

7. $x = -\frac{5}{3}\ln\left(\frac{y+3}{1{,}4}\right)$.

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APPENDIX 2: Sample chapter exam

1. Prove the following statements:
   a. $\log_{\sqrt{a}} x \ = \ 2\log_a x$,
   b. $\log_{\frac{1}{\sqrt{a}}} \sqrt{x} \ = \ -\ \log_a x$,
   c. $\log_{a^4} x^2 \ = \ \log_a \sqrt{x}$.
2. Solve the equation: $\log_{\frac{1}{2}}[\log_2(\log_4 x)] = -1$.
3. Solve the inequality: $\log_{\frac{1}{\sqrt{5}}}(6^{x+1} - 36^x) \geq -2$.
4. Find the domain of the function: $f(x) = \log_{x^2-3}(x^2 + 2x - 3)$.
5. Draw the graph of each of the following logarithmic functions and analyze each of them
   completely (i.e. domain, range, zeros, y −intercept, sign, maximal intervals of
   monotonicity):
   a. $f(x) = \log(-x)$,
   b. $f(x) = -\log(x - 3)$.
6. If $y = 3(\mu e)^k$ show that $k = \frac{\ln y - \ln 3}{\ln \mu + 1}$.
7. If $A = P(1 + i)^n$, find n in terms of A, P and i.

8.* Solve the inequality without using a calculator:

$$\log_{2008}(x^2 - 2007x) \leq 1.$$

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Lesson 8. Square root functions

Name of Unit	Workload	Handbook
Square root functions	**Lecture: 90 min** **Exercises: 90 min**	**Unit 5.4.5.**

DETAILED DESCRIPTION

Square roots are often found in math and science problems. Square roots ask "what number, when multiplied by itself, gives the following result". Students you can easily understand the rules of square roots and answer any questions involving them, whether they require direct calculation or just simplification. Any expression that contains the square root of a number is a radical expression. Both have real world applications in fields like architecture, carpentry and masonry. Radical expressions are utilized in financial industries to calculate formulas for depreciation, home inflation and interest. **Radical equations** in real-life applications may contain two or more variables. Solving such an equation involves isolating the unknown variable, which may be contained within a radical such as a square or cube root, while substituting known values for the remaining variables.

$$S=\sqrt{\frac{x}{32}}\,,\quad T=2\sqrt{L}\,,\qquad r=\sqrt{\frac{V}{h\pi}}$$

Each of the above equations is an example of a radical equation. Students can use the examples given in the lesson to practice solving applications involving radical equations.

A square root asks you which number, when multiplied by itself, gives the result after the " $\sqrt{}$ " symbol. So $\sqrt{4}=\ 2$ and $\sqrt{25}=5$.

The " $\sqrt{}$ " symbol tells you to take the square root of a number and you can find this on most calculators. The symbol " $\sqrt{}$ " is called the <u>radical</u> and x is called the <u>radicand.</u>

We can factor square roots just like ordinary numbers, so $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$ or $\sqrt{6}=\sqrt{2}\,\sqrt{3}$.

AIM: To acquire skills in solving equations and inequalities and also to understand concepts standing behind those calculations. Square root functions are also important for some real-life applications in the maritime field.

Learning outcomes

At the end of this lecture, students should be able to

1. Know the definition of square root function
2. Know how to find a square root of negative number.
3. Sketch the graphs of a square root functions.

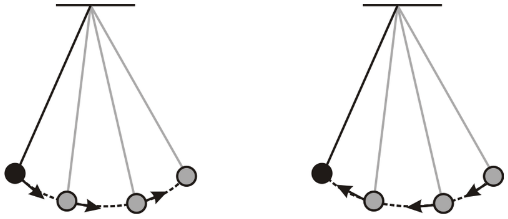
4. Solve square root equations and inequalities
5. Solve practical problems like the root-mean-square velocity, the circular velosity, amount of energy that a moving object (such as a car, roller coaster or boat) possesses.

Key words of this Unit:

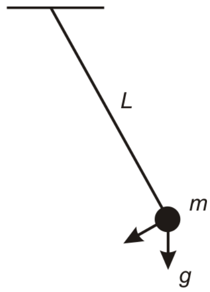
Square root, square root equation, square root inequality.

Previous knowledge of mathematics: Content from basic algebraic operations, functions, graphs, quadratic equations, quadratic inequalities, absolute value.

Relatedness with solving problems in the maritime field: What is a real life example of a radical function? European paper sizes are a good example of real world usage of a radical. The ratio of the length of the longer side of A4 paper to the shorter side is a good approximation of $\sqrt{2}$. As a result, a sheet of A4 can be cut in half to produce two smaller sheets (size A5) with the same proportions as the A4 sheet. Radical expressions are used in real life in carpentry and masonry. Radicals play important roles in biology. Many of these are necessary for life, such as the intracellular killing of bacteria by phagocytic cells such as granulocytes and macrophages. Radicals are involved in cell signalling processes, known as redox signaling. Square roots are important because they show up when we compute areas, which is a fairly practical application. Square roots have real world applications in fields like carpentry and masonry. Radical expressions are utilized in financial industries to calculate formulas for depreciation, home inflation and interest. Electrical engineers also use radical expressions for measurements and calculations. Another job that uses the square roots and the Pythagorean theorem is an architect. Architects need to build large buildings and use right angles in the blue prints. Civil Engineers use square roots when they build roads coming off of a hill side. The road that is flat would be represented by b. Likewise, what is purpose of square root? The principal square root function $f(x) = \sqrt{x}$ is a function that maps the set of nonnegative real numbers onto itself. In geometrical terms, the square root function maps the area of a square to its side length. Mathematicians and physicists have studied the motion of pendulums in great detail because this motion explains many other behaviors that occur in nature. This type of motion is called simple harmonic motion and it is important because it describes anything that repeats periodically. Galileo was the first person to study the motion of a pendulum, around the year 1600. He found that the time it takes a pendulum to complete a swing doesn't depend on its mass or on its angle of swing (as long as the angle of the swing is small). Rather, it depends only on the length of the pendulum.



The time it takes a pendulum to complete one whole back-and-forth swing is called the period of the pendulum. Galileo found that the period of a pendulum is proportional to the square root of its length: $T = a\sqrt{L}$. The proportionality constant, a, depends on the acceleration of gravity: $a = \frac{2\pi}{\sqrt{g}}$. At sea level on Earth, acceleration of gravity is $g = 9{,}81\ \frac{m}{s^2}$. Using this value of gravity, we can find $a = 2.0$ with units of $\frac{c}{\sqrt{m}}$.



Up until the mid $20th$ century, all clocks used pendulums as their central time keeping component.

"Square" TV screens have an aspect ratio of 4:3; in other words, the width of the screen is $\frac{4}{3}$ the height. TV "sizes" are traditionally represented as the length of the diagonal of the television screen.

(https://www.ck12.org/algebra/Applications-Using-Radicals/lesson/Applications-Using-Radicals-ALG-I/).

In conclusion here are some uses of square roots in real life:

- Finance (Rates Of Return Over 2 Years)
- Normal Distributions (Probability Density Function)
- Pythagorean Theorem (Lengths & Distances)
- Quadratic Formula (Height Of Falling Objects)
- Radius Of Circles With A Given Area
- Simple Harmonic Motion (Pendulums & Springs)
- Standard Deviation (Measuring the spread of data).

Contents:

- EXERCISES
- SAMPLE CHAPTER EXAM

Assessment strategies:

Evaluating students activity during lesson.

Teacher Toolkit and Digital Resources:

- Power point presentation to define exponential functions
- GeoGebra https://www.geogebra.org/ https://www.geogebra.org/graphing?lang=en
- https://www.padowan.dk/

Graph

Version 4.4

Graph - plotting of mathematical functions

- Work Sheets
- Websites:

  - https://prezi.com/p/ga_mg_dnoel-/real-life-problem-in-radicals/
  - https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-9.pdf
  - https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#radicals
  - https://jdmeducational.com/what-is-a-square-root-used-for-7-real-life-applications/

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Lesson : Square root functions

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
35 min	Starter/Introduction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use exponential function ?
45 min	Presentation	Definition, Properties of square root function, graphs of square root function	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	Are students understand the idea of square roots, their restrictions and properties?
40 min	Presentation Exercise 1-3	Square root equation Square root inequalities	Frontal Explains task and supports	Active listening and contributing to questions Complete the worksheets	Are students able to solve equations and inequalities with square roots
45 min	Presentation	Applications of square roots in various real problems	Frontal Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
15 min	Summary		Giving homework		

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • **Lesson** https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-5-Functions-9.pdf • **Lesson** https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#radicals
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve square root equations • ***all*** students should be able to solve square root inequality.

A. The first section is a definition of square root function, its properties, graphs. Teacher shows students that they can use Graph or GeoGebra to plot graphs of square root functions

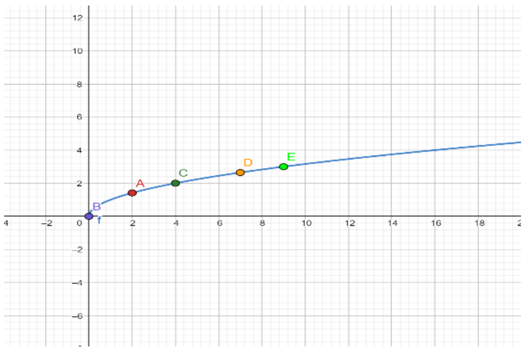
The square root function is a type of power function $f(x) = x^{\alpha}$, with fractional power as it can be written

$$\boldsymbol{f(x) = x^{\frac{1}{2}},\ f(x) = \sqrt{x}}.$$

Its domain is the set of non-negative real numbers: $[0, +\infty)$ or $D = R_+ \cup \{0\}$

Its range is also the set of non-negative real numbers: $[0, +\infty)$

The graph of the square root function is shown in Fig. 5.1. with some points.



The graph of $f(x) = \sqrt{x}$.

Properties of square root function

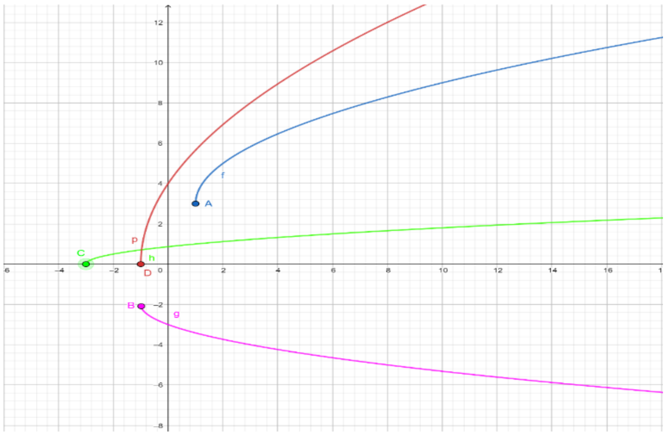
Some of the properties of the square root function may be deduced from Fig.5.1.

1. x and y intercepts are both at (0,0).
2. the square root function is an increasing function
3. the square root function is a one-to-one function and has an inverse.

Square root functions of the general form:

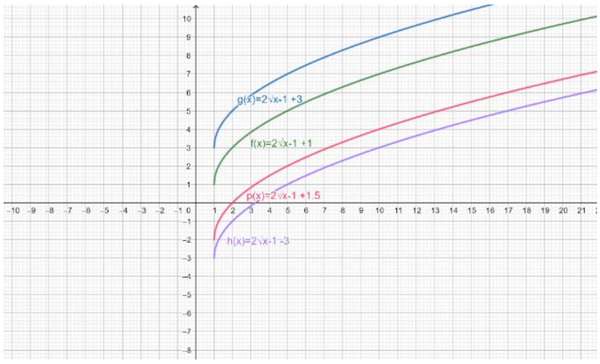
$$\boldsymbol{f(x) = a\sqrt{x - c} + d}.$$

Fig.5.2 presents how the graph of $f(x) = a\sqrt{x - c} + d$ looks like when the parameters a, c, d change.



. Graphs of $\boldsymbol{f(x) = 2\sqrt{x-1} + 3}$, $\boldsymbol{g(x) = -\sqrt{x+1} - 2}$, $\boldsymbol{h(x) = \frac{1}{2}\sqrt{x+3}}$,

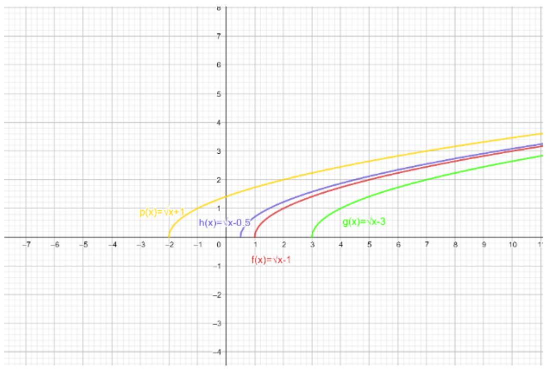
$p(x) = 4\sqrt{x+1}$.



What happens to the graph when the value of parameter d changes?

From graphs we can conclude that

- changes in the parameter d affect the y coordinates of all points on the graph hence the vertical translation or shifting. When d increases, the graph is translated upward and when d decreases the graph is translated downward.



What happens to the graph when the value of parameter c changes.

B. Teacher shows how to solve equations and inequalities and asks students to try to solve some examples. Recalls and asks students about the rules of square roots, how to solve quadratic equations, plot graphs of quadratic functions (students can use GeoGebra or Graph to plot graphs).

Examples.

1. Solve the equation $\sqrt{10x+6} = 9 - x$.

Solution

Assume that $10x + 6 \geq 0$, $x \geq -\frac{3}{5}$.

The left side of the equation is non-negative, so to be sure that the equation is not contradictory we have to assume additionally that $9 - x \geq 0$.

Finally we get the assumption: $x \in \left[-\frac{3}{5}, 9\right]$.

Then using Theorem 5.1 we have

$$\sqrt{10x+6} = 9 - x \Leftrightarrow 10x + 6 = (9-x)^2 \Leftrightarrow x^2 - 28x + 75 = 0.$$

$$\Delta = 28^2 - 4 \cdot 75 = 484, \quad \sqrt{\Delta} = 22, \quad x_1 = \frac{28-22}{2} = 3, \quad x_2 = \frac{28+22}{2} = 25.$$

Due to the assumption $x \in \left[-\frac{3}{5}, 9\right]$, $x_1 = 3$ is the solution of the equation.

2. Solve the inequalities:

a. $\sqrt{x+3} < -2$

b. $\sqrt{2-x} > -5$

c. $\sqrt{5-x} < 3$

d. $\sqrt{11-x} > x - 9$

e. $\sqrt{3-2x-x^2} < 2x^2 + 4x - 3$

Teacher presents a video: https://marematics.pfst.hr/index.php/2021/09/13/5-functions/#radicals . Discuss with students the solutions they see on the video.

C. Teacher shows the possible applications of square roots in some examples.

5. At what temperature, the velocity distribution function for the oxygen molecules will have maximum value at the speed $400\,\frac{m}{s}$?

Solution

The maximum speed for any gas occurs when it is at most probable temperature

$v_{mp} = \sqrt{\frac{2RT}{m}}$, where R is the gas constant, T is the absolute temperature, m is the molar mass of the gass. (Maxwell-Boltzman Distribution).

Given $v_{mp} = 400\ m/s,\ \ m = 32 \cdot 10^{-3} kg$ for 1 mole of oxygen molecules, $R = 8{,}31$.
Then

$$v_{mp}^2 = \frac{2RT}{m},$$

$$T = \frac{m \cdot v_{mp}^2}{2R}$$

$$T = \frac{400^2 \cdot 32 \cdot 10^{-3}}{2 \cdot 8{,}31} \approx 308°\mathrm{C}.$$

6. The temperature of the gas is raised from 27°C to 927°C. What is the root mean square velocity?

Solution

The **root-mean-square velocity** is the measure of the of particles in a gas, defined as the **square root** of the **average velocity-squared** of the molecules in a gas. The **root-mean-square velocity** takes into account both molecular weight and temperature, two factors that directly affect the kinetic energy of a material.
T in Kelvin= °C + 273

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

Change °C into Kelvin:

$$T_1 = 27°\mathrm{C} + 273 = 300$$
$$T_2 = 927°\mathrm{C} + 273 = 1200$$
$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \quad => \quad v_2 = v_1\sqrt{\frac{T_2}{T_1}}$$

and $v_2 = v_1\sqrt{\frac{1200}{300}} = 2v_1$

therefore root mean square velocity will be doubled.

7. Boat builders share an old rule of thumb for sailboats. The maximum speed K in knots is 1.35 times
the square root of length L in feet of the boat's waterline. A customer is planning to order a sailboat
with a maximum speed of 8 knots. How long should the waterline be?

Solution

The *knot* (kn) is *a* unit of speed equal to ***one*** nautical mile per hour, exactly 1.852 km/h
The feet (ft') is a unit of length in the British imperial and United State customary systems of measurement, exactly 0,3048 m.

$$K = 1.35\sqrt{L}$$
$$8 = 1.35\sqrt{L}$$
$$\sqrt{L} = \frac{8}{1.35} \approx 5.926 \quad => \quad L = 35.12$$

The waterline should be 35.12 feet long.

Students Activity

- Teacher asks students to solve the square root equations and inequalities to be sure they understand how to do their task and how to use radicals to real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions: (test on-line:

  https://docs.google.com/forms/d/1CSWvbb7H-Gh01clya9CQHcTJTL1omSrY5aP5m3dS4i4/edit?usp=sharing).

- Students get sample chapter exam – 4 problems. They have to solve their tasks and show the solutions to teacher next lesson or send the solution to teacher by mail or via EduPlatform (an educational platform functioning in PNA).

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APPENDIX 1: Exercises

Exercises:

1. Solve the equations.

a. $(x^2 - 4)\sqrt{1-x} = 0.$

b. $x - \sqrt{x+1} = 5.$

c. $x + \sqrt{10x+6} = 9.$

d. $\sqrt{4+2x-x^2} = x - 2.$

e. $\sqrt{2x-3} + \sqrt{4x+1} = 4.$

f. $x = 15 + \sqrt{9+8x-x^2}.$

g.

2. Solve the inequalities.

a. $\sqrt{5-x} < -2.$

b. $\sqrt{x+3} > -23.$

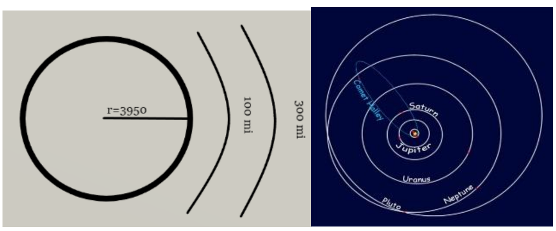
c. $\sqrt{2x+3} > x + 2.$

d. $\sqrt{x+3} + \sqrt{3x-2} \leq 7.$

e. $\sqrt{x-2} + x > 4.$

f. $\sqrt{8-x} > x - 6.$

3. The circular velocity, v , in miles per hour, of a satellite orbiting Earth is given by the formula $v = \sqrt{\frac{1.24 \cdot 10^{12}}{r}}$, where r is a distance in miles from the satellite to the center of the Earth. How much greater is the velocity of a satellite orbitting at an altitude of 100 mi than one orbiting at 300 mi? (Radius of a the Earth is 3950 mi, 1 mi =1,609344 km)



Answers

1.

a. $x_1 = -2, \;\; x_2 = 1.$
b. $x = 8.$
c. $x = 3.$
d. $x = 3.$
e. $x = 2.$
f. No solutions.

2.

a. No solutions.
b. $x \in [-3, \infty)$
c. No solutions.
d. $x \in \left[\frac{2}{3}, 6\right].$
e. $x \in (3, \;\infty).$
f. $x \in (-\infty, 7).$

3. The velocity of a satellite orbiting at an altitude of 100 mi is 1.024 times greater than one orbiting at 300 mi.

https://www.youtube.com/watch?v=ap6O6fsgbqI
https://www.ck12.org/Algebra/Square-Root-Applications/lesson/Square-Root-Applications-ALG-I/

APPENDIX 2: Sample chapter exam

1. Determine in which set the functions are equal:

$f(x) = \sqrt{(x-1)(x-5)}$ and $g(x) = \sqrt{(x-1)} \cdot \sqrt{x-5}$.

2. Solve the equations.

a. $\sqrt{x+3} + \sqrt{x} = 3$.

b. $3 - \sqrt{x-1} = \sqrt{3x-2}$.

c. $(x^2 + x - 6)^{0.5} = \frac{1}{2}x - 1$.

3. Solve the inequalities.

a. $x - 1 < \sqrt{7-x}$.
b. $\sqrt{(x-6)(1-x)} < 2x + 3$.
c. $\sqrt{1 + 10x + 5x^2} \geq 7 - 2x - x^2$.

4.* Find the domain of the given function $f(x) = \log_{17}\left(x + \sqrt{x^2+1}\right)$.

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Lesson 9. Functions of form $\frac{1}{x^p}$.

Name of Unit	Workload	Handbook
Functions	**Lecture + exercise: 30 min**	**Unit 5.5. Functions of form** $\frac{1}{x^p}$.

DETAILED DESCRIPTION

In this lesson we are adding one more function to the list of basic functions.

Learning outcomes

1) Recognize a function $\frac{1}{x^p}$ as a special case of the power function $y = kx^p$
2) Interpret special property's function $\frac{1}{x^p}$ m ay exhibit when p is odd and when p is even.
3) Apply these properties in graphing function $\frac{1}{x^p}$.

Key words of this Unit:

Power function, exponent function, …

Previous knowledge of mathematics:

Knows definition of power and exponent function.

Teacher Toolkit and Digital Resources:

- Video
- Geogebra

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• LESSON FLOW

	Time	Sequence	Content	Teacher activities	Student activities	Points of discussion
VII.	3 min	Introduction	Pre teaching	Motivation	Active listening. Discussions	
VIII.	10 min	Presenting information	Concept of a power function Video	Frontal and questioning	Active listening and contributing to questions	Are students able to tell the difference between odd and even exponent graph.
IX.	15 min	Exercise	Working with Geogebra worksheet	Explain task and supports	Complete the worksheet	
X.	2 min	Summary	Post teaching	Concluding lesson.	Active listening	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• Whiteboard • Lesson (unit 5.11) • Video https://marematics.pfst.hr/?p=3526#funform1xp • Worksheet: 1/x^n **https://marematics.pfst.hr/?p=4022**

I. Introduction

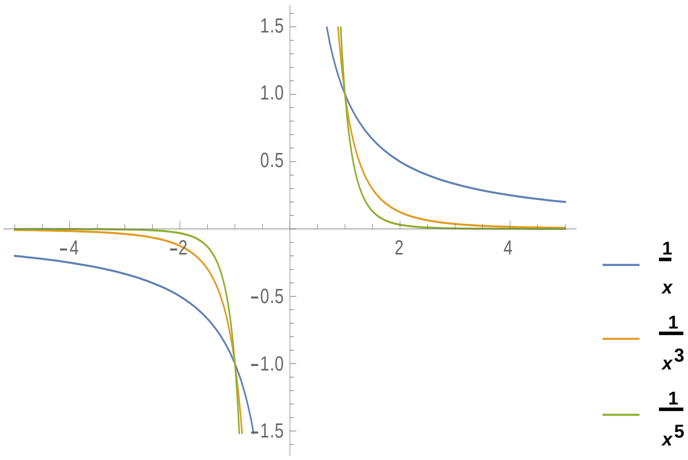
Teacher introduce topic of this lesson. There is a discussion what type of functions students can name and remember.

Students name different functions they know and remember.

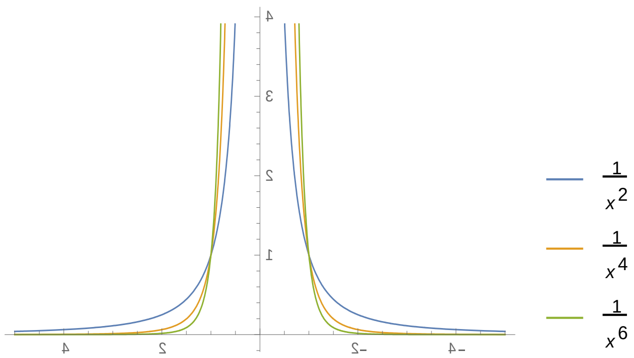
II. Presenting information

Present students definition of a power function and shows graphs of the power functions

- A power function is a single – term function that contains a variable as its base and a constant for its exponent.
- POWER FUNCTIONS WHERE P = - 1, - 3, - 5, …



- POWER FUNCTIONS WHERE P = - 2, - 4, -6, …



P is odd	P is even
• Domain: $X \neq 0$ • Domain of variation: $Y \neq 0$ • Eveness and odness: $y(-x) = -y(x) \rightarrow$ odd • Monotomy: monotonus • Extremes: none • Intersection points with coordinate axes: none • Domain of convexity: $\check{X} = (-\infty; 0)$; $\hat{X} = (0; +\infty)$ • Inflection point: none • Inverse functions: ○ If p = -1, then $x = \frac{1}{y}$ ○ If p < -2, then $x = \frac{1}{\sqrt[\lvert p \rvert]{y}}$	• Domain: $X \neq 0$ • Domain of variation: $Y > 0$ • Eveness and odness: $y(-x) = -y(x) \rightarrow$ odd • Monotomy: $X \uparrow = (-\infty; 0)$; $X \downarrow = (0; +\infty)$ • Extremes: none • Intersection points with coordinate axes: none • Domain of convexity: $\hat{X} = (-\infty; 0)$ • Inflection point: none • Inverse functions: ○ If p = - 2, then $x = \frac{1}{\sqrt{y}}$ ○ If p < -2, then $x = \frac{1}{\sqrt[\lvert p \rvert]{y}}$

- Teacher shows student video about function $\frac{1}{x^p}$

Student write down information and watch video. Time to ask questions if they have any.
What the difference between graphs of y = x^2 and y = $\frac{1}{x^2}$

III. Exercise

Teacher gives student worksheet (link to worksheet) and explains the assignment. If needed assists student with the work.

IV. Assigment

Using Geogebra https://www.geogebra.org/calculator draw the graph of the function y = $\frac{1}{x^p}$ and write down the properties while p is the last number of your student identity code.

Students work on a worksheet and ask questions if needed.

V. Summary

Teacher concludes what students learned today and gives students homework.

Students present their works.

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APPENDIX worksheet

Exercise

Using Geogebra https://www.geogebra.org/calculator draw the graph of the function y = $\frac{1}{x^p}$ and write down the properties while p is the last number of your student identity code.

1) Graph

2) Properties

   - Domain:
   - Domain of variation:
   - Eveness and odness:
   - Monotomy:
   - Extremes:
   - Intersection points with coordinate axes:
   - Domain of convexity:
   - Inflection point:
   - Inverse functions:

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Lesson 10. TRIGONOMETRY

Name of Unit	Workload	Handbook
Trigonometry	Lecture: 90 min	Unit 5.6. Functions: trigonometry

DETAILED DESCRIPTION

The resources are picked from project MareMathics and available on the https://marematics.pfst.hr/ .

AIM: To successfully convert between radian and degree measure and to use trigonometric functions and concepts in maritime problems.

Learning outcomes

1. To convert between radian and degree measure (including coterminal angles)
2. To apply trigonometric concepts in everyday tasks linked to maritime navigation

Prior Knowledge: functions, coordinate geometry, basic geometry, Pythagorean theorem, ratios.

Relationship to real maritime problems: polar coordinates, navigation using polar coordinates, constraints problems (limited docking space)

Contents:

Contents:

5.10.1. Types, graphs, important limits

5.10.2. Equations

5.10.3. Inequalities

5.10.4. Application in maritime affairs

Assessment strategies:

Evaluating students' activity during lesson and quiz.

Teacher Toolkit and Digital Recources:

- Powerpoint presentation (link: https://marematics.pfst.hr/index.php/2022/04/11/presentations/
- Geogebra link: https://www.geogebra.org/m/hjgj9nfc
- Geogebra link: https://www.geogebra.org/m/ehagjzfm
- Geogebra link: https://www.geogebra.org/m/xc4ugska

Quiz: MS Form link: https://docs.google.com/forms/d/1VTfiT9n1C-6f9Li-0iHe4RKnq1LP9VKn-OmrT-1fwa0/viewform?ts=6145db2f&edit_requested=true

TRIGONOMETRY: Application in maritime affairs

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
10 min	Presentatio n	The definition of a radian	Introduction Moderator Using Geogebra	Active listening + discussion Practical work using Geogebra	What is a radian?
5 min	Exercise 1	Converting radians to degrees and vice versa	Frontal and questioning	Active listening and contributing to questions	How to convert degrees to radians?
20 min	Example (maritime problem)	How to read a radar?	Frontal and questioning Using Geogebra	Active listening and contributing to questions Practical work using Geogebra	How to read a radar?
25 min	Exercise 2 - Submarine	Application of sine law and cosine law	Frontal and questioning Using Geogebra	Active listening and contributing to questions Practical work using Geogebra	When can we use sine law and when cosine law?
20 min	Exercise 3 – Signal (Geogebra and Geography)	Application of sine law and cosine law	Frontal and questioning Using Geogebra	Active listening and contributing to questions Practical work using Geogebra	When can we use sine law and when cosine law?
10 min	Final quiz in MS Forms	Functions	Frontal	Individual work	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Activity 1: What is a radian?

Teacher can make a comparison o fan equilateral trinagle and an angle of one radian. Teacher can ask a question: If we bend one side of the triangle what happens to the measure of the opposite angle?

Students can conclude that one radian is equal to the measre o fan angle of slightly less than 60°.

Student can use Geogebra applet Arc and Angle to discuss converting angles from degrees to radians.

Student will solve Exercise 1: converting the angles.

Connections and applications

The teacher introduces the polar coordinate system by showing students pictures of sonars and radars. Then he asks the students how those instruments are used in aeronautics. The teacher should aim to induce words such as “direction” and “distance”. He can ask these questions to get a response:

Of two objects how does the sonar operator know which is closer?

How do we know if the objects are at the same distance relative to the origin?

What is the set of all points with the same distance from the origin?

How would you describe the direction from which an object is approaching?

In aeronautics, **directions** are determined using the **azimuth.** Azimuth is the angle between the object direction and true north. For example, the azimuth of direction North is 0°, the azimuth of direction West is 270°.

Instead of writing angles, the directions on a radar are sets of 3 numbers. For example, the direction of 0° (north) is 000, the direction of East (90°) is 090 (read zero-nine-zero).

What about the circles?

Well, all points on the same circle are equally distant from the circle centre, regardless of the direction.

Students are given table and they have to determine the azimuth depending on direction

direction	azimuth
N	000
E	090
W	270
S	180

The students now ready to solve the following problems using Geogebra.

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Exercise 2

A submarine accompanies a ship on its voyage.
The position of the submarine and ship are given in the following Geogebra file Submarine:
At each point in time, the ships distance is 200 meters in the direction 060.

a) The submarine captain spots a dangerous reef located directly north from the submarines' location, 200 m away.
   1) Plot the point locating the reef in Geogebra
   2) Teacher asks students: What is the distance from the ship to the reef?

Solution: 200 m

   3) Teacher asks students: What would be the reefs direction when observed from the ship?

Solution: 300

b) The ships safe harbour is located 400 m away, in the direction 300 from the current submarine location.
   1) Plot the harbour location

   2) Determine the distance from the boat to the harbour

Students need to determine the distance in two ways by calculation and using Geogebra. Students are given the formulas for Sine law and Cosine law and they have to determine which one to use. After using Cosine law they get the result $d = 529.15$.

   3) Determine the course (azimuth) of the ship if it wants to reach the harbour

Students have to determine the azimuth in two ways by calculation and using Geogebra. The result is 281.

Exercise 3

The following link shows part of the Split sea area, with islands which are close by.

Geogebra and geography

Your ship is in the Rogac port on the island of Solta. Students must find the distance between the Signal and the nearest harbour using Geogebra and by calculating.

Students are given some values and they need to find the missing value and correct a mistake:

Direction Rogac – Split 043

Direction Rogac – Signal ___

Distance Rogac – Split 15.1 km

Direction Signal – Split 345

Direction Signal – Rogac 240

Solution:

Rogac – Signal 060

Direction Signal – Split 015

There can be a discussion about which harbour is the nearest to Signal. After deciding (Split is the nearest) students need to use the correct law (Sine law) and find the distance between Signal and Split.

Distance (Split, Signal) = 6.2 km.

APPENDIX:

Quiz with 5 questions. quiz functions

Lesson 11. Inverse Trigonometric Functions:

Name of Unit	Workload	Handbook
Inverse Trigonometric Functions	**Lecture: 90 min** Exercises: 90 min	**Unit 5.7. Trigonometry**

DETAILED DESCRIPTION

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

Inverse trigonometric functions as the name suggests are the inverse functions of the basic trigomometric functions. Every mathematical function, from the easiest to the most complex, holds an inverse, or opposite function. And for trigonometric functions there exists inverse trigonometric functions. They are used in solving trigonometric equations that arise in finding the angles and sides of triangle. The inverse of any function is important - it provides a way to "get back."

The inverse trigonometric functions are written be applying arc-prefix to the basic functions like arcsin(x), arccos(x), arctan(x).

In this lesson, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behavior through graphical representations. Besides, some elementary properties will also be discussed.

This lesson builds on that understanding of inverse functions by restricting the domains of the trigonometric functions in order to develop the inverse trigonometric functions. In geometry, students used arcsine, arccosine, and arctangent to find missing angles, but they did not understand inverse functions and, therefore, did not use the terminology or notation for inverse trigonometric functions. Students define the inverse trigonometric functions in this lesson. Then they use the notation of inverse trigonometric functions to solve the exercises.

AIM: The students will learn how to interpret and graph an inverse trig. Function and will also learn to solve for an equation with an inverse function.

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Learning outcomes

At the end of this lecture, each student should be able to

1. Understand and use the inverse sine function.
2. Understand and use the inverse cosine function.
3. Understand and use the inverse tangent function.
4. Use a calculator to evaluate inverse trigonometric functions.
5. Find exact values of composite functions with inverse trigonometric functions

Previous knowledge of mathematics: If no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function.

If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of the inverse function, denoted f^{-1}. The graph of f^{-1} is a reflection of the graph of about the line y = x.

Relatedness with solving problems in the maritime field:

Specifically, they are the inverses of the sine, cosine, tangent, cotangent functions, and are used to obtain an angle from any of the angle's trigonometric ratios. Inverse trigonometric functions are widely used in engineering, navigation, physics, and geometry. Inverse Trigonometric Functions can be very useful in the real world and not just in math class. An example can be in the ocean a captain of a ship can determine the wrong course of the ship, always in a straight line, ordering to modify the course in the degree to go directly to the correct destination point. Using inverse trigonometric functions to determine the direction they want to go.

Contents:

1. THE INVERSE SINE FUNCTION
2. THE INVERSE COSINE FUNCTION
3. THE INVERSE TANGENT FUNCTION
4. COMPOSITION OF FUNCTIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS
   - INVERSE PROPERTIES
   - FUNCTIONS AND THEIR INVERSES
5. APPLICATION EXAMPLES
6. PRACTICE EXERCISES
7. APPLICATION EXERCISES

Assessment strategies:

Evaluating students activity during lesson.

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Teacher Toolkit and Digital Resources:

- Power point presentation to define inverse trigonometric function
- Videos: 1,2,3,4, and
- GeoGebra https://www.geogebra.org/
- Work Sheets
- Quiz: InverseTrigFun1, InverseTrigFun2

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Lesson : Inverse Trigonometric Functions

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Starter/Intro duction Presentation	Pre-teaching	Moderator Motivation	Discussion	Where in real life we can use inverse trigonometric functions?
15 min	Presentation Example 1,2 Video 1: arcsin x	Definition of the inverse sine function, finding exact value of arcsin x	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	For each function, select a suitable domain that will make the function invertible. How can we find the equation of the inverse sine? Why is it important for the inverse of sine to be a function?
15 min	Presentation Example 3,4 Video 2: arccos x	Definition of the inverse cosine function, finding exact value of arccos x	Frontal and questioning Group work Use GeoGebra, Graph	Active listening and contributing to questions Complete the worksheets	For each function, select a suitable domain that will make the function invertible. How can we find the equation of the inverse cosine? Why is it important for the inverse of cosine to be a function?
15 min	Presentation Example 5,6 Video 3: arctan x	Definition of the inverse tangent function, finding exact value of arctan x	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	For each function, select a suitable domain that will make the function invertible. How can we find the equation of the inverse tangent?

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					Why is it important for the inverse of tangent to be a function?
20 min	Presentation Example 7,8,9,10 Video4	Composition of functions involving inverse trigonometric functions	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
15 min	Presentation Application examples Video4	Application in engineering, maritime	Frontal and questioning Group work Use GeoGebra, Graph Explains task and supports Discussion using solved examples	Active listening and contributing to questions Complete the worksheets	
5 min	Summary		Giving homework	Complete the quizzes View the video uploaded to OneDrive Complete the worksheets	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• **Whiteboard**
	• Lesson https://marematics.pfst.hr/index.php/2021/09/13/5-functions/
	• **Videos:**
	  • https://marematics.pfst.hr/?p=3526#inv_trig_func
	  • https://marematics.pfst.hr/?p=3526#arcsin
	  • https://marematics.pfst.hr/?p=3526#arccos
	  • https://marematics.pfst.hr/?p=3526#arctan

Learning objectives	By the end of the lesson all students: • Understand and use the inverse sine function. • Understand and use the inverse cosine function. • Understand and use the inverse tangent function. • Use a calculator to evaluate inverse trigonometric functions. • Find exact values of composite functions with inverse trigonometric functions.

A. The first section is a definition of the inverse sine function, finding exact value of arcsin x. Teacher presents and discusses with students a video1 and shows students that they can use GeoGebra to plot graphs of inverse sine function. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
B. The second section is a definition of the inverse cosine function, finding exact value of arccos x. Teacher presents and discusses with students a video2 and shows students that they can use GeoGebra to plot graphs of inverse cosine function. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
C. The third section is a definition of the inverse tangent function, finding exact value of arccos x. Teacher presents and discusses with students a video3 and shows students that they can use GeoGebra to plot graphs of inverse tangent function. Teacher shows how to solve practice exercises. Teacher asks a student to solve the exercises
D. The fourth section is about composition of functions involving inverse trigonometric functions. Teacher presents and discusses with students a video4 and shows how to solve practice exercises. Teacher asks a student to solve the exercises

E. In the fifth section teacher introduces some practical applications Teacher presents and discusses with students parts of following video: video4. Teacher shows how to solve application exercises.

Students Activity

- Teacher asks students to exercises to be sure they understand how to do their task and how to use inverse trigonometric functions to solve real life problems.
- Students ask questions, solve their task on whiteboard or in the copybooks.
- Students work alone or in two-three persons groups, after solving problems shows their solutions to the rest of the whole group.
- Students watch and discuss or comment short videos.
- Students get some examples with answers to check and verify their solutions.
- Students get homework – quizzes. They have to solve their tasks on QUIZIZZ platform.

APPENDIX 1: Exercises

FIND THE EXACT VALUE OF EACH EXPRESSION

1. $\arcsin\frac{1}{2}$;
2. $\arcsin\frac{\sqrt{2}}{2}$;
3. $\arcsin\left(-\frac{1}{2}\right)$;
4. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$;
5. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$;
6. $\arctan\left(-\frac{\sqrt{2}}{2}\right)$;
7. $\arctan(-1)$;
8. $\arctan\sqrt{3}$;
9. $\arcsin 0$;
10. $\arcsin 1$.

USE A CALCULATOR TO FIND THE VALUE OF EACHEXPRESSION ROUNDED TO TWO DECIMAL PLACES

1. $\arcsin(-20)$;
2. $\arcsin 0.3$;
3. $\arccos\frac{1}{8}$;
4. $\arcsin 0.47$;
5. $\arctan(-20)$;
6. $\arctan 30$;
7. $\arccos\frac{\sqrt{5}}{7}$;
8. $\arccos\frac{4}{9}$;
9. $\arctan(-\sqrt{5061})$;
10. $\arcsin(-0.625)$.

FIND THE EXACT VALUE OF EACH EXPRESSION, IF POSSIBLE. DO NOT USE A CALCULATOR

1. $\sin(\arcsin 0.9)$;
2. $\arcsin\left(\sin\frac{\pi}{3}\right)$;
3. $\arcsin\left(\sin\frac{5\pi}{6}\right)$;
4. $\tan(\arctan 125)$;
5. $\arctan\left(\tan\left[-\frac{\pi}{6}\right]\right)$;
6. $\arcsin(\sin\pi)$;
7. $\sin(\arcsin\pi)$;
8. $\cos(\arccos 0.57)$;
9. $\arccos\left(\cos\frac{4\pi}{3}\right)$;
10. $\arccos(\cos 2\pi)$;
11. $\arctan\left(\tan\left[-\frac{\pi}{3}\right]\right)$;
12. $\arctan\left(\tan\frac{3\pi}{4}\right)$.

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USE A SKETCH TO FIND THE EXACT VALUE OF EACH EXPRESSION

1. $\cos\left(\arcsin\frac{1}{2}\right)$;
2. $\tan\left(\arccos\frac{5}{13}\right)$;
3. $\tan\left(\arcsin\left[-\frac{3}{5}\right]\right)$;
4. $\sin\left(\arctan\frac{7}{24}\right)$;
5. $\cot\left(\arcsin\left(-\frac{4}{5}\right)\right)$;
6. $\cos\left(\arcsin\frac{5}{13}\right)$;
7. $\sin\left(\arccos\frac{\sqrt{2}}{2}\right)$;
8. $\sin\left(\arctan\left[-\frac{3}{4}\right]\right)$.

USE A RIGHT TRIANGLE TO WRITE EACH EXPRESSION AS AN ALGEBRAIC EXPRESSION. ASSUME THAT x IS POSITIVE AND THAT THE GIVEN INVERSE TRIGONOMETRIC FUNCTION IS DEFINED FOR THE EXPRESSION in x

1. $\tan(\arccos x)$;
2. $\cos(\arcsin 2x)$;
3. $\cos\left(\arcsin\frac{1}{x}\right)$;
4. $\cot\left(\arctan\frac{x}{\sqrt{3}}\right)$;
5. $\sin(\arctan x)$;
6. $\sin(\arccos 2x)$;
7. $\cot\left(\arctan\frac{x}{\sqrt{2}}\right)$;
8. $\cot\left(\arcsin\frac{\sqrt{x^2-9}}{x}\right)$.

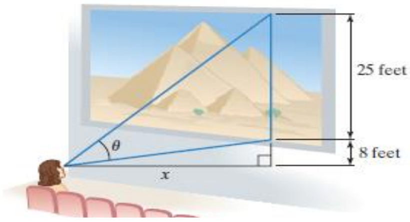
USE TRANSFORMATIONS (VERTICAL SHIFTS, HORIZONTAL SHIFTS, REFLECTIONS, STRETCHING, OR SHRINKING) OF THESE GRAPHS TO GRAPH EACH FUNCTION. THEN USE INTERVAL NOTATION TO GIVE THE FUNCTION'S DOMAIN AND RANGE

1. $f(x) = \arcsin x + \frac{\pi}{2}$;
2. $f(x) = \arccos(x+1)$;
3. $g(x) = -2\arctan x$;
4. $f(x) = \arcsin(x-2) - \frac{\pi}{2}$;
5. $f(x) = \arccos x + \frac{\pi}{2}$;
6. $h(x) = -3\arctan x$;
7. $f(x) = \arccos(x-2) - \frac{\pi}{2}$;
8. $f(x) = \arcsin\frac{x}{2}$.

DETERMINE THE DOMAIN AND THE RANGE OF EACH FUNCTION

1. $f(x) = \sin(\arcsin x)$;
2. $f(x) = \cos(\arccos x)$;
3. $f(x) = \arcsin(\cos x)$;
4. $f(x) = \arcsin(\sin x)$;
5. $f(x) = \cos(\arccos x)$;
6. $f(x) = \arccos(\sin x)$.

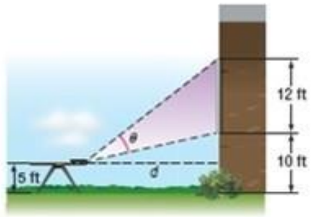
APPENDIX 2: Application exercises



1. Your neighborhood movie theater has a 25-foot-high screen located 8 feet above your eye level. If you sit too close to the screen, your viewing angle is too small, resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit feet back from the screen, your viewing angle θ, is given by

$$\theta = \arctan\frac{33}{x} - \arctan\frac{8}{x}.$$

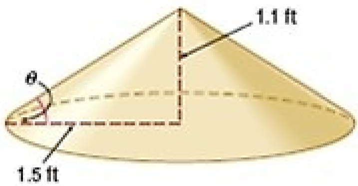
Find the viewing angle in radians, at distance of 5 feet, 10 feet, 15 feet, 20 feet and 25 feet.



2. SPORTS. Steve and Ravi want to project a prosoccer game on the side of their apartment building. They have placed a projector on a table that stands 5 feet above the ground and have hung a 12-foot-tall screen 10 feet above the ground.

a. Write a function expressing θ in terms of distance d.

b. Use a graphing calculator to determine the distance for the maximum projecting angle



3. SAND. When piling sand, the angle formed between the pile and the ground remains fairly consistent and is called the angle of repose. Suppose Jade creates a pile of sand at the beach that is 3 feet in diameter and 1.1 feet high.

a. What is the angle of repose?

b. If the angle of repose remains constant, how many feet in diameter would a pile need to be to reach a height of 4 feet?

APPENDIX 3: Homework

- Video1, video2, video3, video 4
- https://youtu.be/JGU74wbZMLg (author: https://www.khanacademy.org/)
- https://youtu.be/eTDaJ4ebK28 (author: https://www.khanacademy.org/)
- https://youtu.be/Idxeo49szW0 (author: https://www.khanacademy.org/)

Quizzes: InverseTrigFun1, InverseTrigFun2

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Lesson 12. Limit of a function

Name of Unit	Workload	Handbook
Limit of a Function	**Lecture: 90 min**	**Unit 5.1.3 Functions: Limits**

DETAILED DESCRIPTION

A concept of a limit is a fundamental concept of calculus and mathematical analysis. It is the basic for understanding differential and integral calculus.

AIM: To learn concept of a limit of a function and apply it.

Learning Outcomes:

1. Find the limit of a given function
2. Find the limit of a function by observing the graph

Keywords of this Unit: function, limit, converge, diverge

Prior Knowledge: algebraic expressions, algebraic identities, linear equations, and inequalities.

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LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Example: Activity 1	Introduction to limits: determining the function values as the function approaches infinity	Introduction Frontal and questioning Discussion	Active listening + discussion	What happens with the value of a function as x approaches infinity?
10 min	Definition	The definition of a limit	Frontal and questioning	Active listening Contributing to questions	How to define the limit of a function?
10 min	Properties	Properties of limits with examples	Frontal and questioning	Contributing to questions	What properties can be used?
10 min	Example: Activity 4	Finding a limit of a rational function	Frontal and questioning	Active listening and contributing to questions	
40 min	Example: Activity 5	Finding the limits of different functions	Frontal and questioning	Solving the limits and individual work	
10 min	Example: Activity 6	Finding the limits of different functions using excel document	Frontal and questioning	Discussion	
5 min	Appendix: Final quiz in MS Forms	Limit of a function	Frontal	Individual work	

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Contents

Assessment strategies:

Evaluating students' activity during lesson and quiz.

Teacher Toolkit and Digital Recources:

- Lessons: 8. Limits
- Excel document: Limitsxlsx
- Videos:
    - Limit of a Function Graphically
    - Limit of a Rational Function
- Quiz link: https://docs.google.com/forms/d/e/1FAIpQLSfaPH_6iD3mfD5ET2yGrz4zpngIVdyILAfb4WFwI1LeFuqUxQ/viewform?usp=sf_link

Useful Websites

https://www.youtube.com/watch?v=vaJhttcgz2s&list=PL3E167BBDE2E5BCF4&index=1

https://opentextbc.ca/calculusv1openstax/chapter/the-limit-of-a-function/

https://www.cuemath.com/calculus/limits/

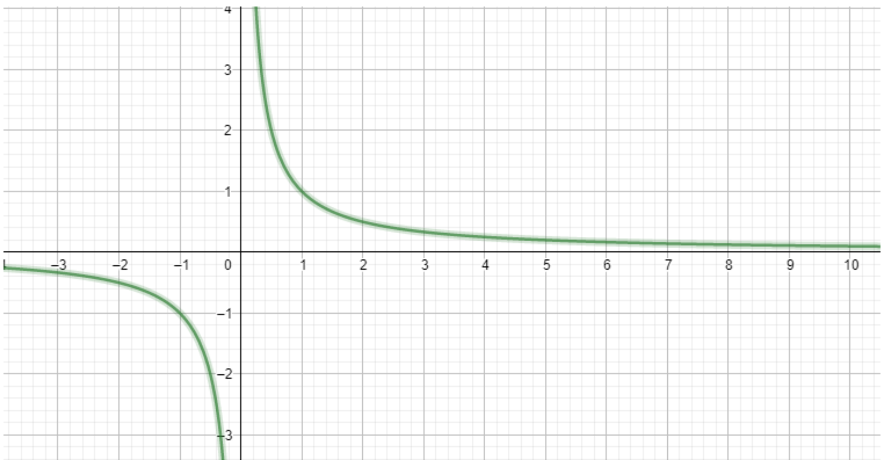
https://www.mathsisfun.com/calculus/limits-evaluating.html

https://www.youtube.com/watch?v=nJZm-zp639s

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

The teacher starts the session with the following task:

ACTIVITY 1.



Given the function $f(x) = \frac{1}{x}$ above, determine the function values as the function *approaches* infinity.

Use the table below:

x	$f(x)$
10	
100	
1000	
10 000	
100 000	

SOLUTION

x	$f(x)$
10	0.1
100	0.01
1000	0.001
10 000	0.0001
100 000	0.00001

The teacher begins the discussion about the function value when x is extremely large. The goal is to guide the students to say that the function **approaches** 0, but it will never actually be zero.

Questions for discussion:
1. Do the function values converge (become nearer to a specific value)? If yes, to which value?
2. Will the function ever obtain the specified value? Explain your reasoning…

The teacher then asks the following questions:

1. What about negative numbers, will the function *approach* a single value when x goes to negative infinity?
2. What about zero? Will the function *approach* a value when x goes to zero?
   Does the direction at which we approach zero matter? If yes, why? Explain your reasoning?

After the discussion, the teacher writes the formal definition of a limit:

If for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, then the function $f(x)$ has a limit L in a point a and we can write $\lim_{x \to a} f(x) = L$

A function has a **limit L in a point a** if the value of the function approaches the value L when the input x **approaches** the value a.

The notation is as follows

$$\lim_{x \to a} f(x) = L$$

The function can also have a limit at infinity when the function values converge at a number when x goes to infinity.

The mathematical definition is:

If for every $\varepsilon > 0$, there exists $m > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x > m$, then the function $f(x)$ has a limit L in infinity and we can write $\lim_{x \to \infty} f(x) = L$.

If the function values gather around a specific value, we say that the function **converges**. When the function values go to infinity, we say that the function **diverges**.

The teacher starts a discussion about the function value of the function $f(x) = \frac{1}{x}$ around x=0.

The point of the discussion is to establish that a limit around 0 does not exist, because the function **diverges.**

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Properties of limits.

The teacher writes the following properties on the blackboard:

Suppose that $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = N$.

Then

P1 $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = M \pm N$

Addition property - the limit of a sum/difference is equal to the sum/difference of limits

P2 $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = M \cdot N$

Multiplication property - the limit of a product is equal to the product of limits

Note that this implies the following:

P2*

$$\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x) \,, c \in R$$

I.e.

$$\lim_{x\to 0} 2x = 2 \lim_{x\to 0} x$$

P3 $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{M}{N}, N \neq 0$

Division property - the limit of a quotient is equal to the quotient of limits

P4 $\lim_{x\to a} f(x)^k = M^k, M > 0$

Power property

The teacher briefly explains each rule, and then continues.

Important limits are also called **common limits** and are usually listed in tables and/or known by heart.

The first common limit to remember is:

$$\lim_{x\to\pm\infty} \frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{0}$$

The second, more general limit is even more important:

$$\lim_{x\to\pm\infty} \frac{\mathbf{1}}{\boldsymbol{x}^{\boldsymbol{n}}} = \mathbf{0}, \qquad \text{where } n \in N$$

Also, as a side note, the limit of a constant is always equal to that constant:

$$\lim_{x \to \pm\infty} \boldsymbol{c} = \boldsymbol{c}$$

Now, the students are equipped with techniques for calculating the most common types of limits.

The teacher then continues to the following activity, explaining in detail the limit solving procedure.
The procedure below is very detailed, helping the teacher touch upon all the key points in calculating limits.

ACTIVITY 4.

Calculate the following:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x}$$

Note that the technique described below is suitable only when calculating the limit at infinity.
The most common technique when dealing with limits is to divide the numerator and denominator of the fraction by the **highest order polynomial in the denominator.**
The expression can be written as:

$$\lim_{x \to \pm\infty} \frac{x^3 + 3x^2 + 3}{x^3 - 7x} : \frac{x^3}{x^3}$$

Divide the numerator and denominator separately:

$$\lim_{x \to \pm\infty} \frac{(x^3 + 3x^2 + 3) : x^3}{(x^3 - 7x) : x^3}$$

Since the parenthesis is divided by x^3, we can divide each term in the parenthesis separately:

$$\lim_{x \to \pm\infty} \frac{x^3 : x^3 + 3x^2 : x^3 + 3 : x^3}{x^3 : x^3 - 7x : x^3}$$

Remember the division property of powers? $x^m : x^n = x^{m-n}$
Now let us apply that property to every term in the numerator and denominator:

$$\lim_{x \to \pm\infty} \frac{x^{3-3} + 3x^{2-3} + 3x^{-3}}{x^{3-3} - 7x^{1-3}}$$

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Calculating the differences, we obtain:

$$\lim_{x\to\pm\infty}\frac{x^0+3x^{-1}+3x^{-3}}{x^0-7x^{-2}}$$

Finally, let us calculate each term, and rewrite using the following power property: $ax^{-n}=\frac{a}{x^n}$. Also, remember that $x^0=1$ if $x\neq 0$.

$$\lim_{x\to\pm\infty}\frac{1+\frac{3}{x}+\frac{3}{x^3}}{1-\frac{7}{x^2}}$$

Only now are we able to use the property P.3. (the limit of a quotient is the quotient of limits)

$$\frac{\lim\limits_{x\to\pm\infty}\left(1+\frac{3}{x}+\frac{3}{x^3}\right)}{\lim\limits_{x\to\pm\infty}\left(1-\frac{7}{x^2}\right)}$$

After using the property P.1. (the limit of a sum is the sum of limits) the expression simplifies further:

$$\frac{\lim\limits_{x\to\pm\infty}1+\lim\limits_{x\to\pm\infty}\frac{3}{x}+\lim\limits_{x\to\pm\infty}\frac{3}{x^3}}{\lim\limits_{x\to\pm\infty}1-\lim\limits_{x\to\pm\infty}\frac{7}{x^2}}$$

We now use P.2*. (the constant can be written before the limit):

$$\frac{\lim\limits_{x\to\pm\infty}1+3\lim\limits_{x\to\pm\infty}\frac{1}{x}+3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty}1-7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}}$$

Finally, we can use the common limits written above:

$$\frac{\lim\limits_{x\to\pm\infty}1+3\lim\limits_{x\to\pm\infty}\frac{1}{x}+3\lim\limits_{x\to\pm\infty}\frac{1}{x^3}}{\lim\limits_{x\to\pm\infty}1-7\lim\limits_{x\to\pm\infty}\frac{1}{x^2}}=\frac{1+3\cdot 0+3\cdot 0}{1-7\cdot 0}=\frac{1+0+0}{1-0}=\frac{1}{1}=1$$

Therefore, we conclude that:

$$\lim_{x\to\infty}\frac{x^3+3x^2+3}{x^3-7x}=1$$

The procedure for calculating limits may seem complex right now but the procedure is quite straightforward:

1) Divide by the greatest order polynomial of the denominator
2) Simplify the expression in the numerator and denominator.

This part was most of the work, but the work done has nothing to do with limits, just carefully dividing polynomials and using power properties. Here steps may be omitted if the student is confident with basic polynomial algebra.

3) Using the properties of limits, reduce the limit to a series of **common limits**, and calculate separately each common limit.

ACTIVITY 5.

Find the limits

1. $\lim\limits_{x\to 3}(x^2-2)$
2. $\lim\limits_{x\to\infty} x^2$
3. $\lim\limits_{x\to 0}\frac{1}{x^2}$
4. $\lim\limits_{x\to\infty}\frac{2}{x^3}$
5. $\lim\limits_{x\to\infty}\frac{x-6}{2x+1}$
6. $\lim\limits_{x\to\infty}\frac{3x^2+2x-6}{4x^2-5}$
7. $\lim\limits_{x\to\infty}\frac{5x^3+2}{3x^2+x-3}$
8. $\lim\limits_{x\to 1}\frac{x-1}{x^2-1}$

SOLUTIONS

1. $\lim\limits_{x\to 3}(x^2-2)=(3^2-2)=7$

This exercise was simple. We can just substitute 3 instead of x and calculate. Property P1 is used in this example

2. $\lim\limits_{x\to\infty} x^2=[\infty^2]=\infty$

If x approaches infinity, we can assume that x is a huge number, e.g. 1000 or even greater and substitute 1000 instead of x in our expression. Since $1\,000^2=1\,000\,000$ is even higher, we can conclude that the result is infinity.

Notice that we put ∞^2 in brackets $[\infty^2]$. That is because ∞^2 is not a regular mathematical expression.

3. $\lim\limits_{x\to 0}\frac{1}{x^2}=\left[\frac{1}{0^2}\right]=\left[\frac{1}{0}\right]=\infty$

Dividing by zero is not defined, so we can imagine that x is a very small number, e.g. x=0,001 or even smaller. In that case the value of expression $\frac{1}{x^2}=\frac{1}{0{,}001^2}=\frac{1}{0{,}000001}=1\,000\,000$ which is pretty large number. We can conclude that as the x approaches to zero, the value of expression gets greater and approaches to infinity.

4. $\lim\limits_{x\to\infty}\frac{2}{x^3}=\left[\frac{2}{\infty}\right]=0$

When we divide some number (e.g. 2) with some big number, e.g. 1000^3 or even greater the result is a very small number, close to zero.

5. $\lim\limits_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]$

As x gets greater, the values of denominator and numerator are approaching to infinity. If denominator and numerator are polynomials we can solve this problem by dividing all through by the highest power of x. In this exercise we divided by x.

$$\lim_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{1-\frac{6}{x}}{2+\frac{1}{x}}$$

Since $\lim_{x\to\infty}\frac{6}{x}=0$ and $\lim_{x\to\infty}\frac{1}{x}=0$ we get

$$\lim_{x\to\infty}\frac{x-6}{2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{1-\frac{6}{x}}{2+\frac{1}{x}}=\frac{1}{2}$$

6. $\lim_{x\to\infty}\frac{3x^2+2x-6}{4x^2-5}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{6}{x^2}}{\frac{4x^2}{x^2}-\frac{5}{x^2}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{6}{x^2}}{4-\frac{5}{x^2}}=\frac{3}{4}$

Since both denominator and numerator are polynomials, we divided all through by the x^2.

7. $\lim_{x\to\infty}\frac{5x^3+2}{3x^2+x-3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{5+\frac{2}{x^3}}{\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}}=\left[\frac{5}{0}\right]=\infty$

8. $\lim_{x\to 1}\frac{x-1}{x^2-1}=\left[\frac{0}{0}\right]$

If we divide denominator and numerator by x^2 it will not be helpful.

In cases when the values of denominator and numerator both approach to zero, we can solve the problem by using algebra to simplify the expression.

We know that $x^2-1=(x-1)(x+1)$

$$\lim_{x\to 1}\frac{x-1}{x^2-1}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

ACTIVITY 6.

Teacher can show how to use excel document Limits.xlsx to better understand the limits of a function.

APPENDIX 1:

Quiz with 6 questions. quiz functions

Limits Teaching and learning plan.

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The resources are picked from project ***MareMathics*** and available on the https://marematics.pfst.hr/.

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Lesson 13. Important Limits

Name of Unit	Workload	Handbook
Limits	*Lecture + exercise: 45 min*	*Unit 5.8 Important limits*

DETAILED DESCRIPTION

In this lesson we are learning about important limits.

Learning Outcomes:

1) To apply first remarkable limit in solving tasks.
2) To identify different consequences of this limit.
3) To express the definition of second important limit.
4) To apply second remarkable limit in solving tasks.

Key words of this Unit:

- Limits

Previous knowledge of mathematics:

limit, basic trigonometric formulas

Teacher Toolkit and Digital Resources:

- Video
- Presentation
- Quiz
- Worksheet

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Lesson 13. Important Limits

• LESSON FLOW						
	Time	Sequence	Content	Teacher activities	Student activities	Points of discussion
I.	5 min	Introduction	Pre teaching	Motivation	Active listening and contributing to questions	What students know about limits?
II.	5 min	Presentation	Video	Frontal	Active listening and contributing to questions	
III.	10 min	Teaching	Presentation	Frontal . Moderator		
IV.	20 min	Worksheet	Solving exercises	Explain task and supports	Complete the worksheet	
V.	10 min	Quiz	Doing a quiz	Moderator		
VI.	5 min	Summary	Post teaching	Frontal Concluding lesson. Giving homework	Active listening	What did we learn today?

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources

- Whiteboard
- Lesson (unit 5.13) ?
- Video **Important Limits**
- WorkSheet **https://maremathics.pfst.hr/?p=4022**
- Quiz https://quizizz.com/admin/quiz/624156af66030e00207dbf55

I. Introduction

Teacher states topic of todays lesson. Starts asking students what they have previously learned about limits.

Students write down topic of the lesson and name things they have previously learned.

II. Presentation

Teacher shows students video about limits. Answer questions if needed.

Students watch video and ask questions if necessary.

III. Teaching

Teacher provides students new material and gives examples.

Students listen to explanations, write down material and ask questions.

Example 1: Find the limit of the function $\lim\limits_{x\to 0}\frac{\sin 7x}{5x}$.

Solution: As you can see, the function under the limit is close to the first remarkable limit, but the limit of the function itself is not equal to one. In such tasks for the limits, it is necessary to select the variable in the denominator with the same coefficient that is contained in the variable under the sine. In this case, divide and multiply by 7.

$$\lim_{x\to 0}\frac{\sin 7x}{5x}=\lim_{x\to 0}\frac{7\sin 7x}{5\cdot 7x}=\frac{7}{5}\lim_{x\to 0}\frac{\sin 7x}{7x}=\frac{7}{5}.$$

Example 2: Find the limit of the function $\lim\limits_{x\to 0}\frac{\sin 6x}{\tan 11x}$.

Solution: To understand the result, we write the function as $\frac{\sin 6x}{\tan 11x}=\frac{\sin 6x}{\frac{\sin 11x}{\cos 11x}}=\frac{\cos 11x\cdot\sin 6x}{\sin 11x}$.

To apply the wonderful limit rules, we multiply and divide by factors. Further, we write the limit of the product of functions in terms of the product of the limits.

$$\lim_{x\to 0}\frac{\sin 6x}{\tan 11x}=\frac{6}{11}\lim_{x\to 0}\cos 11x\cdot\frac{11x}{\sin 11x}\cdot\frac{\sin 6x}{6x}=\frac{6}{11}\cdot\lim_{x\to 0}\cos 11x\cdot\lim_{x\to 0}\frac{\sin 6x}{6x}\cdot\lim_{x\to 0}\frac{11x}{\sin 11x}=\frac{6}{11}\cdot 1\cdot 1\cdot 1=\frac{6}{11}$$

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Consequences

1) $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$
2) $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = 1$
3) $\lim\limits_{x\to 0} \frac{\arctan x}{x} = 1$
4) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

I. Worksheet

Teacher gives students worksheet with exercises and explains the assignment. Asks some students to start solving exercises on a whiteboard. It helps to get some feedback if the new material was understood.

Students listens when teacher explains the assignment and start working on a worksheet. Some students solve exercises on the whiteboard while others work in their notebooks.

II. Quiz

Teacher gives students instructions where to find a quiz and helps if someone have problems opening the link.

Students follow instructions and work on the quiz.

III. Summary

Teacher gives students some feedback about the quiz were overall results good or not. Concludes what new were learned today and gives homework.

Students listen to the teacher and give feedback. Write down homework.