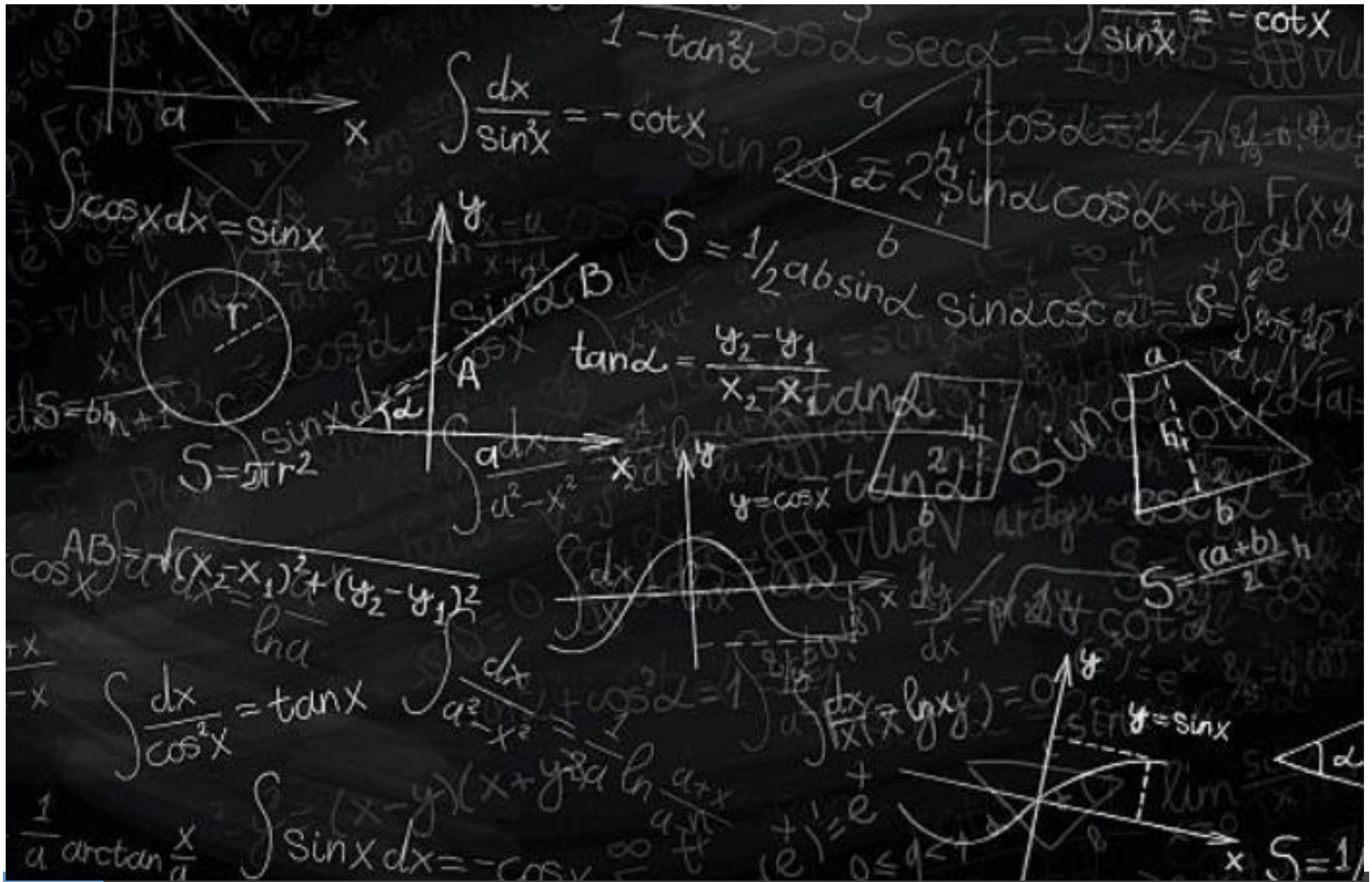
Innovative Approach in Mathematical Education for Maritime Students



Teacher's Manual

Integrals

Author: Ingrida Veilande

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MareMathics

Innovative Approach in Mathematical Education for Maritime Students

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https://marematics.pfst.hr/

Manual for teachers

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Reviewed by partners from Faculty of Maritime studies in Split

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The Manual is the outcome of the collaborative work of all the Partners for the development of the MareMathics Project.

Partners in the project:

University of Split Faculty of Maritime Studies **CROATIA**

Tallinn University of Technology, Estonian Maritime Academy **ESTONIA**

Latvian Maritime Academy **LATVIA**

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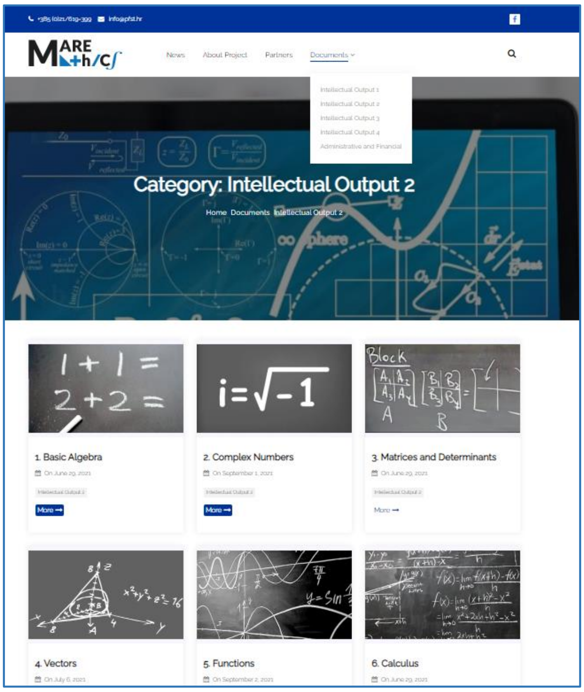
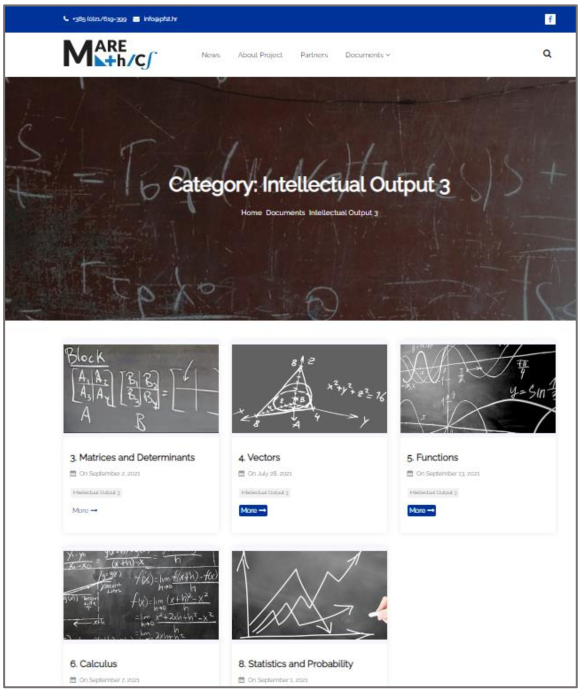
INTEGRALS: Teaching and Learning Plan

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The topic INTEGRALS consist of several lessons. The sets of exercises at the end of lessons are recommended for classes to acquire the methods of solution of indefinite integrals and application of definite integrals.

The resources are picked from project ***MareMathics*** and available on the https://maremathics.pfst.hr/.





Lesson 1. Introduction to Indefinite Integrals

Name of Unit	Workload	Handbook
Part 1: Introduction to Indefinite Integrals	**Lecture: 45 min**	**Unit 1. Introduction**
Part 2: Basic Rules of Integration	**Lecture: 45 min**	**Unit 2. Basic rules**

DETAILED DESCRIPTION

The first part contains introductory questions about the indefinite integrals. The relation between integrals and derivatives will be considered. The reverse procedure of differentiation of functions will be discussed. The definition of the indefinite integral will be formulated. The section contains examples with geometric interpretation of antiderivatives. The section ends with exercises and their solutions.

The second part introduces basic formulas of integration of elementary functions and the main properties of indefinite integrals. The section explains how to derive integration formulas from well-known differentiation rules. Several examples with explanations are discussed. Exercises for individual learning of integration are presented. At the end of the section there is an example on how to check the correctness of the solution of an integral.

For construction of graphs GeoGebra, DESMOS graphing calculator, MS Excel, or other tools can be applied.

AIM of FIRST PART: To learn the relationship between indefinite integrals and derivatives of elementary functions. To interpret the family of antiderivatives geometrically

AIM of SECOND PART: To learn basic formulas and properties of integrals; to introduce methods of integration

Learning Outcomes:

1. Perform the reverse procedure of differentiation for simple elementary functions
2. Express the function with a given rate of change
3. Construct the graph of the specified function
4. Learning the basic integration formulas
5. Application of the properties of indefinite integrals
6. Computing simple integrals of elementary functions
7. Transformation of integrands if necessary

Prior Knowledge: properties of elementary functions; graphs of elementary functions, algebra and trigonometry knowledge; rules of differentiation.

Key words of this Unit: antiderivative, differentiation, elementary functions, graphs

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content Part 1

1. Conceptions of the antiderivative and definition of the indefinite integral
2. Geometric interpretation of the indefinite integral
3. Uniqueness of antiderivatives
4. Exercises
5. Solutions

Content Part 2

1. Integration formulas
2. List of basic integration formulas
3. Properties of indefinite integrals
4. Alteration of the integrand
5. Exercises
6. Solutions
7. Additional note

Assessment strategies:

Assessing students' knowledge about the differentiation during the lesson

Teacher Toolkit and Digital Resources:

- Powerpoint presentation to introduce indefinite integrals and basic formulas
- Geogebra Classic to demonstrate the set of the antiderivatives (MS Excel; Desmos graphing calculator or other tools)
- Videos
- WorkSheet
- Useful websites

https://www.geogebra.org/t/indefinite-integral
https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/v/antiderivatives-and-indefinite-integrals
https://www.wolframalpha.com/calculators/integral-calculator/
https://www.symbolab.com/solver/indefinite-integral-calculator

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Discussion	Introduction to the concept of antiderivative	Frontal and questioning Recall of main prerequisites	Active contributing to questions	What does it means reverse process?
5 min	Introduction Presentation	Some history facts about the development of Calculus	Motivation	Active listening	Which famous mathematicians are the founders of Calculus?
10 min	Presentation	Concept of antiderivative	Frontal Explain's task and supports (videos, GeoGebra files)	Active listening	
10 min	Presentation Examples 2.1	Geometric interpretation of antiderivative	Frontal Construction of graphs by GeoGebra applet	Active listening Discussion	How to choose one definite function from the whole set of prime functions
10 min	Example 2.2	Connectedness and application in the maritime field: Antiderivative and the problem of the height of the flare	Frontal Videos	Active listening Contributing to questions	Maritime safety, the role of the flares
5 min	Presentation	Property of antiderivatives	Frontal	Active listening	Can we have two different solutions for the given integral?
5 min	Presentation	Proof of formula for power function	Frontal Solution of example	Active listening	Can we prove all integration formulas?
5 min	Presentation	List of basic integration formulas	Demonstration	Active listening and contributing to questions	
15 min	Presentation Proofs of some properties	Properties of indefinite integrals	Frontal Explain's task and supports	Active listening Contributing in proving	

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			(videos, GeoGebra files)		
15 min	Presentation Exercises	Alteration of the integrand	Frontal discussion using solved examples	Active listening	
5 min.	Video Problem posing	Application of indefinite integral	Posing the problem; recall of knowledge; solving	Discussion Contributing the solving process	Can we apply indefinite integrals in real life?

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• Whiteboard • Lesson 1 (unit 1 and unit 2) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (flare): https://www.youtube.com/watch?v=LTfVxpiVnP4 • Video (flare): https://www.youtube.com/watch?v=7FejZcTxUYM • Video (forklift): https://www.youtube.com/watch?v=-IDzcXuHRYQ
Learning objectives	By the end of the lesson: • ***all*** students should understand the meaning of the indefinite integral • ***all*** students should understand the relation between the derivatives and antiderivatives of the elementary functions

Part 1

A. The lesson starts with a challenge about the function, which is a derivative of some unknown function. *"We are interested in the reverse procedure: what function have we differentiated to get function $\boldsymbol{f}(\boldsymbol{x})$?"* Lecturer discusses some examples with students.

B. Historical facts about the integration can be presented. https://encyclopediaofmath.org/wiki/Integral_calculus;

C. Introduction of concept of antiderivative. Examples. Definition of Indefinite integral, explanation of new terms

D. Geometric interpretation of indefinite integral; demonstration of GeoGebra constructions of graphs to show the family of prime functions

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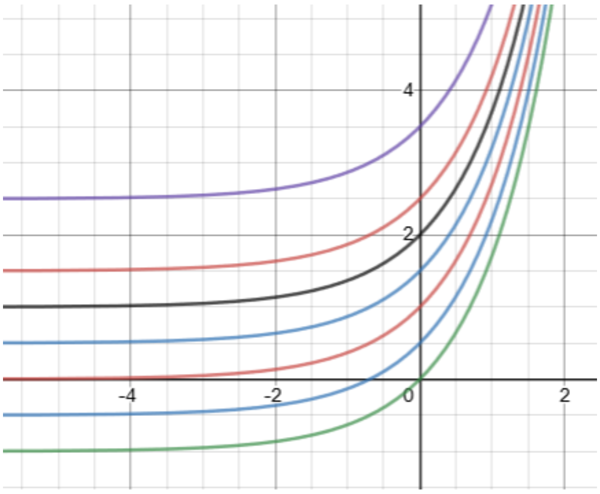

Figure 1 Representors of the set $y = e^x + C$

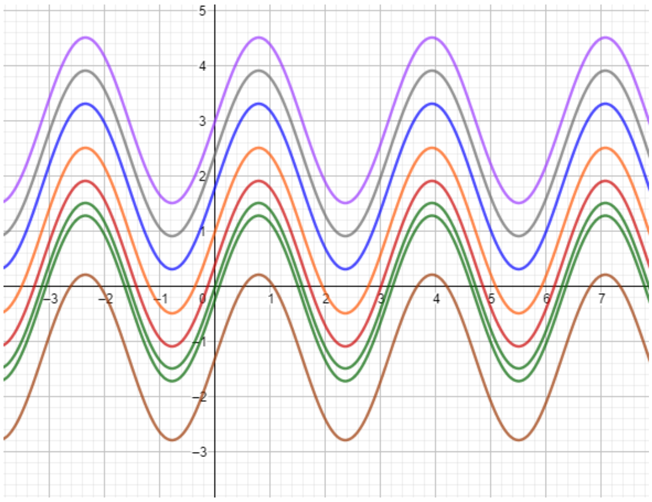

Figure 2 Representors of the set $y = 1.5\sin x + C$

E. Posing the question: "*How we can choose one definite function from the whole set of prime functions?*" Discussion about the specific initial condition to get a definite answer.

F. Examples of real application of indefinite integrals. Recall of meaning of derivative as a change of rates. Video demonstration about the flare and discussion about the maritime safety.

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URL: https://www.youtube.com/watch?v=LTfVxpiVnP4
URL: https://www.youtube.com/watch?v=7FejZcTxUYM

Solution of the example 2.2 from Unit 1:

Example 2.2 A flare is ejected vertically upwards from the ground at 15 m/s. Find the height of the flare after 2.5 seconds.

Suggestion: Additional examples of application of indefinite integrals lecturer can find for instance on the homepage:

https://www.intmath.com/applications-integration/1-apps-indefinite-integral.php

G. Explanation of the uniqueness of the antiderivatives. Lecturer formulates the theorem about the two different antiderivatives.

Part 2

H. Lector emphasizes that integration is the reverse procedure of differentiation and poses the question: "How we can get the integration formulas?" Proof of integral formula for power function:

For instance, let us prove the formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

The derivative of the right side of the formula

$$\left(\frac{x^{n+1}}{n+1} + C\right)' = \left(\frac{x^{n+1}}{n+1}\right)' + C' = \frac{1}{n+1} \cdot (n+1)x^n + 0 = x^n$$

The challenge for students: "Whether the formula is completely correct?"

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Students should notice the restriction – division by zero is not allowed (if $n = -1$): "How to proceed in this case?"

Introduction of special case

$$\int x^{-1}dx = \int \frac{dx}{x} = \ln x + C$$

And correction of integral for power function

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C; \quad n \neq -1$$

I. Presentation of the list of basic integrals

J. Introducing main properties of the indefinite integrals. Proving of some properties.

For example, property

$$\left(\int f(x)dx\right)' = f(x)$$

derives from the definition of the indefinite integral.

$$\int f(x)dx = F(x) + C$$

Then

$$\left(\int f(x)dx\right)' = (F(x) + C)' = F'(x) + C' = f(x)$$

K. Solving examples with application of properties of integrals. Examples 3.1, 3.2, 3.3 (Unit 2).

L. Application of algebraic transformation or trigonometry formulas to change the integrand. Examples 4.1 – 4.5 (Unit 2).

M. Feedback – what we did learn today. Application of indefinite integral to solve the problems in real life. Velocity, acceleration, distance. Video: accident with forklift. URL: https://www.youtube.com/watch?v=-IDzcXuHRYQ

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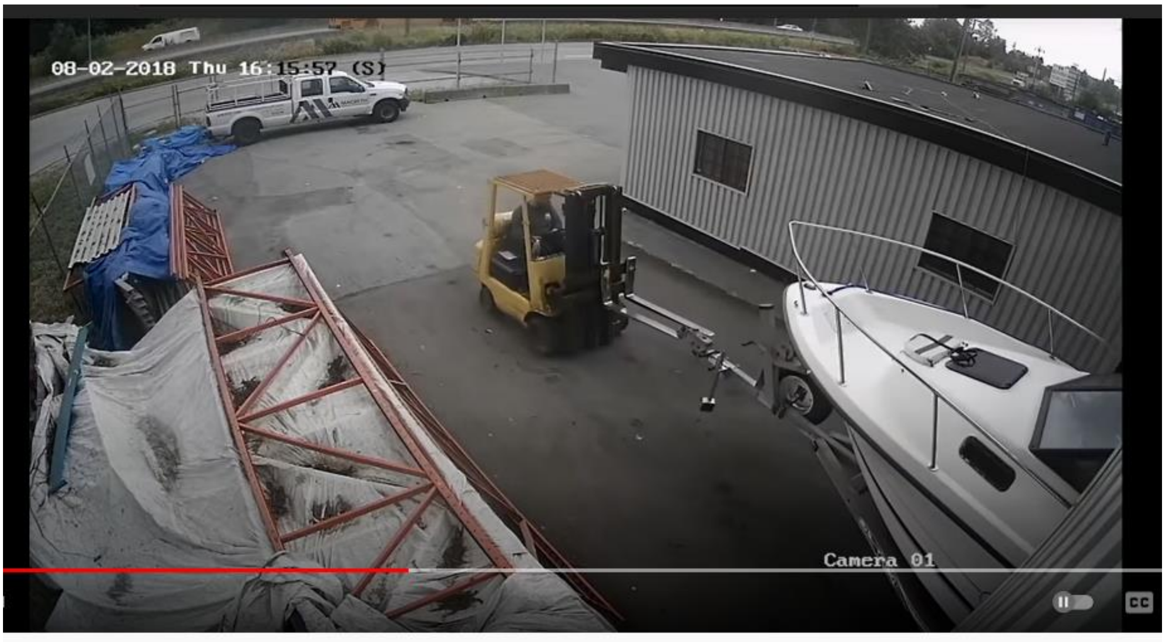

N. Solving the problem. Exercise 6 Unit 1:
Car starts from the origin and has acceleration $(t) = 2t - 5\ {}^{m}/_{s^2}$. Find the function of velocity of the car!

O. Recommendation for students to solve the exercises included at the end of the first and second unit.

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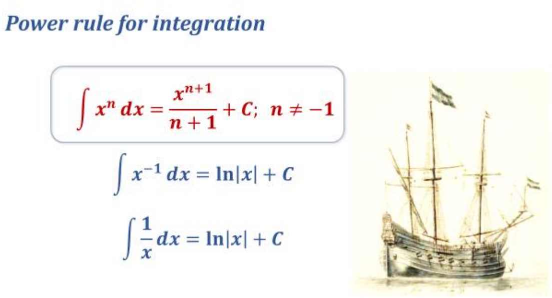
APPENDIX: Video Clip

Integration of power function

Video clip explains some special cases of integration of power function. The viewer's attention is focused to the correct application of formula. It demonstrates the wrong application of the formula and shows the correct solution of problem. The challenge for viewer is given at the end of video clip where is necessary to recall the knowledge about properties of exponents in the problem solving. The correct solution is explained.



If the power function is in the denominator

$$\int \frac{1}{x^n}\,dx$$

$$\int \frac{1}{x^n}\,dx \neq \frac{1}{n+1} \cdot \frac{1}{x^{n+1}} + C \quad ? \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x^n}\,dx = \int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1} + C = \frac{1}{1-n} \cdot \frac{1}{x^{n-1}} + C$$

Additional formulas

$$\int \frac{1}{x^n}\,dx = \frac{1}{1-n} \cdot \frac{1}{x^{n-1}} + C$$

Specialcase

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

Function with a fractional exponent

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^{4/7}\,dx = $$

$$= \frac{x^{4/7+1}}{4/7+1} + C = \frac{x^{11/7}}{11/7} + C =$$

$$= \frac{7}{11} x^{11/7} + C$$

Integral of square root

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \sqrt{x}\,dx =$$

$$= \int x^{1/2}\,dx =$$

$$= \frac{x^{3/2}}{3/2} + C = \frac{2}{3} x\sqrt{x} + C$$

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Challenge

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{3\sqrt[4]{x \cdot x^{-1/2}}}{4x^{0.25} \cdot \sqrt{x}}\,dx =$$

$$= \frac{3}{4}\int x^{\frac{1}{4}\left(1-\frac{1}{2}\right)-\frac{1}{4}-\frac{1}{2}}\,dx = \frac{3}{4}\int x^{\frac{1}{8}-\frac{2}{8}-\frac{4}{8}}\,dx = \frac{3}{4}\int x^{-\frac{5}{8}}\,dx =$$

$$= \frac{3}{4} \cdot \frac{x^{-\frac{5}{8}+1}}{-\frac{5}{8}+1} + C = \frac{3}{4} \cdot \frac{x^{\frac{3}{8}}}{\frac{3}{8}} + C = \frac{3}{4} \cdot \frac{8}{3} \cdot x^{\frac{3}{8}} + C = 2\sqrt[8]{x^3} + C$$

Additional note

Add a couple of useful formulas to your ***special list***:

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$$

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Lesson 2: Integration Techniques – Substitution

Name of Unit	Workload	Handbook
Integration Techniques – Substitution	**Lecture: 90 min**	**Unit 3.1. Substitution**

DETAILED DESCRIPTION:

In this chapter we will investigate the methods of integration of composite functions. The Chain Rule of derivation will be presented as an argument for integration of composite functions. The reverse process will be presented that introduces the formula of the Reverse Chain Rule. The method of substitution can help to simplify the notation of an integral. It will be called u-substitution. Several examples are presented in the chapter. There are integrals of composite functions given where the inner function is either linear or non-linear.

AIM: To master the skills of substitution to solve the integrals of composite functions.

Learning Outcomes:

1. Students will acquire the method of changing the differential to compute integrals
2. Students will be able to carry out integration by making substitution
3. Students will recognize that the method of substitution is useful with integrals of composite functions

Prior Knowledge: rules of differentiation; basic rules of integration; algebraic formulas; knowledge of elementary mathematics.

Key words of this Unit: chain rule, composite function, elementary function, indefinite integral, reverse chain rule, u-substitution

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

1. Integration of composite functions
2. Reverse Chain Rule
3. Application of Reverse Chain Rule
4. The change of differential
5. Method of substitution
6. Examples

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7. Exercises
8. Solutions

Assessment strategies:

Assessing students' knowledge about the integration of power function; assessing students' knowledge about the composite functions during the lesson

Teacher Toolkit and Digital Resources:

- PowerPoint presentation to introduce reverse chain rule for integration of composite functions and u-substitution
- Whiteboard to solve the examples
- Videos
- Useful websites

To create the multiple choice tests:
https://quizizz.com/
https://kahoot.com/
https://www.socrative.com/
To find the theoretical explanation of the substitution method
a) Chapter Integration/ Integration by substitution:
https://www.mathsisfun.com/calculus/index.html
b) https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-9/v/u-substitution
To see the solution of an interesting example:
https://www.youtube.com/watch?v=KIRRxmxw4b4
To check the answers:
https://www.wolframalpha.com/calculators/integral-calculator/
https://www.symbolab.com/solver/indefinite-integral-calculator

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Integration Techniques: Substitution

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Discussion	Recall the definition of indefinite integral	Frontal and questioning	Active contributing to questions	
5 min	Test	Short quiz about the basic integration formulas	Check students' knowledge	Test their knowledge	Can you evaluate the integrals just applying derivative formulas?
5 min	Discussion	Recall the concept of composite function	Recall of main prerequisites	Active contributing to questions	What composite functions do you know?
10 min	Presentation	Concept of reverse chain rule	Frontal Explain's task and gives examples	Active listening	
15 min	Solving examples 3.1 – 3.3	Change of differential	Frontal explanation	Solving of examples	
20 min	Solving real-life problem	Rate of change as composite function	Frontal Videos	Active listening Contributing to questions	Which kind of compressed air and gas systems can be found on the vessel?
20 min	Presentation	Method of substitution	Frontal explanation	Active listening	
10 min	Discussion	Examples	Solution of examples, Ask for suggestions	Discussion Contributing the solving process	What are the difference between the methods of substitution and change of differential?

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson 2 (unit 3) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (main engine starting air system): **https://www.youtube.com/watch?v=fwy6UyoWQC0** • Software • Multiple choice test: Socrative or Quizizz, or Kahoot!, or other
Learning objectives	By the end of the lesson: • ***all*** students should understand the method of substitution • ***all*** students should understand how to substitute composite function

A. The lecturer starts the lesson by short survey about main concepts introduced in the previous lesson. He/she can ask students to formulate the definition of the indefinite integrals, asks, for example, *"What is a prime function? What means the letter C added to the prime function?"*

B. Students have short multiple choice test about the simplest integration formulas, for instance, about the integration of power function (see Appendix 1). The test can be organized applying some software (Kahoot!, Socrative, etc.) or presented on slides.

C. Discussion about the composite functions. Lecturer gives different examples like as:

$$y = \sin 2x;\;\; y = (1+x)^3;\;\; y = \tan x;\;\; y = \ln\left(\ln\left(3\sqrt{x^2-4}\right)\right);\; r = 2\varphi$$

The challenging question is *"Which of the functions are elementary, basic elementary, and composite?"*

D. Concept of the reverse chain rule. Lecturer asks students to define the chain rule for differentiation of composite functions. Lecturer demonstrates some examples. Then he/she explains the reverse procedure - integration of the gained result of differentiation and points out that the integrand must contain the derivative of the argument function.

E. Lecturer demonstrates some simple integrals of composite functions and asks the students to call the answers. The integrals can be, for instance,

$$\int \sin(x^2)\cdot 2x\,dx\,; \qquad \int \frac{\ln^5 x}{x}dx$$

F. Next topic is about the case if the argument of composite function is linear

$$\int f(ax+b)dx$$

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Lecturer recalls the rule for calculation of differential

$$dy = y'dx \text{ if } y = y(x)$$

and explains how to change the differential just using well known method "*multiplication by 1*"

$$dx = \frac{a}{a}dx = \frac{1}{a}d(ax + b)$$

G. The class is solving some examples applying now introduced method.

H. Lecturer starts discussion about the compressed air units on the vessel "*Which kind of compressed air and gas systems can be found on the vessel?*". Teacher demonstrates the video about the main engine starting air system: https://www.youtube.com/watch?v=fwy6UyoWQC0

I. Lecturer solve the real-life problem of the inflating the balloon with air (see Appendix 2). This task considers a composite function.

J. The method of u-substitution has to be explained

$$\int f(u)u'dx = \int f(u)du$$

Lecturer has to underline that by applying u-substitution the expression of integrand becomes simpler. He/she evaluates several integrals.

K. At the end part of the lesson must be a discussion. Lecturer presents integrals of composite functions and asks the students on the advice for evaluating those. For instance, *"What function we will substitute by u?"; "What is the differential of the chosen function?"; "How we can replace the integrand and differential dx?"*:

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$$

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L. The summary of the lesson must be given at the end recalling main terms – composite function, reverse chain rule, linear argument of composite function, u-substitution.

M. The set of exercises should be recommended for students' individual work (see Unit 3, Chapters "Exercises" and "Solution").

N. The video clip "Tips and Tricks of Integration: Part 2: Integration of composite functions – change of differential" on MareMathics website can be recommended: **https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/**

(see Appendix 3)

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APPENDIX 1: Creation of Multiple Choice Test

The screenshot shows how to create one of the questions for the test about the integration of power functions in software Quizizz.

APPENDIX 2: Example of the inflating the balloon with air

Problem. An empty spherical balloon is given. The air is pumped into the balloon. The filling rate is given as a function of the change of the volume as a function of time t:

$$\frac{dV}{dt} = \frac{4}{2t+1}\ cm^3/_{sec}$$

Detect the rate of change of the radius!

Solution

The volume of the balloon we can express in two ways

By geometry formula $V = \frac{4}{3}\pi r^3$ and as the integral of rate of change of the volume

$$V(t) = \int \frac{4}{2t+1} dt$$

At first we will find the radius as a function of time. We integrate

$$V(t) = \int \frac{4}{2t+1} dt = 2\ln(2t+1) + C$$

As the balloon is empty at the initial moment we get $C = 0$.

Now we can express radius as a function of the time

$$r(t) = \sqrt[3]{\frac{3V(t)}{4\pi}} = \sqrt[3]{\frac{3 \cdot 2\ln(2t+1)}{4\pi}}$$

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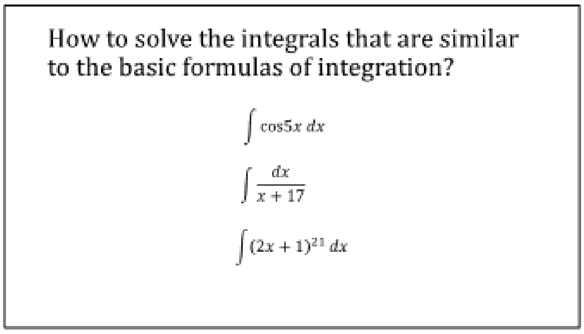
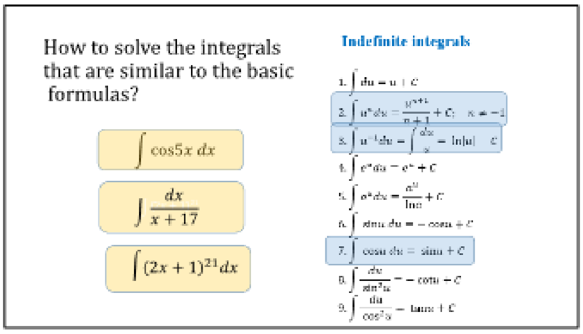
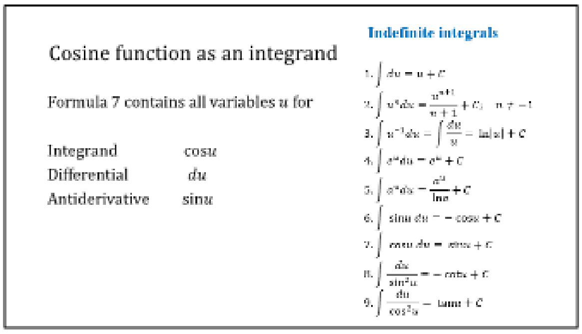
The rate of change of the radius is

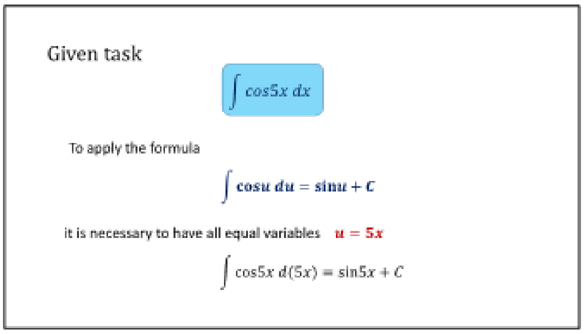
$$\frac{dr}{dt} = \sqrt[3]{\frac{3}{2\pi}} \cdot \frac{1}{3}(ln(2t+1))^{-\frac{2}{3}} \cdot \frac{2}{2t+1}$$

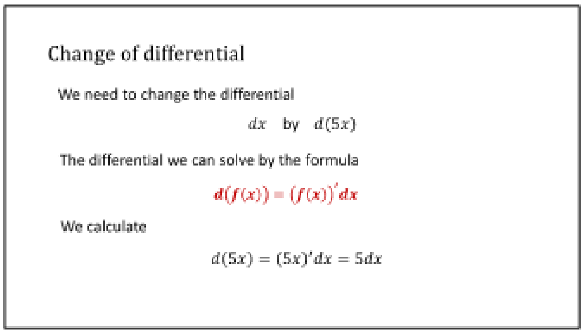
APPENDIX 3: Example Tips and Tricks of Integration











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Solving of given integral

$$\int \cos 5x \, dx =$$

Multiplication by 1

$$= 1 \cdot \int \cos 5x \, dx = 5 \cdot \frac{1}{5} \int \cos 5x \, dx =$$

$$= \frac{1}{5} \int \cos 5x \, d(5x) =$$

Moving the constant 5 under the integral and then under the differential

$$\int \cos 5x \, d(5x) = \sin 5x + C$$

$$= \frac{1}{5} \sin 5x + C$$

Summary

For composite functions with a linear argument we apply the formula

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(ax+b)\,d(ax+b) = \frac{1}{a}F(ax+b) + C$$

In shorter form

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

Conclusion

To solve the integral of composite function with a linear argument

$$\int f(ax+b)\,dx$$

1. - find the approriate basic formula $\int f(u)\,du = F(u) + C$
2. - detect the multiplier a of the argument x for $u = ax + b$
3. - multiply the antiderivative by an inverse constant $\frac{1}{a}$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

Example 1

$$\int \frac{dx}{x+17}$$

1. $\int \frac{1}{u}\,du = \ln|u| + C$
2. $u = x + 17; \quad a = 1$
3. $\int \frac{dx}{x+17} = \ln|x+17| + C$

Example 2

$$\int (2x+1)^{21}\,dx$$

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

   $n = 21 \qquad \int x^{21}\,dx = \frac{x^{22}}{22} + C$
2. $u = 2x + 1; \quad a = 2$
3. $\int (2x+1)^{21}\,dx = \frac{1}{2} \cdot \frac{(2x+1)^{22}}{22} + C$

Challenge

Try to solve the following exercise using here described method:

$$\int \frac{4dx}{\sqrt{8 - \frac{x}{4}}}$$

Answer

$$-32\sqrt{8 - \frac{x}{4}} + C$$

Additional note

Add some useful formulas to your *special list*:

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \frac{dx}{x+a} = \ln|x+a| + C$$

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Lesson 3: Integration Techniques - Integration by parts

Name of Unit	Workload	Handbook
Integration Techniques - Integration by parts	**Lecture: 90 min**	**Unit 3.2. Integration by parts**

DETAILED DESCRIPTION:

The section starts by recalling the Product rule for differentiation of multiplication of two functions. The integration of this formula produces the method of integration by parts. The application of this method is useful if the integrand is a product of two functions of special type. The most popular cases are discussed and are complemented by examples.

AIM: to learn the method of integration by parts and to recognize the types of integrals for which the method is useful.

Learning Outcomes:

1. Students recognize the integrals for that the integration by parts is useful.
2. Students can apply the method of partial integration to compute the integrals of different type.

Prior Knowledge: rules of differentiation; rules for integration; the method of substitution; algebra and trigonometry formulas.

Key words of this Unit: differentiation, indefinite integral, integration by parts

Relationship to real maritime problems: a well-known application of the method of integration by parts is the calculation of Fourier coefficients of the Fourier series. Fourier series have broad applications in many disciplines. They are used to describe periodical physical phenomena, for instance, in signal processing, to detect and correct sources of vibration in mechanical devices.

Content

1. Formula for integration by parts
2. Special cases
3. Examples
4. Repeated application of the method
5. Exercises
6. Solutions

Assessment strategies:

Assessing students' knowledge about the basic integrals and about formulas of derivatives during the lesson.

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Teacher Toolkit and Digital Resources:

- PowerPoint presentation to introduce the method of integration by parts
- Whiteboard to solve the examples
- Worksheet about the differentials
- Video
- Useful websites
  https://www.mathsisfun.com/calculus/integration-by-parts.html

  To see several examples of solving integrals by parts:
  https://www.youtube.com/watch?v=sWSLLO3DS1I
  Application of special methods to solve the integrals partially
  https://www.youtube.com/watch?v=Scl_r00DPwA

Integration Techniques: Integration by Parts

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Discussion	Recall of derivative of the product of functions	Frontal and questioning	Active contributing to questions	
10 min	Presentation	Introduction of the formula of partial integration	Frontal	Active listening	
20 min	Solving examples 3.1 – 3.3	Special cases for application partial integration	Solving examples, explaining, asking questions	Active contributing to questions	
15 min	Test	Recall of differentiation rules	Present worksheet about the calculation of differentials	Completing worksheet, self-assessment	What is the meaning of the differential?
15 min	Solving real-life problem	Problem about the sewage treatment unit (application of integrals, example 7)	Frontal discussion Video	Discussion Contributing to questions	How can ships be more environmentally friendly?
15 min	Presentation Examples	Repeated application of method	Frontal explanation	Active listening	
10 min	Discussion	Examples	Solution of examples, Ask for suggestions	Discussion Contributing the solving process	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson 3 (unit 3.2) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (sewage treatment plant): https://www.youtube.com/watch?v=u0Zwl7LdavM • Video (tabular method): https://www.youtube.com/watch?v=spsFNH2pNj4
Learning objectives	By the end of the lesson: • ***all*** students should understand how to apply the method of partial integration

Lesson 3:

A. The lecturer recalls the formula for derivation of a product of functions.

B. The method of integration by parts is introduced and explained applying the integration of the derivation formula for functions' product.

C. The example of integral is presented and wrong choice of the *u*-function is presented. The gained result should be discussed: *"What happens with the polynomial if it is differentiated? What happens with exponent and trigonometric functions?"* The example, for instance, can be:

$$\int x\sin x\,dx = \begin{vmatrix} u = \sin x & du = \cos x\,dx \\ dv = x dx & v = \frac{x^2}{2} \end{vmatrix} = \frac{x^2}{2}\sin x - \frac{1}{2}\int x^2 \cos x\,dx = \cdots$$

D. After the comparison of the given integral the correct application of the given method is underlined – second integral should not be more complicated as given one.

E. Lecturer solves the example using the right choice of the *u*-function.

F. Lecturer presents standard integral cases where it is advantageous to apply integration by parts. He/she discusses the correct choice of *u*-function in any of standard cases.

G. The LIATE method about how to memorize which function should be chosen as *u*-function is presented. See:

Herbert E. Kasube, A Technique for Integration by Parts, The American Mathematical Monthly, 90 (1983). 210–211

H. Examples 3.1 – 3.3 can be solved with active participation of students.

I. Emphasizing the importance of the differentials, the worksheet on the calculation of the differentials of functions is given for students' self-assessment (see Appendix 1)

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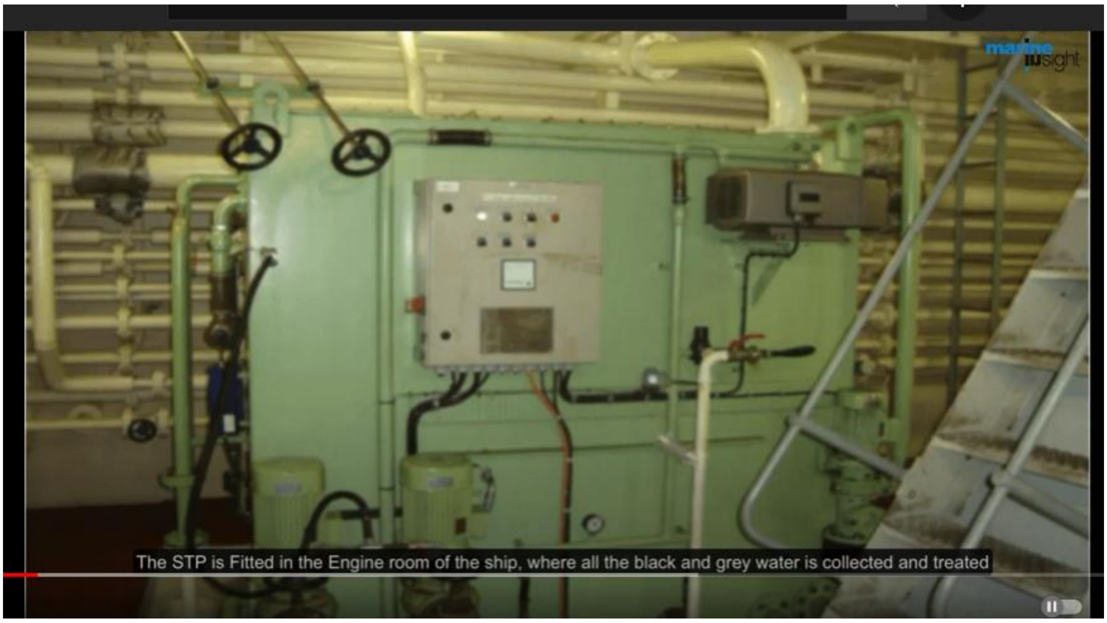
J. Short discussion about the environmental protection requirements in maritime is conducted. The video clip about the sewage treatment unit is demonstrated. https://www.youtube.com/watch?v=u0Zwl7LdavM



K. The problem of the decay time of sewage can be solved (see Appendix 2). It is recommended to solve the given improper integral as an indefinite integral just to demonstrate the application of integration by parts.

L. Lecturer continues to discuss the given method and shows the evaluation of integral where it is necessary to use integration by parts repeatedly. For instance,

$$\int x^2 e^x \, dx$$

M. In the cases where the repeated application of partial integration is necessary the integrals can be evaluated by tabular method. Tabular method is useful if the polynomial is of higher order than one. Teacher can show the video clip of the tabular integration. https://www.youtube.com/watch?v=spsFNH2pNj4

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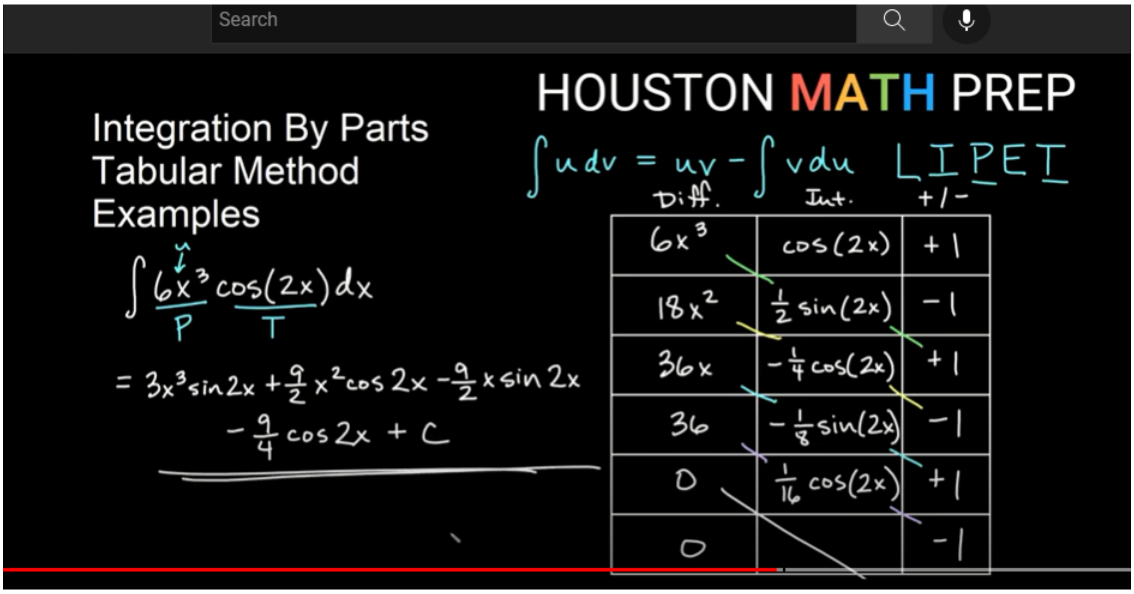

N. The exercises should be solved with active contributing of students. Lecturer asks for the advices which of the functions can be used as a u-function.

O. At the end of the lesson lecturer recommends to evaluate the remaining exercises form the Unit 3.2. Lecturer gives a challenge for students – to solve the following integral:

$$\int \cos x \, e^x \, dx \text{ or } \int \sqrt{1 + x^2} \, dx$$

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APPENDIX 1: Worksheet: Test about the differentials

Part I Calculate the given differentials and mark the correct answer!

1. $d(\sin x)$

A	$(\cos x)'dx$	**B**	$\cos x$	**C**	$\cos x\,dx$	**D**	$\sin 2x\,dx$

2. $d(4x^3 - 8x)$

A	$3 \cdot 4(x^2 - 8)\,dx$	**B**	$(12x^2 - 8)\,dx$	**C**	$(x^4 - 8)\,dx$	**D**	$4(3x^2 - 2)dx$

3. $d(5^x)$

A	$5^x ln5\,dx$	**B**	$x5^{x-1}\,dx$	**C**	$\frac{5^x}{5}\,dx$	**D**	$5\,dx$

4. $d(\cos 3x)$

A	$3\cos x\,dx$	**B**	$-3\sin x\,dx$	**C**	$-\sin 3x\,dx$	**D**	$-3\sin 3x\,dx$

5. $d(\text{artctan}x)$

A	$\frac{dx}{1+x^2}$	**B**	$\frac{1}{cos^2x}dx$	**C**	$\frac{1}{1+x}dx$	**D**	$\frac{dx}{1+\cos x^2}$

Answers

Part I

1 – C; 2 – B; 3 – A; 4 – D; 5 – A

Solution

Part I Apply the formula $\boldsymbol{dy = y'\,dx}$ $if\;\; y = y(x)$

Example 1

$$d(\sin x) = (\sin x)'dx = \cos x\,dx$$

Example 2

$$d(4x^3 - 8x) = (4x^3 - 8x)'dx = (12x^2 - 8)\,dx$$

Example 3

$$d(5^x) = (5^x)'dx = 5^x \ln 5 dx$$

Example 4

$$d(\cos 3x) = (\cos 3x)'dx = -\sin 3x \cdot 3\,dx = -3\sin 3x\,dx$$

Example 5

$$d(\arctan x) = (\arctan x)'dx = \frac{1}{1+x^2}dx$$

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APPENDIX 2: Decay time of sewage

Unit: Application of definite integrals: Example 7. According to the requirements of MARPOL on every vessel has been installed sewage treatment unit. Air compressors blow the air through the sewage continuously. Bio-active substances must be supplied in the unit periodically. The time period of supplement can be computed according to the mean decay time of sewage

$$\bar{t} = \int_0^{\infty} t \cdot ke^{-kt} dt,$$

where k is the constant that characterises the velocity of the decay. In the formula is used an improper integral (see the topic: Improper integrals).

Solution

$$\bar{t} = \int_0^{\infty} t \cdot ke^{-kt} dt = \lim_{T\to\infty} \int_0^{T} t \cdot ke^{-kt} dt =$$

$$= \left| \begin{array}{l} \text{let} \quad u = t; \qquad dv = ke^{-kt} dt \\ \text{then } du = dt; \quad v = k\int_0^T e^{-kt} dt = -\dfrac{ke^{-kt}}{k} \end{array} \right| =$$

$$= \lim_{T\to\infty} \left(-te^{-kt}\Big|_0^T + \int_0^T e^{-kt} dt \right) =$$

$$= \lim_{T\to\infty} \left(Te^{-kT} - \frac{e^{-kt}}{k}\Big|_0^T \right) =$$

$$= \lim_{T\to\infty} \left(Te^{-kT} - \frac{e^{-kT}}{k} + \frac{1}{k} \right) = \frac{1}{k}$$

Answer. Expected decay time of sewage per time unit is $\bar{t} = \frac{1}{k}$.

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Lesson 4 and Lesson 5: Integration Technics – Integration of Partial Fractions

Name of Unit	Workload	Handbook
Part 1: Integration Techniques – Integration of rational functions	**Lecture: 90 min**	**Unit 3.3. Integration of rational functions**
Part 2: Continuation	**Lecture: 90 min**	**Unit 3.3. Integration of rational functions**

DETAILED DESCRIPTION

In this chapter we will discuss the problem-solving methods for indefinite integrals of rational functions. We will introduce proper rational functions and improper rational functions, and algebraic methods on how to decompose them into rational fractions or powers and rational fractions. Appropriate integral formulas will be considered. Examples of integration of simple rational functions will be demonstrated.

Students can investigate the examples presented by Symbolab Step-by-Step Calculator; Algebra; Rational Fractions

(URL: https://www.symbolab.com/solver/partial-fractions-calculator). With this software it is also possible to check their own solutions by applying the Symbolab calculator for integrals.

AIM: To acquire the technique of integration of rational functions.

Learning Outcomes:

1. Perform the expansion of proper rational function in partial fractions
2. Perform the long division of polynomials to get a polynomial plus a proper rational function
3. Solve the integrals of rational functions

Prior Knowledge: algebraic identities; completing the square; factorising of polynomials; roots of polynomials; basic integration and derivation formulas.

Keywords: decomposition of partial fractions; long division of polynomials; rational function

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

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Content

1. Rational functions, proper and improper rational functions
2. Basic integrals for simple cases
3. Decomposition of partial fractions
   3.1. Case 1. Denominator can be factorised in all linear multipliers
   3.2. Case 2. Denominator contains an irreducible quadratic
   3.3. Case 3. Denominator contains the repeated linear factor
4. Computation of improper rational functions
5. Summary
6. Exercises
7. Solutions of the exercises

Assessment strategies:

Assessing students' algebraic knowledge during the lesson; check students' understanding on application of partial integration as e results of the previous lesson

Teacher Toolkit and Digital Resources:

- PowerPoint presentation to introduce different algebraic methods for decomposing partial fractions
- Whiteboard to solve the examples
- Videos: short video Mercator map creation: https://www.youtube.com/watch?v=CPQZ7NcQ6YQ
- Useful websites

To find some historical facts about the application on integrals of partial functions:

https://en.wikipedia.org/wiki/Integral_of_the_secant_function#History

To find the theoretical explanation of the integration of rational fractions:

a) MITOPENCOURSEWARE: https://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/unit-4-techniques-of-integration/
b) Khan Academy: https://www.khanacademy.org/math/ap-calculus-bc/bc-integration-new/bc-6-12/v/integration-with-partial-fractions

To check the answers:

https://www.wolframalpha.com/calculators/integral-calculator/
https://www.symbolab.com/solver/indefinite-integral-calculator

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Lesson 4: Integration Techniques: Integration of Partial Fractions- Part 1

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Recall of the previous topic	Recall the method of integration by parts	Frontal and questioning	Active contributing to questions	
10 min	Worksheet	Answering on the question on applying the method of partial integration	Check students' knowledge	Show their knowledge of the acquired topic	Whether you selected the u-function correctly?
5 min	Presentation	Concept of rational function	Recall of main prerequisites	Active listening	
5 min	Presentation	Definitions of proper and improper rational part	Frontal Explains task and gives examples Ask questions	Active contributing to questions	
10 min	Solving basic cases	Integrals of elementary rational parts	Frontal Explanation Solving of examples	Solving of examples	
10 min	Presentation	Introduction of decomposition method	Frontal explanation	Active listening	
20 min	Presentation Solving example	Case 1: Denominator includes all linear multipliers	Explanation, solving examples	Active participation in solving example	What are equivalent polynomials?
10 min	Discussion	Example of real-life application	History facts, discussion, solution of example, video	Discussion	"What is Mercator map and how it is used in navigation"
20 min	Presentation Solving example	Case 2: Denominator contains an irreducible quadratic	Frontal explanation	Active participation in solving example	

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Lesson 5: Integration Techniques: Integration of Partial Fractions-Part 2

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Recall of the problem statement	Recall the first two cases of decomposition of rational part	Frontal and questioning	Active contributing to questions	
20 min	Presentation Solving example	CASE 3: DENOMINATOR CONTAINS THE REPEATED LINEAR FACTOR	Explanation, solving examples	Active participation in solving example	
10 min	Recall of algebra knowledge	Improper rational parts: long division of polynomials	Frontal explanations	Active contributing in the formulation of algebraic relations	
10 min	Solving	Several examples of long division	Solving examples, asking questions to students	Active participation in solving the tasks	
10 min	Summary	The summary of possible transformation of rational fraction and integration of i	Summarising highlighting the most important steps	Active listening	What basic integration formulas we can use for evaluation elementary rational parts?
20 min	Solving examples	Solution of examples where the long division of polynomials and decomposition of proper rational part is necessary	Solving, explaining, commenting students' solutions	Solving examples	
15 min	Solving exercises	The set of exercises for students	Solving, explaining, commenting students' solutions	Solving exercises, asking questions to lecturer	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lessons 4 an 5 (unit 3.3) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (Mercator map): https://www.youtube.com/watch?v=CPQZ7NcQ6YQ • Worksheet • Integration by Parts
Learning objectives	By the end of the lessons: • ***all*** students can distinguish the proper rational part from improper rational part • ***all*** students should carry out the long division of polynomials • ***all*** students know the methods of decompose of rational fractions

Lesson 4:

A. The lecturer should control the quality of students' knowledge about the previous topic on integration by parts. He/she recalls the formula of partial integration and asks for standard cases of application of it.

B. Students have to perform the tasks given in worksheet (see Appendix 1). After that teacher can collect the answers to evaluate students' knowledge. Short discussion about the more complicated cases will take a place.

C. Lecturer introduces new topic what types of integrals have to be evaluated:

$$\int \frac{P_n(x)}{Q_m(x)}dx$$

Lecturer recalls the analytic expression of the polynomial and gives some examples.

D. He/she defines proper and improper rational part. The examples of different type are given for student to discuss which of the functions are proper and which of the functions improper rational parts are. For instance,

$$g(x) = \frac{2x-1}{x^2-4}; \quad t(x) = \frac{x+1}{x-3}$$

E. The basic elementary rational parts and their integrals are presented. Students are advised to add the formula for integration of the linear expression in the denominator in their formula list:

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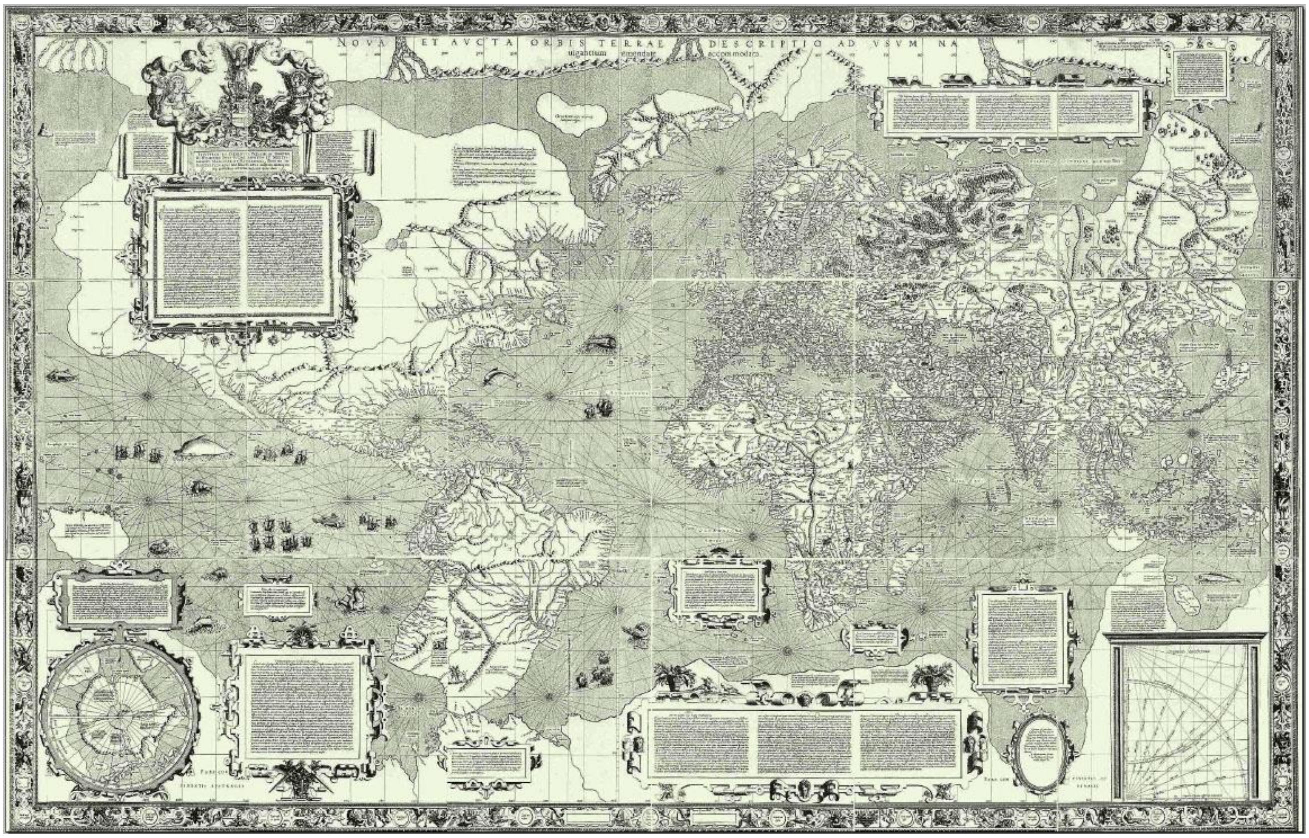
$$\int \frac{dx}{x+B} = \ln|x+B| + C; \quad B \in \mathbb{R}$$

F. Lecturer explains the decomposition of the proper rational parts in different cases. Case 1: *Denominator can be factorised in all linear multipliers*.

The question *"What are equivalent polynomials?"* should be discussed. The insertion method (or "plug-in" method) for calculation of unknown coefficients are explored.

G. Lecturer explains the example of evaluation of proper rational function step by step (see example 3.2 in Appendix 2).

H. The discussion about the application on integrals of rational functions should take a place: *"What is Mercator map and how it is used in navigation".* Lecturer gives some history facts about the integral of secant function that is important for accurate construction of Mercator projection:



A Mercator projection created by Gerardus Kremer (Gerardus Mercator, 1569)

https://upload.wikimedia.org/wikipedia/commons/b/b2/Mercator_1569.png

Short video clip can be demonstrated – animation of Mercator map creation https://www.youtube.com/watch?v=CPQZ7NcQ6YQ

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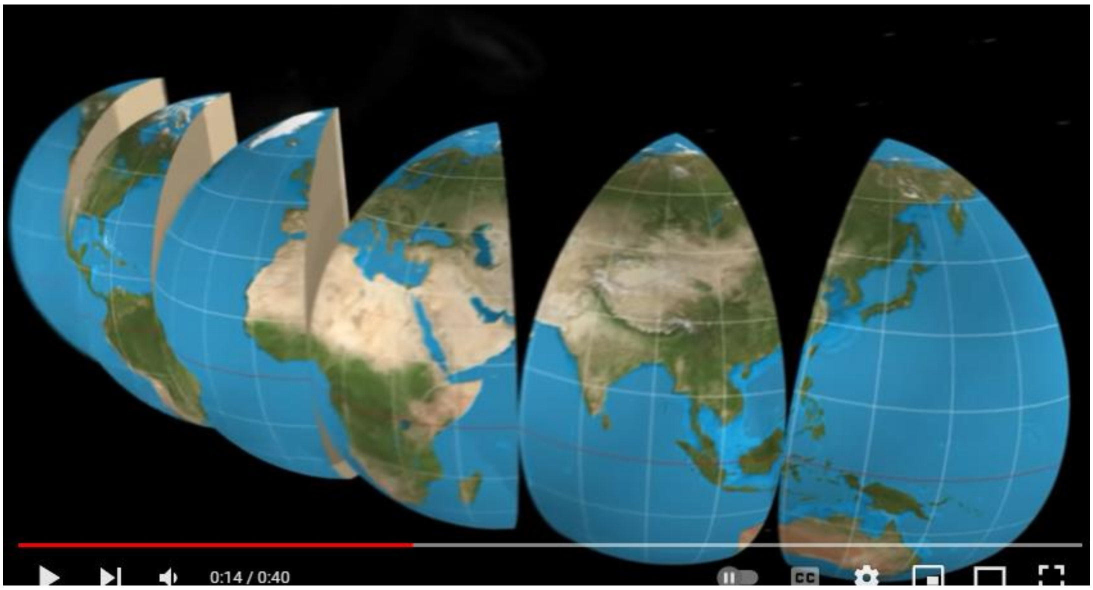

G. What is the relationship between the secant function and rational function? Integral: $\int \sec x \, dx$ – an *"outstanding open problems of the mid-seventeenth century"* (see https://en.wikipedia.org/wiki/Integral_of_the_secant_function#History;). There are different methods how to solve this integral. For instance, using appropriate substitution we get the integral of rational function (see Appendix 3).

H. Lecturer introduces next topic: Case 2: *Denominator contains an irreducible quadratic* (see Unit 3.3) and explains the method of decomposition of the rational fraction.

I. The example 3.3 from Unit 3.3 is demonstrated and explained step by step. The method of unknown coefficients is introduced.

Lesson 5

J. The lecturer recalls the previous topic about the decomposition of proper rational part, first two cases.

K. He/she introduces case 3: *Denominator* ***contains the repeated linear factor*** and explains that unknown coefficients can be find by mixed method – using "plug-in" method in the beginning and then applying the method of unknown coefficients.

L. The example 3.4 from Unit 3.3 is demonstrated and explained step by step.

M. Presentation of improper rational parts. Recall of algebra knowledge on long division of polynomials.

N. Students solve the examples of long division. The examples are recommended of the division of polynomials with some zero coefficients too, for instance,

$$\mathbf{(2x^4 - 8x + 11) : (2x + 1)}$$

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O. Lecturer gives the summary of the evaluation of the integral of the rational function – what is an action if the integrand is improper rational part or if the integrand is a proper rational part.

P. The example 5.1 from Unit 3.3 is demonstrated and explained step by step.

Q. The set of exercises should be recommended for students' individual work (see Unit 3.3, Chapters "Exercises" and "Solution").

R. The video clip "Tips and Tricks of Integration: Part 4: Struggle with a rational function" on MareMathics website can be recommended:

https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/

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APPENDIX 1: Worksheet Integration by Parts

Integration by parts

$$\int \boldsymbol{u}\,\boldsymbol{dv} = \boldsymbol{u}\cdot\boldsymbol{v} - \int \boldsymbol{v}\,\boldsymbol{du}$$

For given integrals **choose** the functions for $\boldsymbol{u}$**-substitution**:

x; $8x$; $2x+1$; x^2; $\sin x$; $\cos x$; $\cos^2 6x$; e^x; $\ln x$; $\lg x$; $\arcsin x$; $\arctan x$

Nr	Integral	$\boldsymbol{u}$-function
1	$\int x^2 \cos x\, dx$	$u = x^2$
2	$\int 8x \sin x\, dx$	
3	$\int x \arctan x\, dx$	
4	$\int (2x+1)e^x dx$	
5	$\int 8x \ln x\, dx$	
6	$\int \frac{\lg x}{x^2}\, dx$	
7	$\int e^x \sin x\, dx$	
8	$\int \frac{6x}{\cos^2 6x} dx$	
9	$\int (2x+1)\ln x\, dx$	
10	$\int \arcsin x\, dx$	
11	$\int x^2 \ln x\, dx$	

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Answers

Nr	Integral	u-function
1	$\int x^2 \cos x \, dx$	$u = x^2$
2	$\int 8x \sin x \, dx$	$u = 8x$
3	$\int x \arctan x \, dx$	$u = \arctan x$
4	$\int (2x+1)e^x dx$	$u = 2x + 1$
5	$\int 8x \ln x \, dx$	$u = \ln x$
6	$\int \frac{\lg x}{x^2} \, dx$	$u = \lg x$
7	$\int e^x \sin x \, dx$	$u = \sin x \; or \; u = e^x$
8	$\int \frac{6x}{\cos^2 6x} dx$	$u = x$
9	$\int (2x+1) \ln x \, dx$	$u = \ln x$
10	$\int \arcsin x \, dx$	$u = \arcsin x$
11	$\int x^2 \lg x \, dx$	$u = \lg x$

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APPENDIX 2: Example of Integration of proper rational function; Case 1

Unit 3.3: Integration Technics: Integration of Rational Functions

Example 3.2

Compute the integral

$$\int \frac{4x+7}{x^2+x-6}dx$$

Solution

First part: decomposition of partial fractions

Step 1. Factorise the denominator

$$x^2+x-6=(x-2)(x+3)$$

Step2. Write partial fractions with unknown constants

$$\frac{4x+7}{x^2+x-6}=\frac{4x+7}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$$

Step 3. Equalize the denominators, equate the numerators, and discard denominators

$$4x+7=A(x+3)+B(x-2)$$

Step 4. Plug in $x=2$ to calculate constant A

$$4\cdot 2+7=A(2+3)+B(2-2)$$

$$8+7=5A$$

$$A=3$$

Step 5. Plug in $x=-3$ to calculate constant B

$$4\cdot(-3)+7=A(-3+3)+B(-3-2)$$

$$-12+7=-5B$$

$$B=1$$

Second part: integration

$$\int \frac{4x+7}{x^2+x-6}dx=\int\left(\frac{3}{x-2}+\frac{1}{x+3}\right)dx=$$

$$=3\ln|x-2|+\ln|x+3|+C$$

Answer

$$\int \frac{4x+7}{x^2+x-6}dx=3\ln|x-2|+\ln|x+3|+C$$

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APPENDIX 3: Integral of secant function

$$\int \sec x\, dx = \int \frac{1}{\cos x} dx = \int \frac{1}{\cos x} \cdot \frac{\cos x}{\cos x} dx = \left| \begin{matrix} u = \sin x; & du = \cos x\, dx \\ \cos^2 x = 1 - u^2 & \end{matrix} \right| =$$

$$= \int \frac{1}{1-u^2} du = \frac{1}{2}\int \frac{1}{1+u} du - \frac{1}{2}\int \frac{1}{1-u} du = \text{ (fraction expantion)}$$

$$= \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| + C =$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

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Lesson 6: Integration Techniques - Integration of Trigonometric Functions

Name of Unit	Workload	Handbook
Integration Techniques - Integration of Trigonometric Functions	**Lecture: 90 min**	**Unit 3.4. Integration of Trigonometric functions**

DETAILED DESCRIPTION

Various oscillation processes can be described by trigonometric functions. The research of such processes requires the calculation of integrals where integrands are composite functions. Trigonometric identities are useful to modify these integrals. In this chapter we will present the application of trigonometric formulas for more common cases and the appropriate substitution for solving integrals. The method of trigonometric substitution will be introduced additionally.

AIM: **To learn the use of trigonometric identities and special cases of substitution for trigonometric integrands.**

Learning Outcomes:

1. Students will be able integrate the integrals of trigonometric functions applying some trigonometric identities.
2. Students can apply the trigonometric substitution.

Prior Knowledge: rules of integration and differentiation; substitution methods for integrals; algebra and trigonometry formulas.

Keywords: composite functions, trigonometry, substitution

Relationship to real maritime problems: trigonometric integrals are useful for describing and for solving different problems on sinusoidal processes – for example, to construct an effective shape of a ship propellers' blades, or to calculate wave resistance for steady motion in ship's control equipment.

Content

1. Composite trigonometric functions of a linear argument
2. Product of sines and cosines
3. Powers of trigonometric functions
4. Double-angle trigonometric identity
5. Trigonometric substitution
6. Exercises
7. Solutions

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Assessment strategies:

Assessing students' knowledge about proper and improper rational parts by test; the knowledge about the trigonometry during the lesson.

Teacher Toolkit and Digital Resources:

- PowerPoint presentation of examples for quick test
- Presentation about the integration of trigonometric functions
- GeoGebra tool
- Whiteboard to solve the examples
- Useful websites

    Explanation of some simple examples

        https://www.zweigmedia.com/RealWorld/trig/trig4.html

    Methods

        https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-intusingtrig-2009-1.pdf

    Set of exercises

        https://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/trigintdirectory/TrigInt.html

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Integration Techniques: Integration of Trigonometric Functions

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
10 min	Test	Quick yes/no questions about proper and improper rational part	Present examples one by one	Students' self-assessment	
10 min	Presentation	Introduction to the integrals of trigonometric functions	Frontal, construction of graphs, recall the method of integration of composite functions	Active listening, answering the questions	
10 min	Presentation	Product of trigonometric functions with different variables	Recalling trigonometry formulas, solving examples	Participation in the solving of examples	What are the ways to change trigonometric expressions?
20 min	Presentation Solving examples	Powers of trigonometric functions (with an odd exponent)	Introducing substitution methods, solving examples	Active listening, solving examples	
15 min	Presentation Solving examples	Double angle formulas	Explanation of the transformation of integrand	Active listening, solving examples	
5 min	Discussion	Trigonometry in navigation	Discussion	Discussion Contributing to question	Where are used trigonometry in real-life and in navigation?
20 min	Presentation Examples	Trigonometric substitution	Frontal explanation, ask students for advices	Active listening, contributing the questions	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • GeoGebra software • Lesson 6 (unit 3.4) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (bearing and trigonometry): https://www.youtube.com/watch?v=jnjRBawGJ_c • Video (police boat and criminal boat): https://www.youtube.com/watch?v=qAYd9FPYubk
Learning objectives	By the end of the lesson: • ***all*** students should know how to use the trigonometry formulas to transform the integrand • ***all*** students understand the application of trigonometric substitution

Lesson 6:

A. In the beginning lecturer gives the yes/no test about the proper and improper rational parts. He/she shows some examples of rational fractions and ask students' immediate reaction. Every separate example is displayed only for 10 or 20 seconds and then changed by other example (some 10 examples can be presented). Students have to mark answer "yes" or "no" (see Appendix 1). After the test students get correct answers and can evaluate their knowledge about the previous topic.

B. The lecturer starts the lesson with some simplest integrals of trigonometric functions – integrals of composite functions of a linear arguments. He/she recalls the method of integration of composite functions and recommends to students to add following formulas to the basic formula list:

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C\,; \quad \int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

The lecturer asks „Do the students remember the graphs of trigonometric functions and what is the difference of graphs of elementary functions and composite functions“. Some examples of composite and standard sine function should be construct using GeoGebra and the graphs of tangent and cotangent functions as well.

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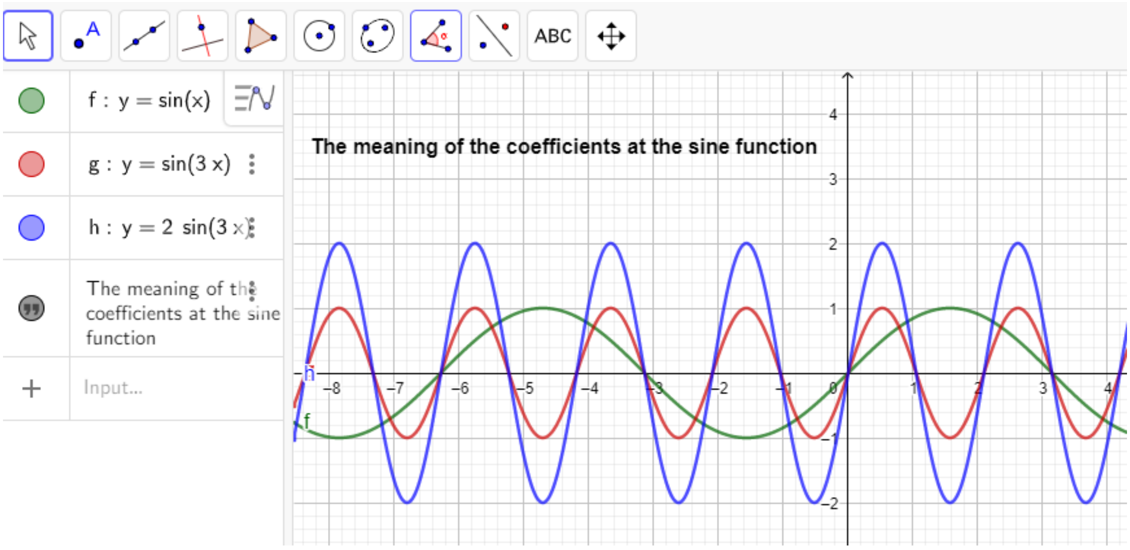

C. Next question is about the product of trigonometric functions with different variables. Lecturer recalls trig formulas – argument addition theorems

$$\sin ax \cdot \cos bx = \frac{1}{2}(\sin(ax+bx)+\sin(ax-bx))$$
$$\cos ax \cdot \cos bx = \frac{1}{2}(\cos(ax+bx)+\cos(ax-bx))$$
$$\sin ax \cdot \sin bx = \frac{1}{2}(\cos(ax-bx)-\cos(ax+bx))$$

He/she asks students give the advices how to apply formulas of composite sine and cosine functions to evaluate the integrals.

D. Next cases are about the integrand of trigonometric functions of the same argument. The first case is on the powers of sin and cosine functions

$$\int \sin^n x \cdot \cos^m x \, dx$$

Lecturer separate the cases if at least one of the exponent is an odd number or both exponents are even.

E. Lecturer explains that in the case of odd exponent the u-substitution takes a place, for example,

$$\int \cos^5 x \cdot \sin^3 x \, dx$$

The students have to advice which of the functions should be substitute. They have to recall the trigonometric identity – Pythagorean formula for sines and cosines.

F. The examples of negative exponents should be presented too. Lecturer demonstrates two calculation ways of such integral (see appendix 2).

G. The examples where trigonometric functions have only even exponents must be linearized and simplifies applying formulas

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$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin 2x = 2\sin x \cdot \cos x$$

Lecturer demonstrates example of one simple case and one more complicated case where different types of double angle formulas can apply (see Unit 3.4 examples 4.1 and 4.2).

H. The discussion about the application of trigonometry to describe the real-life processes take a place (examples to find the distance between the objects in the environment; to describe fluctuation; GPS navigation to find precise location of the object; trigonometry in architecture; trigonometry in industry to produce parts of special type; in navigation etc.)

The lecturer can recommend the video clip about the traveling of the boat – bearing (angle) and trigonometry for students' individual work: https://www.youtube.com/watch?v=jnjRBawGJ_c; and other problem about the police and criminal boats: https://www.youtube.com/watch?v=qAYd9FPYubk

I. The lecturer demonstrates 3 special cases for that the trigonometric substitution is useful:

Case 1. For $\sqrt{a^2 - x^2}$ we substitute $x = a\sin u$ or $x = a\cos u$

Case 2. For $\sqrt{a^2 + x^2}$ we substitute $x = a\tan u$

Case 3. For $\sqrt{x^2 - a^2}$ we substitute $x = \frac{a}{\cos u}$

J. Lecturer discuses with students how to make transformation of expressions for chosen u-substitution and solves one example underlying the necessity to return to the variable $\boldsymbol{x}$ (see Unit 3.4 example 5.1).

K. The exercises at the end part of the Unit 3.4 are recommended for students' individual work.

L. The video clip "Tips and Tricks of Integration – Part 5 – This difficult, difficult trigonometry" should be recommended for students

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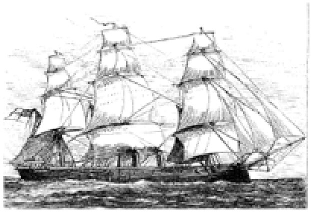

MareMathics: Innovative Approach in Mathematical Education for Maritime Students

Tips & Tricks of Integration
Part 5: This difficult, difficult trigonometry

Latvian Maritime Academy

APPENDIX 1: Quick Test about the proper and improper rational parts

The main question: "Is that a proper rational part?"

Examples:

$$\frac{x^2}{x^3+2};\quad \frac{5}{2x-1};\quad \frac{4x-7x^4+7}{3x^2+9};\quad \frac{5+3x}{4x+2};\quad \frac{3x^3-2x^2+5x+4}{x(x-3)(x+6)};\ \dots..$$

Answers: Yes ☐ No ☐

APPENDIX 2: Two methods of u-substitution

Evaluate the integral

$$\int \frac{\cos^3 x}{\sin^5 x}\,dx$$

Solution

Method 1:

$$\int \frac{\cos^3 x}{\sin^5 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^5 x}\,dx =$$

$$= \left|\begin{matrix} u = \sin x \\ du = \cos x\,dx \end{matrix}\right| = \int \frac{1-u^2}{u^5}\,du =$$

$$= \int u^{-5}\,du - \int u^{-3}\,du =$$

$$= \frac{u^{-4}}{-4} - \frac{u^{-2}}{-2} + C =$$

$$= -\frac{1}{4\sin^4 x} + \frac{1}{2\sin^2 x} + C$$

Method 2:

$$\int \frac{\cos^3 x}{\sin^5 x}\,dx = \int \frac{\cos^3 x}{\sin^3 x}\cdot\frac{1}{\sin^2 x}\,dx =$$

$$= \left|\begin{matrix} u = \cot x \\ du = -\dfrac{1}{\sin^2 x}\,dx \end{matrix}\right| = -\int u^3\,du = -\frac{u^4}{4} + C$$

$$= -\frac{\cot^4 x}{4} + C$$

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Lesson 7: Definite Integral – Introduction

Name of Unit	Workload	Handbook
Definite Integral – Introduction	**Lecture: 90 min**	**Unit 4. Definite integral**

DETAILED DESCRIPTION

This chapter introduces the definite integral. The description starts with the question: how to calculate the area of a region that is bounded by a curve and straight lines? The method of approximate calculation is discussed. Then this method is generalized, and the definition of the definite integral is formulated. Basic properties and the Newton-Leibniz formula are presented for calculations of integrals. For students, we can recommend the "Integral Calculator" https://www.integral-calculator.com/. This software shows the solution of an integral step by step and comes with an interactive graph of integrand and antiderivative. With a help of such software students can check their individual solutions of the task. The graphs can be constructed using free software GeoGebra Classic; DESMOS Graphing Calculator, and Microsoft Excel.

AIM: Learn the definition of the definite integral as the limit of a sum.

Learning Outcomes:

1. Understand the meaning of the definite integral
2. Understand and apply the rules for calculating definite integrals

Prior Knowledge: basic rules of integration and differentiation; knowledge of the properties of elementary functions and their graphs; algebra and trigonometry formulas.

Keywords: definite integral, limit, Newton-Leibniz formula

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculate the area of different shapes, volumes of solids, and to solve other geometric problems. Definite integrals are used for various calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in theory of cargo transport, in economics, in classical signal theory, and in other specialities.

Content

1. Statement of area problem
2. Definition of the definite integral
3. Properties of the definite integral
4. Calculation of a definite integral
5. Exercises
6. Solutions

Assessment strategies:

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Assessing students' knowledge about trigonometric integrals; the knowledge about the methods of integration during the lesson.

Teacher Toolkit and Digital Resources:

- Presentation about the definite integral
- Worksheet on trigonometric functions
- Whiteboard to solve the examples
- Useful websites

    Integrals to solve some simple real-life tasks
    https://www.khanacademy.org/math/old-ap-calculus-ab/ab-applications-definite-integrals

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Definite Integral - Introduction

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
10 min	Worksheet	Test on trigonometric integrals	Assesses students' knowledge	Students' self-assessment	
10 min	Presentation	Statement of area problem	Frontal, construction of graphs, solving of example	Active listening, answering the questions	What would you do if you had to calculate the area between curved lines?
10 min	Presentation	Definition of the definite integral	Frontal explanation	Active listening	
10 min	Discussion	Application of integrals	Asking questions, giving examples of applications	Participation in discussion	
20 min	Presentation Explanation	Properties of the definite integral	Explanation of the transformation of integrand	Active listening, solving examples	What is the difference between definite and indefinite integrals?
10 min	Solving examples	Calculation of definite integrals	Solving examples, explaining	Active participating in solving of examples	
20 min	Solving the real-life problem	APPLICATION OF MEAN VALUE THEOREM	Video demonstration, problem posing and solving	Participation in the discussion, solving example What are the types on the ship motion?	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • Lesson 6 (unit 4) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (ship's motion): https://www.youtube.com/watch?v=3408T5A-ApU Video (types of the motion of the ship) ://www.youtube.com/watch?v=aK3G8n53D6A
Learning objectives	By the end of the lesson: • ***all*** students should know how to calculate definite integral • ***all*** students understand the difference between the definite and indefinite integral

Lesson 7:

A. The lesson starts with the completing worksheet on the trigonometric integrals (see Appendix 1) where the task has been connected with the correct answer by drawing a line. The lecturer can check students' knowledge about the given topic.

B. The lecturer poses the question: "*What would you do if you had to calculate the area between curved lines?*" The principle of approximate calculation are discussed. Lecturer recalls the Archimedes' method of inscribed and circumscribed polygons to calculate the length of a circle.

C. Lecturer demonstrates the approximate calculations of the area under the parabola. Lecturer shows that more detailed cat of the figure gives better result.

D. Main principles of calculation of the area under some general curve of a function $\boldsymbol{f}(\boldsymbol{x})$ are introduced step by step. He/she introduces the integral-sum and explains what happens if we take a limit from that.

E. The definite integral is defined. Lecturer explains the term "integrable" function on interval [a, b]. He/she underlines that from definition we understand that the integral is a summation where some object is partitioned in small parts, the necessary calculation is completed for every such part, and the results are added.

F. Discussion about the application of definite integrals takes a place. Some fields of application are named – geometry (areas, volumes, surface's area, length of the arc); physic (work, mass, force, etc.).

G. Lecturer demonstrates the real-life problem about the work of the pump – how many gallons were pumped during the hour. Here the integral is expressed by the trapezoidal rule (see Appendix 2) – as a sum.

H. The properties of the definite integral are important to calculate that integrals. Some of these theorems can be proven by the definition and properties of the limits. The property of equal limits of integration can explain the students themselves. Interchange of the

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limits has a simple explanation too. The lecturer has to show what happens if the interval of the integration is divided into two parts.

I. Lecturer introduces Newton-Leibniz formula for calculation of definite integrals

$$\int_a^b f(x)dx = F(b) - F(a)$$

Several examples should be solved by the active participation of students. They have to recall different methods of integration to find appropriate antiderivative.

J. Lecturer returns to the properties of the definite integral. The significance of the mean value theorem is explained. Possible application of this property is discussed – the use of the detection of average values of the functions. Lecturer gives the example of the average speed of the ship (see Appendix 3) and asks students if they know "*What are the types on the ship motion?".* A couple of video clips are demonstrated.

K. At the end of lesson lecturer recommends the exercises given at the end o unit for students' individual work.

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APPENDIX 1: Worksheet Find correct answer

Worksheet: Trigonometric integrals

Connect the integral with a correct answer by drawing the line!

1.	$\int \sin 3x\, dx$		1.	$\frac{1}{2}\sin^{10} x + C$
2.	$\int \frac{\cot^3 x}{\sin^2 x} dx$		2.	$0.5(x + 0.25\sin 4x) + C$
3.	$\int 5\sin^9 x \cdot \cos x \mathrm{dx}$		3.	$8\sin x$
4.	$\int 2\cos\frac{x}{4} dx$		4.	$-\frac{\cot^4 x}{4} + C$
5.	$\int \cos^2 2x\, dx$		5.	$\frac{1}{3\cos^3 x} + C$
6.	$\int \frac{\tan x}{\cos^3 x} dx$		6.	$8\sin\frac{x}{4} + C$
			7.	$3\cos 3x + C$
			8.	$\frac{\sin^{10} x}{10} + C$
			9.	$5(\sin^{10} x + \cos x) + C$
			10.	$-\frac{1}{3}\cos 3x + C$

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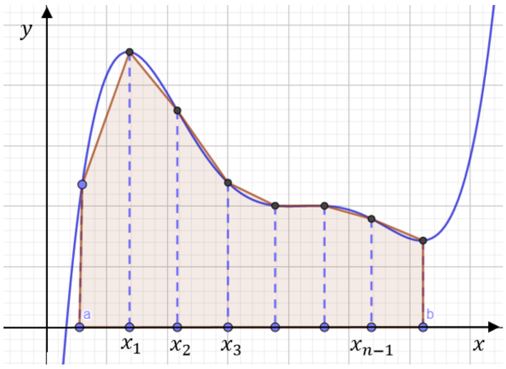
APPENDIX 2: Application of trapezoidal rule

Unit: Application in Real Situations. Example 6. Definite integrals are useful to describe a real situation. Many real applications begin with data that are not represented by a function but that are stored in a table. In this case, the definite integral used in the calculation formula is approximated by the Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(a)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(b)\right)$$

Given interval $[a,b]$ is divided in n equal parts to split the region under the graph of a function. Every part is replaced by a trapezoid (see figure).



Figure

Task. A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown in a table. How many gallons were pumped during that hour?

Pumping rates

time	0	5	10	15	20	25	30	35	40	45	50	55	60
gallons	0	40	45	52	44	46	55	52	50	48	46	50	52

Solution. Let $R(t),\ \ 0 \le t \le 60$ be the pumping rate as a continuous function of time for the hour. We can partition the hour into short subintervals of length $\Delta t = 5$ on which the rate is nearly constant and form the sum

$$\sum_{i=0}^{12} R(t_i)\Delta t$$

as an approximation to the amount pumped during the hour. This reveals the integral formula for the number of gallons pumped to be

$$gallons = \int_0^{60} R(t)\,dt$$

We have no formula for $R(t)$ in this instance, but the 13 equally spaced values in table enable us to estimate the integral with the Trapezoidal Rule:

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$$\int_0^{60} R(t)\,dt \approx \frac{5}{2}(0 + 2\cdot 40 + 2\cdot 45 + \cdots + 2\cdot 50 + 52) =$$

$$= \frac{5}{2}\cdot 2(40 + 45 + 52 + 44 + 46 + 55 + 52 + 50 + 48 + 46 + 50 + 26) =$$

$$= 5\cdot 554 = 2770$$

Answer. The total amount pumped during the hour is about 2770 gallons.

APPENDIX 3: Application of Mean value theorem

Unit: Application in Real Situations. Example 1. In a real situation, the movement of the ship at any time moment is affected by the energy of the water waves (heave, pitch and roll motions), the wind, as well as the speed of the ship itself. The resulting motion of a ship is sinusoidal. Suppose that the speed of a ship is given by the function $v(t) = \rho\cos\left(\frac{\pi}{2}t\right)$ knots per hour, where ρ is some specific constant. Compute the average speed of the ship between 13 hours.

Solution. The formula for calculation of the average value of a function $f(x)$ on the interval $[a, b]$ is

$$\boldsymbol{AVR = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

The given time interval is $t \in [0,13]$. We apply the formula to calculate the average speed of a ship

$$v_{avr} = \frac{1}{13}\int_0^{13} \rho\cos\left(\frac{\pi}{2}t\right)\,dt =$$

$$= \frac{\rho}{13}\cdot\frac{1}{\pi}\sin\left(\frac{\pi}{2}t\right)\Big|_0^{13} = \frac{\rho}{13\pi}\ \text{(kph)}$$

Answer. The average speed of a ship is $\frac{\rho}{13\pi}$ knots per hour.

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Lesson 8: Methods for Calculation Definite Integrals . Improper Integrals

Name of Unit	Workload	Handbook
Part 1: Methods for Calculation Definite Integrals **Part 2: Improper Integrals**	**Lecture: 90 min**	**Unit 5. Methods for calculation** **Unit 6. Improper Integrals**

Part1: Methods for Calculation Definite Integrals

DETAILED DESCRIPTION

The basic rules for calculation of the definite integral were discussed in the previous section. In this section we present methods of integration of composite functions and the method of integration by parts for the definite integral.

AIM: to introduce certain methods of calculation of the definite integral if the integrand is non-trivial.

Learning Outcomes:

Students can evaluate different definite integrals using various integration methods

Prior Knowledge: basic rules of integration and differentiation; methods of integration of indefinite integrals; the Newton-Leibniz formula.

Keywords: definite integrals, integration by parts, Newton-Leibniz formula

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculated the area of various shapes; the volumes of solids, and to solve other geometric problems. Definite integrals are used for different calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in the theory of cargo transport, in economics, in classical signal theory, and in other specialities.

Content

1. Integration by parts
2. Substitution method for the definite integral
3. Exercises
4. Solutions

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Assessment strategies:

Assessing students' knowledge about the basic integration formulas and methods of integration of indefinite integrals during the lesson.

Teacher Toolkit and Digital Resources:

- Presentation about the methods of integration
- GeoGebra software to construct graphs
- Whiteboard to solve the examples
- Useful websites

    u-substitution for definite integrals (Khan Academy)
    https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-9/a/u-substitution-definite-integrals

Part 1: Methods for Calculation Definite Integrals

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Presentation	Integration by parts	Frontal, explanations	Active listening	
10 min	Discussion	Application of integrals, example of Fourier series	Giving example of application, video, constructing graphs	Participation in discussion	What signal reception equipment are in your daily use?
10 min	Presentation Solving examples	Substitution for definite integral	Solving examples	Active listening, answering the questions	
10 min	Solving examples	Calculation of definite integrals by substitution method	Solving examples, explaining	Active participating in solving of examples	

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Part 2: Improper Integrals

DETAILED DESCRIPTION

In this section, we will extend the concept of the definite integral. Special integrals are investigated over infinite intervals. We will introduce two types of improper integrals: integrals with infinite limits and integrals with an infinite discontinuity in the region of integration. Such integrals are defined using the notion of the limit. The ways of calculation of improper integrals are presented. Examples of convergent and divergent improper integrals are discussed.

The software GeoGebra, DESMOS, Excel are recommended for construction of graphs. To check their solutions and for deeper understanding, students can use *Definite and Improper Integral Calculator* (https://www.emathhelp.net/calculators/calculus-2/definite-integral-calculator/)

AIM: to learn about improper integrals and methods of their evaluation, to understand the concepts of convergence and divergence of integrals.

Learning Outcomes:

2. Acquire the methods of evaluation of improper integrals of type I.
3. Distinguish the improper integrals of type II and acquire the methods of their calculation.

Prior Knowledge: definite integrals; limits; detection of the domain of function; elementary functions and their graphs.

Keywords: elementary functions, integrals, limits

Relationship to real maritime problems: By describing the shape of the hull of a ship mathematically, it is possible to research the ship's wave resistance that can be presented by an improper integral. Improper integrals are used to express the electrical potential of a given field. A probability density function for a continuous random variable can be described by an improper integral.

Content

1. Improper integrals with infinite upper limit
2. Improper integrals with an infinite discontinuity in the region of integration
3. Exercises
4. Solutions

Assessment strategies:

Assessing students' knowledge about integration methods and about the elementary functions during the lesson.

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Teacher Toolkit and Digital Resources:

- Presentation about the improper integral
- Whiteboard to solve the examples
- GeoGebra software to construct graphs
- Useful websites

  Improper integrals of type I examples:
  https://www.youtube.com/watch?v=GWJcxx9dx0k
  Improper integrals of type II examples
  https://www.youtube.com/watch?v=4mDJxBIlSLI
  GeoGebra applet Improper integrals
  https://www.geogebra.org/m/TWHMwwdR

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Part 2: Improper Integrals

LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
10 min	Presentation	Improper integral - introduction	Frontal explanation, drawing graphs	Active listening	
10 min	Solving examples	Examples of improper integrals	Solving examples, construction of graphs	Active solving, answering the questions	What are the graphs of elementary functions?
20 min	Solving real-life problems	Application of improper integral, examples	Frontal explanation, discussion	Active participation in discussion	Do improper integrals have any practical significance?
5 min	Presentation	Improper integral with an infinite discontinuity	Frontal explanation, drawing graphs	Active listening	
10 min	Solving examples	Examples of improper integral	Construction of graphs, solving examples	Active participating in solving of examples	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Part 1: Methods for Integration Definite Integrals

RESOURCES	• Whiteboard • GeoGebra software • Lesson 6 (unit 4) https:// https://marematHics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (travel of information wireless): https://www.youtube.com/watch?v=Ax7dYaRiY6o Video (data transmission explained) https://www.youtube.com/watch?v=wi0GnTW5QAE
Learning objectives	By the end of the lesson: • ***all*** students know how to apply integration by parts for definite integral • ***all*** students understand the method of u-substitution

Lesson 8. Part 1:

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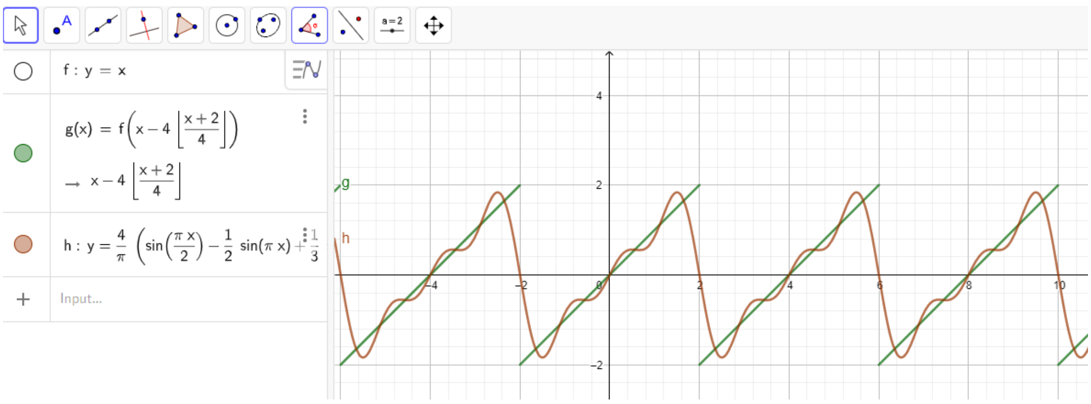
A. The lecturer recalls the formula of integration by parts for indefinite integrals. Then explains how to apply the formula for definite integrals and demonstrates the solution of an example. He/she can ask the students for advice whish of the functions is necessary to substitute by u.

B. Discussion starts about the application of integrals. Lecturer tells about the approximation of functions. Shows the graph of "saw tooth" function and names the application of such waves in the signal theory. The question arises *"What signal reception equipment are in your daily use?"* Lecturer tells that by the signal transmission the trigonometric approximation takes a place and shows the animation how the piecewise function is approximated by the trigonometric Fourier series. The periodic function $f(x) = x, \;\; x \in [-2, 2]$ is approximated by the integral

$$\int_0^{-2} x \sin\frac{\pi n x}{2} dx$$

The task is solved applying partial integration and the graphs are constructed:



C. Lecturer recommends to students see video clip "How information travel wirelessly" https://www.youtube.com/watch?v=Ax7dYaRiY6o. Or "Data transmission" https://www.youtube.com/watch?v=wi0GnTW5QAE

D. Next topic is about the substitution method for definite integrals. Lecturer formulates and explains the theorem about the substitution of differentiable function on the interval [a, b]. He/she draws students' attention to the need to change the boundaries of the integral according to the new variable. It is not necessary to return to the previous argument.

E. Several examples are solved with the active participation of students.

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Part 2: Improper Integrals

	• Whiteboard • GeoGebra software • Lesson 6 (unit 4) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (to escape gravity): https://www.youtube.com/watch?v=YlxKh4oCKhw • GeoGebra applet about the improper integral https://www.geogebra.org/m/TWHMwwdR
Learning objectives	By the end of the lesson: • ***all*** students should understand the meaning of the improper integral • ***all*** students can calculate improper integrals

Lesson 8. Part 2:

F. The lecturer introduces the integral with an infinite upper limit, draw the graph for general function. He/she explains that choosing increasing definite values for upper limit of the integral a sequence of numbers is formed. The limit of such number sequence can be calculated.

G. Lecturer explains the meaning of terms "convergence" and "divergence" of improper integral. The way of calculation of such integral is presented.

H. The examples of improper integral are solved. The graphs of integrands are constructed to show the behavior of these functions on the domain. Some properties of the limits are recalled. The lecturer underline how important is to know the graphs of elementary functions.

I. Lecturer starts discussion by the question *"Do improper integrals have any practical significance?"* He/she gives the example on overcoming the gravitational force (see Appendix 1). The video can be demonstrated about how to escape gravity https://www.youtube.com/watch?v=YlxKh4oCKhw. More practical example is from the field of probability theory about the last of the telephone call (see Appendix 2).

J. The integrals that have the discontinuity in the interval of integration are discussed. The improper integral is defined using one-sided limit.

K. Several examples are solved with the active participation of students.

L. At the end of lesson lecturer recommends the exercises given at the end of unit for students' individual work.

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APPENDIX 1: Application of improper integral: to overcome the gravity

Example 1.4 We would like to escape from the ground of the Earth. What must be the minimum velocity to overcome the gravitational force?

Solution An object with a mass m must be moved from the ground of the Earth. The object must have enough energy to move arbitrarily far away from Earth into universe. It is necessary to calculate the amount of work needed to move it.

Here are some constants useful for calculation:

$M \approx 5.97 \cdot 10^{24}\ kg$ – the mass of the earth

$G \approx 6.67 \cdot 10^{-11} Nm^2/kg^2$ – the constant of gravity

$R \approx 6.37 \cdot 10^6\ m$ – the distance between the centre of the Earth and the centre of object

We will calculate the energy needed to overcome the force of gravity

$$\int_R^\infty \frac{mMG}{x^2}dx = -\frac{mMG}{x}\bigg|_R^\infty = mMG\left(-\frac{1}{\infty}+\frac{1}{R}\right) = \frac{mMG}{R}$$

The energy needed to move an object is kinetic energy that can be calculated

$$E_{kin} = \frac{mv^2}{2}$$

So we have the equation

$$\frac{mv^2}{2} = \frac{mMG}{R}$$

Let us calculate the velocity by inserting well known constants in the equation

$$v = \sqrt{\frac{2MG}{R}} \approx \sqrt{\frac{2 \cdot 5.97 \cdot 10^{24} \cdot 6.67 \cdot 10^{-11}}{6.37 \cdot 10^6}} \approx 11.2\ m/s$$

The result is minimum velocity to overcome the Earth's pull. It is called ***escape velocity***.

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APPENDIX 2: Application of Improper integral: telephone call

Problem (probability theory): Let X be a random variable that measures the duration of telephone calls in a certain city and suppose that a probability density function for X is

$$f(x) = \begin{cases} 0.1e^{-0.2x}, & if \;\; x \geq 0 \\ 0, & if \;\; x < 0 \end{cases}$$

where x denotes the duration (in minutes) of a randomly selected call. Find the probability that a randomly selected call will last at least 5 minutes.

Solution

$$P(x \geq 5) = P(5 \leq x < \infty) = \int_5^{\infty} 0.1e^{-0.2x}\, dx =$$

$$= -\frac{0.1}{0.2}\int_5^{\infty} e^{-0.2x}\, d(-0.2x) = -\frac{1}{2}\left(\lim_{x\to\infty} e^{-0.2x} - e^{-1}\right) = \frac{1}{2e} \approx 0.184$$

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Lesson 9 and Lesson 10: Application of Definite Integrals:

Name of Unit	Workload	Handbook
Lesson 9: Application of Definite Integrals: Areas of Plane Regions I	**Lecture: 90 min**	**Application of Definite integrals Area**
Lesson 10: Application of Definite Integrals: Areas of Plane Regions II	**Lecture: 90 min**	**Continuation**

DETAILED DESCRIPTION:

Definite integrals are used to solve various problems. One of the usual applications is the calculation of the area of a plane region bounded by curves. This chapter presents different types of regions and gives the methods to calculate their areas. Formulas of definite integrals are given for curves expressed analytically, expressed by parametrical equations, as well as for curves given in the polar coordinate system. To construct the curves, the software programs GeoGebra Classic or Desmos Graphing Calculator, or others, can be used. Students can check their solutions with the integral calculator (https://www.integral-calculator.com/) that also constructs graphs of the integrand and the antiderivative.

AIM: to explore the methods of calculation of the area of plane regions of different types.

Learning Outcomes:

1. Students understand the geometrical meaning of the definite integral.
2. Students can calculate the area of plane regions enclosed by curves.
3. Students distinguish the cases if a region must be divided into two or more parts.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Keywords: Cartesian coordinate system; definite integrals; polar coordinate system

Relationship to real maritime problems: Calculation of the area of various specific construction parts is one of the core questions in shipbuilding. However, the shapes are so complex that mostly numerical calculations are used. Calculation of the area of a region is part of solving physics problems: for instance, to detect the pressure that is applied to an object it is necessary to calculate the area of the object's surface.

Content

1. Area under the graph of a function
2. Area between two curves
3. The problem of the compound region

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4. Area under a parametric curve
5. Curve in a polar coordinate system
6. Exercises
7. Solutions

Assessment strategies:

Assessing students' knowledge about the properties of elementary functions, about the knowledge of trigonometric formulas.

Teacher Toolkit and Digital Resources:

- PowerPoint presentation for application of definite integrals for calculation of area
- Presentation about the figures described by the parametrical equations and figures in polar coordinate system
- GeoGebra tool
- Whiteboard to solve the examples
- Useful websites

Explanations

a) https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Active_Calculus_(Boelkins_et_al)/06%3A_Using_Definite_Integrals/6.01%3A_Using_Definite_Integrals_to_Find_Area_and_Length

b) https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-areas-2009-1.pdf

Explanations and exercises with answers if areas are in polar form

https://www.whitman.edu/mathematics/calculus_online/section10.03.htmlSet of exercises

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Lesson 9: Application of Definite Integrals: Area of Plane Regions I

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Discussion	Recall the definition on definite integral	Frontal, explanation	Active participation recalling the definition of definite integral	
15 min	Presentation Examples	Area under the curve	Frontal, construction of graphs, solving of examples	Active listening, answering the questions	
15 min	Presentation Examples	Area between two curves	Frontal, construction of graphs, solving of examples	Contributing to questions, solving examples	What are the graphs of elementary functions?
20 min	Presentation Examples	The problem of compound region	Frontal, construction of graphs, explaining	Active listening, solving examples	
20 min	Discussion about real life examples	Application of definite integral to calculate the area of the region	Example of real life, video, explanation, using GeoGebra tool	Discussion, solving examples	In which practical cases is it necessary to calculate the area of the region?
15 min	Test	Finding cross points	Check solutions of test, discussion, demonstration of correct answers	Self-assessment	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • GeoGebra software • Lesson 6 (unit 3.4) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (creating stained glass window) https://www.youtube.com/watch?v=j-k0i7zY9X0

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Learning objectives	By the end of the lesson: • ***all*** students know how to create the definite integral for calculation of the figure's area • ***all*** students can calculate the area of the given region in Cartesian coordinate system

Lesson 9:

A. Lecturer recalls the formula for definite integral, recalls how that formula was obtained. Explains the connectedness of the formula with the graphical interpretation.

B. The simplest cases of figures should be demonstrated – the area of the figure under the curve. A couple of examples can be solved. Lecturer uses the GeoGebra software to construct the graphs. He/she asks students to characterize the graphs of the given functions. He/she recalls main properties of elementary functions. Examples can be found in Unit: Application of Definite Integrals: Areas of Plane Regions Examples 1.1 – 1.4.

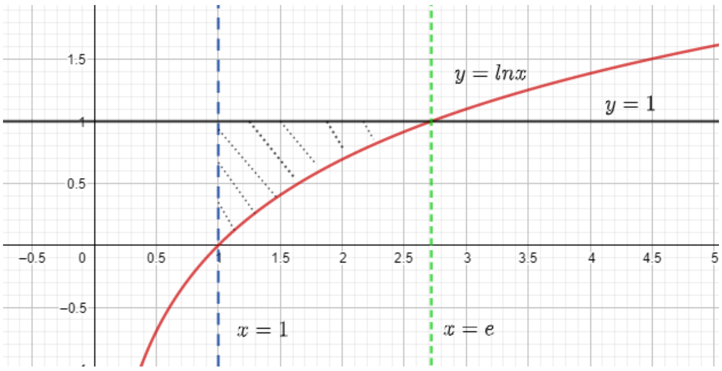
C. Lecturer explains the method to calculate the area between to curves. The example has to be solved (see Example 2.2)

D. Teacher demonstrates the case when the integral can be calculated more easily if the integration takes place along the y-axis. For instance,

Example. Calculate the area of the figure enclosed by lines

$$y = \ln x, \qquad y = 1, \qquad x = 1$$

Graph:



The projection of shaded region on the x-axis is $[1, e]$. The integral for calculation of area along x-axis is

$$S = \int_1^e (1 - lnx)\, dx = \int_1^e dx - \int_1^e lnx\, dx$$

Here we see that second integral is quite complicated. It can be calculated by the integration by parts.

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If we make the projection on the y-axis than we reverse the function and get $x = e^y$. Integral along the y-axis can be calculated easy

$$S = \int_0^1 e^y dy = e^y \Big|_0^1 = e - 1$$

- **E.** Following examples are about the calculations of compound region has to be researched. The example that has been calculated in two ways – along x-axis and along y-axis should be discussed (see Example 3.2).
- **F.** The discussion about the application of integrals on calculation of areas of the regions takes a place. Lecturer demonstrates the video about the creation of stained glass windows. An example of calculating figure's area is shown using GeoGebra (see Appendix 1).
- **G.** At the end part of the lesson lecturer gives the test for students about the calculation of the cross points of given lines (see appendix 2).
- **H.** The results of test are discussed, correct solutions of the exercises are presented by using GeoGebra.

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Lesson 10: Application of Definite Integrals: Area of Plane Regions II

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
15 min	Presentation Examples	Parametrically given functions; area of figures	Present and explains the area calculation for parametrically given functions	Active listening, contributing to the questions	
15 min	Quiz	Trigonometry functions and formulas	Testing students' knowledge	Recalling the formulas, self-assessment	
15 min	Presentation Examples	Curves in polar coordinate system	Frontal, construction of graphs, examples	Active listening, answering the questions	
20 min	Discussion about real life examples	Application of integrals for regions described by polar curves	Example of real life, video, explanation, using GeoGebra tool	Participation in the solving of examples and in discussion	Where you can find the forms that can be described by polar curves?
25 min	Solving exercises	Solving exercises about the areas of differently given regions	Construction of graphs, asking question to students, giving hints to students	Active solving of exercises	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • GeoGebra software • Lesson 6 (unit 3.4) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (explanation on construction of bow thruster) https://www.youtube.com/watch?v=HTr9C1D7L60
Learning objectives	By the end of the lesson: • ***all*** students should understand how to calculate area for regions described parametrically • ***all*** students understand how to calculate the area enclosed by polar curve

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Lesson 10:

A. Lecturer asks to students what parametrically given functions they know. The integral formula for calculation the area of the figure is to modified for the parametrically given function. Lecturer underline, that the limits of the integral have to be calculated according to the figure's projection to the x-axis.

B. The example how to get the formula for calculation on the area of an ellipse is demonstrated. The canonical equation of the ellipse is shown. Lecturer explains that directly expressing the function in the analytic form will lead to the difficult integral:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \quad y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\int b\sqrt{1-\frac{x^2}{a^2}}dx$$

The advantage of area calculation of parametrically given ellipse is presented (see Unit: Application of Definite Integrals: Areas of Plane Regions, Example 4.1)

C. Before to start the topic about the curves in polar coordinate system, students need to remember trigonometric formulas necessary for integration and methods how to integrate trigonometric functions. Lecturer can compose the quiz where students have to complete the formula (see Appendix 3).

D. Lecturer shows a different approach of area calculation for figures in polar plane coordinate system. The example has to be explained (see example 5.1).

E. Lecturer asks "*What forms do you know in the real life that can be described by polar curves?"* The video about the construction of the bow thruster can be demonstrated. Lecturer presents the example of the case where the blades of the bow thruster can be described by the polar curves (see Unit: Application of Definite Integrals, Example 8).

F. At the end part of the lesson students have to solve the exercises. Lecturer helps students to construct the graphs and gives necessary hints.

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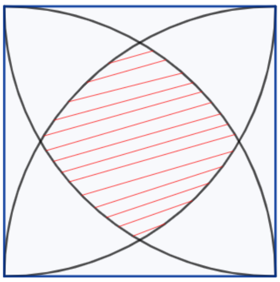
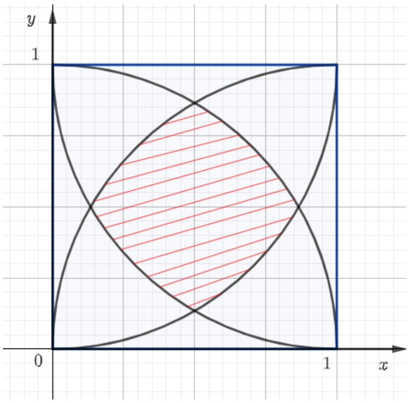
APPENDIX 1: Application of definite integral

Problem. The production of stained glass is very expensive. The master must calculate the area of the central part of the window (see shadowed region in the center of the figure) to determine the amount of the row material.

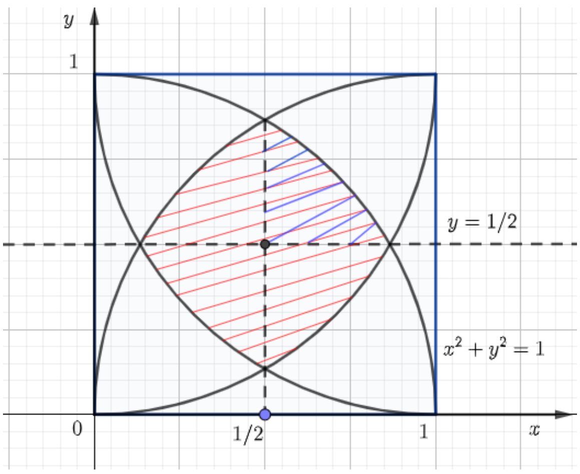


Solution

Let us suppose that the side of the square is one unit and put it in the coordinate system.



Now we can describe the arcs by functions. At first we note that shadowed part is centrally symmetric. It is enough to calculate a fourth part of the figure:



The upper line of the region we can describe as an arc of a circle

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$$x^2+y^2=1$$

The lower line is a horizontal line

$$y=\frac{1}{2}$$

The projection of figure on the x-axis we calculate by the system of equations

$$\begin{cases} x^2+y^2=1 \\ y=\dfrac{1}{2} \end{cases}$$

The projection is

$$x\in\left[\frac{1}{2};\frac{\sqrt{3}}{2}\right]$$

The area of the whole figure we can calculate by the integral

$$S=4\int_{1/2}^{\frac{\sqrt{3}}{2}}\left(\sqrt{1-x^2}-\frac{1}{2}\right)dx$$

Let us calculate this integral by trigonometric substitution

$$S=4\int_{1/2}^{\frac{\sqrt{3}}{2}}\left(\sqrt{1-x^2}-\frac{1}{2}\right)dx=\begin{vmatrix} x=\text{sin}t & dx=\text{cos}t\,dt \\ x_1=\dfrac{1}{2} & t_1=\dfrac{\pi}{6} \\ x_2=\dfrac{\sqrt{3}}{2} & t_2=\dfrac{\pi}{3} \end{vmatrix}=$$

$$=4\int_{\pi/6}^{\frac{\pi}{3}}\left(\sqrt{1-\text{sin}^2t}-\frac{1}{2}\right)\text{cos}t\,dt=$$

$$=4\int_{\pi/6}^{\frac{\pi}{3}}\left(\text{cos}t-\frac{1}{2}\right)\text{cos}t\,dt=4\int_{\pi/6}^{\frac{\pi}{3}}\text{cos}^2t\,dt-\frac{4}{2}\int_{\pi/6}^{\frac{\pi}{3}}\text{cos}t\,dt=$$

$$=4\int_{\pi/6}^{\frac{\pi}{3}}\frac{1+\text{cos}2t}{2}\,dt-2\int_{\pi/6}^{\frac{\pi}{3}}\text{cos}t\,dt=$$

$$=\left(4\cdot\frac{1}{2}t+4\cdot\frac{1}{2}\cdot\frac{1}{2}\text{sin}2t-2\text{sin}t\right)\Bigg|_{\pi/6}^{\pi/3}=$$

$$=2\cdot\frac{\pi}{3}+\frac{\sqrt{3}}{2}-\frac{2\sqrt{3}}{2}-\left(2\cdot\frac{\pi}{6}+\frac{\sqrt{3}}{2}-2\cdot\frac{1}{2}\right)\approx 0.32\text{ square units}$$

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APPENDIX 2: Test on calculation of cross points

Comment: It is reccommended to give for test some 2 or 3 tasks

Test

Find the cross points of the lines and sketch the graphs of given lines:

1. $y = 5x - x^2; \;\; y = -x$
2. $y = -\frac{4}{x}; \;\; y - x = 5$
3. $y = (x+1)^2; \;\; y = 1 - x^2$
4. $y = 2x - x^2 + 3; \;\; y = x^2 - 4x + 3$
5. $y = x^2 - 7x + 10; \;\; y = 4x$
6. $y = 2^x; \;\; 2y = -x + 10$
7. $y = \sqrt{x+1}; \;\; x - 2y + 2 = 0$

Example of solution of the task 6:

Let us solve the system of equations

$$\begin{cases} y = 2^x \\ 2y = -x + 10 \end{cases}$$

We put the first equation into the second equation

$$2 \cdot 2^x = -x + 10$$

$$2^{x+1} = 10 - x$$

We can easily gess the value of the root

$$x = 2$$

Let us check this root

$$2^{2+1} = 10 - 2 \quad \text{or} \;\; 2^3 = 8$$

Graphs of the increasing exponent function and the decreasing straight line can have only one crosspoint. Let us sketch the graph:

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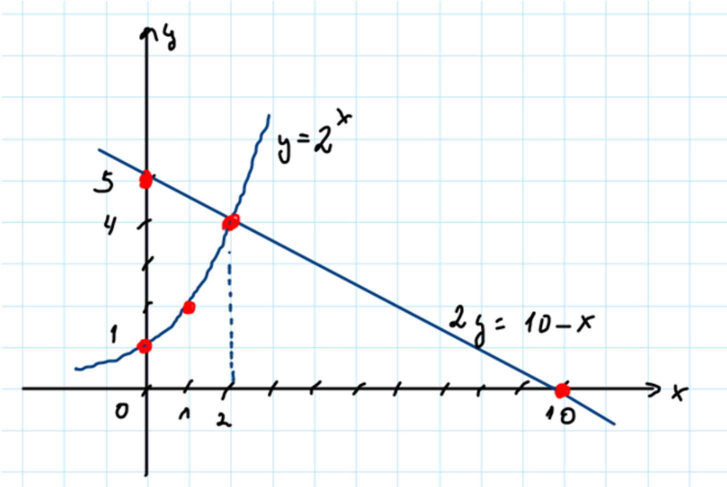
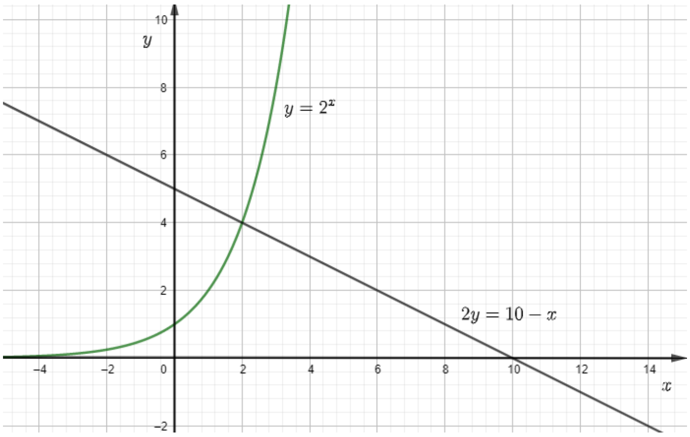


Construction by GeoGebra tool to compare the drawing by an accurately constructed graphs:



By the GeoGebra command ***Intersect*** it is possible to demonstrate correct answer to the question about the crosspoint of lines.

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APPENDIX 3: Quiz about the trigonometric formulas

Quiz

Fill in the empty places:

1. $\sin^2 \sqcup + \cos^2 x = \sqcup$
2. $\int \cos\frac{x}{5} dx = \sqcup \sin\frac{x}{5} + C$
3. $\sin^2(2x) = \frac{1-\cos(\sqcup\ x)}{2}$
4. $\int \frac{\sin^3 x}{\cos^2 x} dx = -\int \frac{(\sqcup - \sqcup)}{t^2} dt$
5. $\sin 2x = 2\ \sqcup\ \cdot \cos x$
6. $\int 3x \cdot sinx\, dx = \sqcup\ x \cdot cosx + \int \sqcup\sqcup\, dx$

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Lesson 11: Application of Definite Integrals – Arc Length

Name of Unit	Workload	Handbook
Application of Definite Integrals – Arc Length	Lecture: 90 min	Unit Application of definite integrals – arc length

DETAILED DESCRIPTION:

Definite integrals can be applied to calculate the length of various curves. This chapter explains the creation of the formula for calculation of arc length. The formula can be transformed for curves that are given as parametric equations or in polar form. The content is supplemented with examples of graphs constructed with GeoGebra and Desmos.

AIM: to demonstrate the calculation of the arc length for curves given in the Cartesian coordinate system and for curves given in the polar coordinate system.

Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can calculate the arc length of given curves.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of a functions; the construction of the graph of a function; algebra and trigonometry formulas.

Keywords: definite integrals, length of the arc, parametrically defined curves, polar curves

Relationship to real maritime problems: With the help of definite integrals it is possible to calculate the lengths of different objects that can be described by functions. For instance, it is possible to calculate the length of a rope hanging between two supports by integration.

Content

1. The formula for calculation of the length of an arc
2. The length of an arc given by parametric equations
3. The arc length of a polar curve
4. Exercises
5. Solutions

Assessment strategies:

Assessing students' knowledge about the basic integration formulas and methods of integration during the lesson.

Teacher Toolkit and Digital Resources:

- Presentation about calculation of arc length

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- GeoGebra software to construct graphs
- Whiteboard to solve the examples
- Useful websites

  Theoretical explanations and some problems
  https://www.mathsisfun.com/calculus/arc-length.html
  Exercises with answers
  https://www.wolframalpha.com/examples/mathematics/calculus-and-analysis/applications-of-calculus/arc-length/
  Video by Numberphile „Mathematics around us“ explains what is a catenary
  https://www.youtube.com/watch?v=AYIQYZuQNMw

Application of Definite Integrals – Arc Length

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
15 min	Presentation	The way of calculation the length of an arc	Frontal explanation, drawing graphs	Active listening	
10 min	Solving examples	Example of calculation the arc length in Cartesian coordinate system	Solving example, construction of graphs	Active solving, answering the questions	
10 min	Presentation Examples	Calculation of the length of parametrically defined arc	Frontal explanation, discussion	Active listening, answering the questions,solving examples	How we can transform the formula of arc length?
15 min	Presentation Example	Polar arc	Frontal explanation, drawing graphs	Active listening, solving examples	
10 min	Discussion	Applications of integrals	Discussion, videos	Participation in discussion	What examples of calculation of arc length do you know in the real life?
15 min	Solving real life examples	Real life examples	Solving, explaining	Active solving of examples	
15 min	Solving exercises	Solving different exercises	Giving hints for students, commenting solutions	Solving exercises themselves	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • GeoGebra software • Lesson 6 (unit 4) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (Catenary-mathematics around us): https://www.youtube.com/watch?v=AYIQYZuQNMw • Video (think logically – hanging cable) https://www.youtube.com/watch?v=l_ffdarcJiQ • GeoGebra applet (arc length exploration) https://www.geogebra.org/m/qSSPdkjy • GeoGebra applet(integral and arc length calculator) https://www.geogebra.org/m/Y583muJe
Learning objectives	By the end of the lesson: • ***all*** students know how to calculate the length of an arc

Lesson 11:

A. The lecturer starts the lesson with the explanation how to calculate the length of an arc. He/she constructs a broken line to approximate the given arc and uses Pythagorean theorem to calculate the lengths of one particular chord. Lecturer underlines that the precise length of the arc can be calculated as a limit of the integral.

B. The formula for the arc length is applied to calculate the example.

C. Lecturer recalls the meaning of the parametrically defined curve. He/she ask to students *"How we can transform the formula of calculation the arc length for parametrically given curve?"* The substitution is demonstrated and the derivation formula for parametric function is used. Teacher explains the solution of an example and draw a figure by GeoGebra.

D. To get the formula for arc length the transitional formulas for the change of coordinate system are used (see Appendix):

$$\begin{cases} x = r(\varphi)\cos\varphi \\ y = r(\varphi)\sin\varphi \end{cases}$$

E. An example is solved constructing the graph of a curve by GeoGebra.

F. Lecturer starts discussion asking *"What examples of calculation of arc length do you know in the real life and in maritime?"* The video "Mathematics around us" is demonstrated. Teacher solves the example of the length of hanging rope (see Unit Applications in real life Example 4). After the solution other video about the problem on catenary is shown. Here is solution that needs the logical thinking only.

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G. Another real life problem about cylindrical container is solved (see Unit Applications in real life Example 3).

H. At the end part of lesson students are solving exercises themselves. They can ask the questions to lecturer. He/she gives hints and comments students' solutions.

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APPENDIX: How to design the formula for calculation of the polar arc's length

Formula for calculation the arc length in polar coordinates

We know the formula for calculation of arc length if the curve is defined parametrically

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$$L = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

The arc is given by the polar equation $r = r(\varphi)$ between the rays α and β. If we would like to express the arc in Cartesian coordinate system, we use the formulas for transition from one coordinate system to another

$$\begin{cases} \boldsymbol{x = r(\varphi)\cos\varphi} \\ \boldsymbol{y = r(\varphi)\sin\varphi} \end{cases}$$

Now we have the parametric equations and can apply the formula for arc length. At first we differentiate both function with respect to the argument φ:

$$\dot{x} = r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi$$

$$\dot{y} = r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi$$

Let us simplify the expression that we have to write under the square root

$$\dot{x}^2 + \dot{y}^2 = (r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi)^2 + (r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi)^2 =$$

$$= (r'(\varphi))^2\cos^2\varphi - 2r'(\varphi)r(\varphi)\sin\varphi\cos\varphi + (r(\varphi))^2\sin^2\varphi +$$

$$+(r'(\varphi))^2\sin^2\varphi + 2r'(\varphi)r(\varphi)\sin\varphi\cos\varphi + (r(\varphi))^2\cos^2\varphi =$$

$$= (r'(\varphi))^2(\cos^2\varphi + \sin^2\varphi) + (r(\varphi))^2(\cos^2\varphi + \sin^2\varphi) =$$

$$= (r(\varphi))^2 + (r'(\varphi))^2 = r^2 + r'^2$$

The formula for calculation the length of the polar arc is

$$\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{r^2 + r'^2}\, d\varphi}$$

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Lesson 12: Application of Definite Integrals: Volume of a solid of revolution

Name of Unit	Workload	Handbook
Application of Definite Integrals: Volume of a solid of revolution	**Lecture: 90 min**	**Unit . Application of Definite integrals – volume of solid of revolution**

DETAILED DESCRIPTION

This chapter introduces the main principles for calculation of the volume of solids of revolution. Different examples are discussed where solids are generated by elementary curves and lines. Some composite constructions are explained. The case of the parametrically given curve is included to describe the solid of revolution. The content is supplemented with examples of graphs and surfaces constructed with GeoGebra tools.

AIM: **to show the methods of calculation of the volume of solids of revolution.**

Learning Outcomes:

1. Students understand the application of the definite integral in solving geometry tasks.
2. Students can construct regions of a revolution and understand what surfaces they form.
3. Students can calculate the volume of solids of revolution.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Keywords: curves; definite integrals; solids of revolution; volume

Relationship to real maritime problems: Volume is a very important concept if we are speaking about the capacity of cargo holds, the capacity of fuel oil tanks or ballast water tanks, tanks of lubricating oil, or others. It is important to know the amount of material required for producing a specific part with a definite volume. Calculations of the volume of containers, cauldrons, and tanks are among the necessary premises for designing a ship's engineering equipment.

Contents

1. Volume of a solid of revolution obtained by rotating an area about x-axis
2. Volume of a solid of revolution generated by two curves
3. Rotation about the y-axis
4. Revolution of parametrically given curves
5. Exercises
6. Solutions

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Assessment strategies:

Assessing students' knowledge about the integration methods during the lesson.

Teacher Toolkit and Digital Resources:

- Presentation about the calculation of volumes of a solids of revolution
- GeoGebra tool
- Whiteboard to solve the examples
- Useful websites

Explanations

a) https://www.mathsisfun.com/calculus/solids-revolution-disk-washer.html

b) https://opentextbc.ca/calculusv1openstax/chapter/volumes-of-revolution-cylindrical-shells/

disc method explained: https://www.youtube.com/watch?v=cVu9ZYP55Dk

washer method explained: https://www.youtube.com/watch?v=v2kp-WsJerA

shell method: https://www.youtube.com/watch?v=SmPUUzn5Njs

presentations of creation and approximation of solids of revolution:

https://www.youtube.com/watch?v=JrRniVSW9tg

https://www.youtube.com/watch?v=-4vgUfQdi2E

how ships float – Archimedes principle: https://www.youtube.com/watch?v=b7UOmMfAbPs

Ship stability: https://www.youtube.com/watch?v=AC9EULljVbo

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Application of Definite Integrals: Volume of a solid of revolution

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
5 min	Presentation	Introduction on the solid of revolution	Frontal, explanation, construction by GeoGebra	Active listening	
10 min	Presentation Examples	Explanation of disc method to get the formula	Frontal, construction of graphs, videos	Active listening	
10 min	Examples	Calculation of examples	Frontal, construction of graphs, solving of examples	Contributing to questions, solving examples	
15 min	Presentation example	Revolution of compound region	Frontal, construction of graphs, explaining, video	Active listening, solving examples, answering questions	
20 min	Discussion about real life examples	Application of definite integral to calculate the area of the region	Solving examples of real life, video, explanation	Discussion, solving examples	What kinds of tanks are on the ships, for what reasons it is necessary to calculate their volume?
15 min	Presentation Examples	Formula for case if the curve is rotating around the y-axis	Frontal, construction of graphs, solving of examples	Active listening, solving examples, answering questions	
15 min	Presentation	Revolution of parametrically given curve	Frontal, construction of graphs, solving of examples	Solving examples	How to calculate the volume of a solid formed by the polar curve revolving around the polar axis?

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

RESOURCES	• Whiteboard • GeoGebra software • Homepage - Lesson 12 (unit 9) https:// https://marematics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (stability of a ships) https://www.youtube.com/watch?v=AC9EULljVbo • Video (explanation of calculation of volume) https://www.youtube.com/watch?v=cVu9ZYP55Dk https://www.youtube.com/watch?v=-4vgUfQdi2E https://www.youtube.com/watch?v=JrRniVSW9tg
Learning objectives	By the end of the lesson: • ***all*** students know how to create the definite integral for calculation of the volume of a solid of revolution • ***all*** students understand the specific of the cases if the curves are given by the equation of one variable or by parametric equations

Lesson 12:

A. Lecturer presents the case of revolving of an arc around the axis. He/she defines the solid of revolution and gives some examples applying GeoGebra 3D calculator. (The instruction is given in YouTube video clip by Tim Brzezinski:

https://www.youtube.com/watch?v=c6SXST9pDfk)

B. Lecturer explains the approximate calculation of the volume of a solid of revolution by the disc method to explain the formula. Additional videoclip can be demonstrated (see: https://www.youtube.com/watch?v=cVu9ZYP55Dk)

C. The example of the curve rotating around the x-axis can be solved. At first teacher constructs the curves and after constructs the solid by GeoGebra 3D calculator. Then the integral is created and calculated (see Example 1.1 in the unit Volume of a solid of revolution).

D. Lecturer discusses with students how to solve a case if the compound figure bounded by two curves revolves around the x-axis. Videos about the washer method (https://www.youtube.com/watch?v=v2kp-WsJerA) and shell method

(https://www.youtube.com/watch?v=SmPUUzn5Njs) can be demonstrated.

E. The example of the compound figure is being solved. Solving includes the construction of figure and the construction of the solid of revolution too (see Example 2.1).

F. The discussion about the necessity to calculate the volume of a tanks for ships stability takes a place. Lecturer asks “What kinds of tanks are on the ships, for what reasons it is

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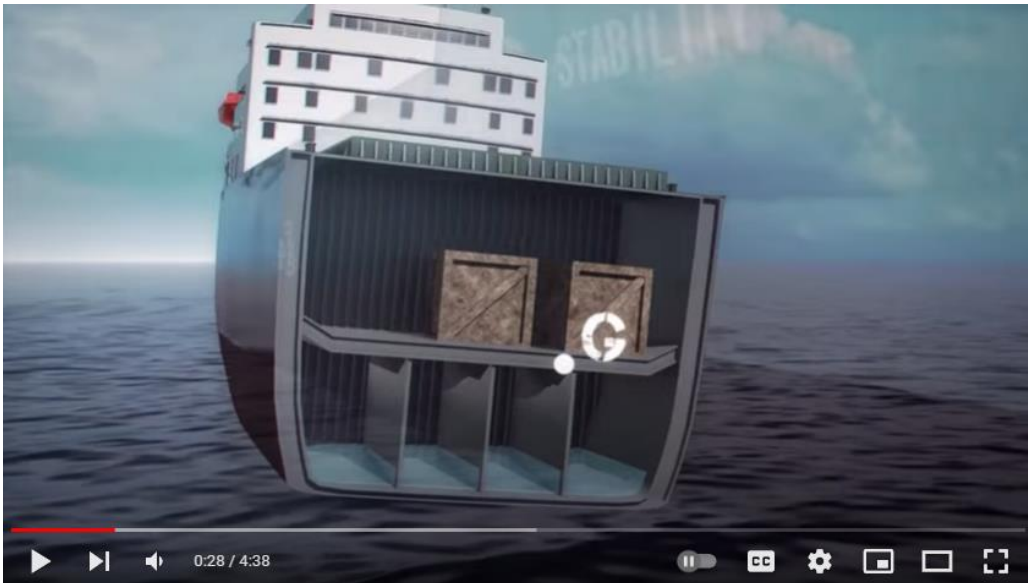
necessary to calculate their volume?”. The video clips about the ship stability can be watched (https://www.youtube.com/watch?v=AC9EULljVbo). The examples of calculation of the volume of the tanks can be solved (see examples 14 and 15 of unit Application of definite integrals).



G. Next topic is about the curve that revolves around the y-axis. The lecturer asks whether the volume of the solids formed by the curve revolving around the different axes are the same. The examples 3.2 and 3.3 are solved comparing the solids with the solids that rotates around the x-axis recalling examples 1.1 and 2.1. Lecturer
H. The real life example 13 from unit Application of definite integrals can be solved.
I. The curves given by parametrical equations are discussed. The formula for calculation of volume is being modified. Lecturer demonstrates the example of the parametrically given curve that revolves around the x-axis and then around the y-axis. The solids are being compared.
J. Lecturer recommends to students to solve the exercises (see Appendix) given at the end of unit and to watch the video about the Archimedes principle – how ships float (https://www.youtube.com/watch?v=b7UOmMfAbPs). The challenge is given for students, asking *”How to calculate the volume of a solid formed by the polar curve revolving around the polar axis?“* (see Appendix).

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APPENDIX: Challenge

Solve the problem!

Calculate the volume of the solid obtained by rotating the cardioid $r = 2(1 + \cos\varphi)$ around the polar axis!

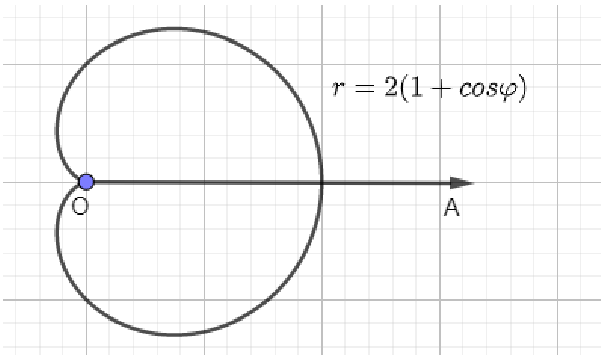

Figure 1

Solution

The curve that is given in polar form we can express in the Cartesian coordinate system using the formulas for transition from one coordinate system to another

$$\begin{cases} \boldsymbol{x = r(\varphi)\cos\varphi} \\ \boldsymbol{y = r(\varphi)\sin\varphi} \end{cases}$$

We have

$$\begin{cases} \boldsymbol{x = 2(1 + \cos\varphi)\cos\varphi} \\ \boldsymbol{y = 2(1 + \cos\varphi)\sin\varphi} \end{cases} \qquad \textbf{(1)}$$

These are parametrical equations for the given curve therefore we can apply the formulas for calculation of volume of a solid that is created by parametrical curve revolving around the x-axis

$$\boldsymbol{V = \pi \int_{t_1}^{t_2} (y(t))^2 x'(t) dt}$$

In the given case the parameter is an angle φ and due to symmetry, we can choose the interval $0 \le \varphi \le \pi$ (see figure 2)

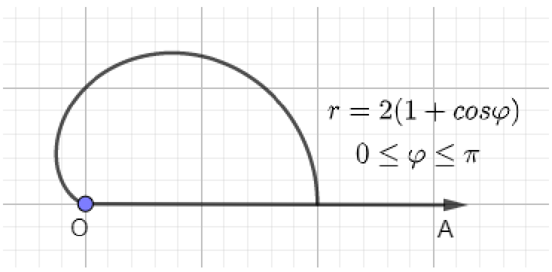

Figure 2

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Let us calculate the derivative of the function $x(\varphi)$

$$x' = (2(1+\cos\varphi)\cos\varphi)' = \left(2\left(\cos\varphi + \cos^2\varphi\right)\right)' =$$
$$= -2(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)$$

We will make the transformation of integrand

$$y^2\dot{x} = (2(\sin\varphi + \sin\varphi\cos\varphi))^2 \cdot (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi) =$$
$$= -8(\sin^2\varphi + 2\sin^2\varphi cos\varphi + \sin^2\varphi\cos^2\varphi)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi) =$$
$$= -8(\sin^3\varphi + 4\sin^3\varphi cos\varphi + 5\sin^3\varphi\cos^2\varphi + 2\sin^3\varphi\cos^3\varphi)$$

According to the minimal value of $x(\varphi) = 0$ the lower limit of integral is π, but the upper limit is 0. The integral for calculation of the volume is

$$V = -8\pi \int_{\pi}^{0} (\sin^3\varphi + 4\sin^3\varphi cos\varphi + 5\sin^3\varphi\cos^2\varphi + 2\sin^3\varphi\cos^3\varphi)\, d\varphi =$$
$$= 8\pi\left(\int_0^\pi \sin^3\varphi\, d\varphi + 4\int_0^\pi \cos\varphi \cdot \sin^3\varphi\, d\varphi + 5\int_0^\pi \cos^2\varphi \cdot \sin^3\varphi\, d\varphi \right.$$
$$\left. + 2\int_0^\pi \cos^3\varphi \cdot \sin^3\varphi\, d\varphi\right)$$

Let us evaluate every integral separately as an indefinite integral

$$\int \sin^3\varphi d\varphi = -\int (1-\cos^2\varphi) d(\cos\varphi) = -\cos\varphi + \frac{1}{3}\cos^3\varphi + C$$

$$4\int \cos\varphi \cdot \sin^3\varphi\, d\varphi = 4\int \sin^3\varphi\, d(\sin\varphi) = 4\frac{\sin^4\varphi}{4} + C = \sin^4\varphi + C$$

$$5\int \cos^2\varphi \cdot \sin^3\varphi\, d\varphi = -5\int \cos^2\varphi(1-\cos^2\varphi)\, d(\cos\varphi) =$$
$$= -\frac{5\cos^3\varphi}{3} + \cos^5\varphi + C$$

$$2\int sin^3\varphi cos^3\varphi\, d\varphi = 2\int sin^3\varphi(1 - sin^2\varphi)\, d(sin\varphi) =$$
$$= 2\int sin^3\varphi\, d(sin\varphi) - 2\int sin^5\varphi\, d(sin\varphi) = \frac{2sin^4\varphi}{4} - \frac{2sin^6\varphi}{6} + C$$

Then the volume is

$$V = 8\pi\left(-cos\varphi + \frac{1}{3}cos^3\varphi + sin^4\varphi - \frac{5cos^3\varphi}{3} + cos^5\varphi + \frac{sin^4\varphi}{2} - \frac{sin^6\varphi}{3}\right)\Bigg|_0^\pi =$$

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$$= 8\pi\left((-(-1) - \frac{1}{3} + \frac{5}{3} - 1) - (-1 + \frac{1}{3} - \frac{5}{3} + 1)\right) = \frac{64}{3}\pi \approx 67.02 \text{ cub. un.}$$

The solid of revolution is presented in figure 3.

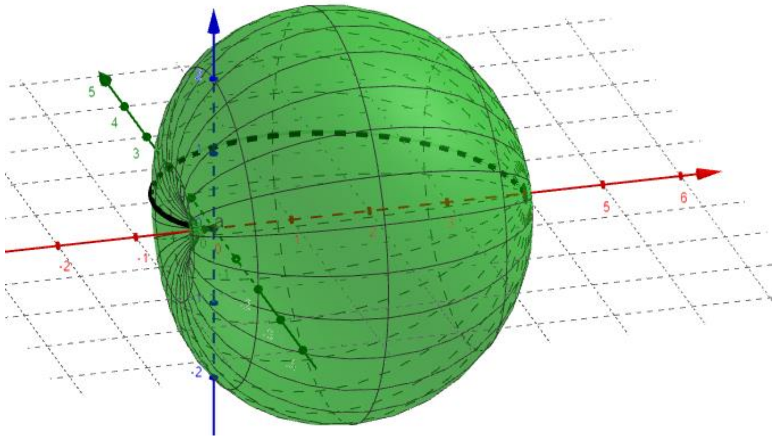

Figure 3

Explanation

Let us investigate the case closer. If we transform the coordinate system from the polar system to the Cartesian coordinate system it is necessary to have the functions that are well-defined or unambiguous with the respect of an argument x. Having the curve in Cartesian coordinate system we need to split the curve in parts (see picture 4). Here are three curves that represent well – defined functions y_1, y_2, y_3.

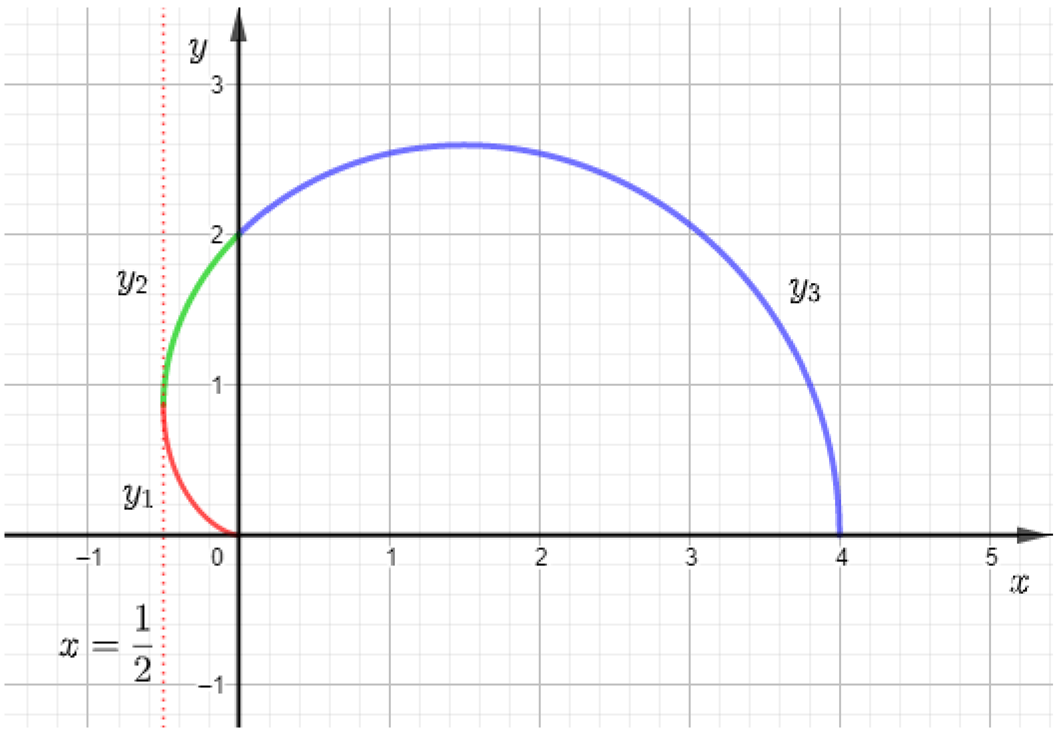

Figure 4

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The curves y_2, y_3 revolving around the x-axis separately make convex solid. The curve y_1 makes concave part of the solid. For every such part we can construct separate integral.

If $-\frac{1}{2} \leq x \leq 4$, we will use parts y_2 and y_3 and have the integral for calculation of volume with a positive result (we apply the same parametric expression of the given curve (1) given in the solution of the example):

$$V_1 = \pi \int_{\frac{2\pi}{3}}^{0} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi$$

The limits of the integral we choose according to the direction of growth of argument x.

The integral of the third part y_2 is:

$$V_2 = \pi \int_{\frac{2\pi}{3}}^{\pi} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi$$

We have

$$V_1 > 0 \quad and\ V_2 > 0.$$

Common value of the volume of revolution is

$$V = V_1 - V_2$$

Applying the properties of the definite integral we can add both integrals:

$$V = V_1 - V_2 =$$

$$= \pi \int_{\frac{2\pi}{3}}^{0} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi -$$

$$-\pi \int_{\frac{2\pi}{3}}^{\pi} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi =$$

$$= \pi \int_{\frac{2\pi}{3}}^{0} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi +$$

$$+\pi \int_{\pi}^{\frac{2\pi}{3}} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi =$$

$$= \pi \int_{\pi}^{0} (2(\sin\varphi + \sin\varphi\cos\varphi)^2 (-2)(\sin\varphi + 2\cos\varphi \cdot \sin\varphi)\, d\varphi$$

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Lesson 13: Application of Definite Integrals – Area of a Surface of Revolution

Name of Unit	Workload	Handbook
Application of Definite Integrals – Area of a Surface of Revolution	**Lecture: 90 min**	**Unit. Application of definite integrals – surface area**

DETAILED DESCRIPTION

Definite integrals can be applied to calculate the area of surface of revolution. The chapter demonstrate the application of a special formula of calculation of this area for cases if an arc revolves about the x-axis or about the y-axes. The formula can be transformed for curves that are given by the parametric equations. The content is supplied by the examples with a graph constructed with GeoGebra applet. The exercises that relay to the topic are attached at the end of the lesson.

AIM: to demonstrate the calculation of the area of surface of revolution in Cartesian coordinate system.

Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can apply computer aids to construct geometric shapes and surfaces.
3. Students can calculate the area of surface of revolution.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of a functions; the construction of the function graphs; the three dimensional construction of surfaces; algebra and trigonometry formulas.

Keywords: definite integral; differential of an arc; surface of revolution; area of a surface of revolution

Relationship to real maritime problems: Calculation of surface of revolution is an important part at the design of different parts of mechanical equipment. For instance, to increase the operational efficiency of centrifugal pump it is useful the calculation of surfaces of revolution for blade construction as an integral part of pump. The satellite dish has the shape of a solid of revolution. The calculation of its surface is necessary to detect the amount of paint required to cover the surface.

Content

1. Formula for calculation of a surface of revolution
2. Revolution about the y-axis

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3. The surface area of a solid of revolution for parametrically given curve
4. Exercises
5. Solutions

Assessment strategies:

Assessing students' knowledge about the basic integration formulas and methods of integration during the lesson.

Teacher Toolkit and Digital Resources:

- Presentation about calculation of the area of a surface revolution
- GeoGebra software to construct the graphs of curves and surfaces of revolution
- Whiteboard to solve the examples
- Useful websites

  Theoretical explanations and some problems
  https://math.libretexts.org/Courses/University_of_California_Davis/UCD_Mat_21B%3A_Integral_Calculus/6%3A_Applications_of_Definite_Integrals/6.4%3A_Areas_of_Surfaces_of_Revolution
  Theory, examples and exercises
  https://www.stewartcalculus.com/data/ESSENTIAL%20CALCULUS%20Early%20Transcendentals/upfiles/topics/ess_at_06_asr_stu.pdf
  Different types of surfaces of revolution
  https://mathworld.wolfram.com/SurfaceofRevolution.html
  GeoGebra applet – how to fast create the surface of revolution
  https://www.geogebra.org/m/gxngyyrb

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Application of Definite Integrals – Area of a Surface of Revolution

LESSON FLOW					
Time	**Sequence**	**Content**	**Teacher activities**	**Student activities**	**Points for discussion**
10 min	Presentation	Introductory part	Discussion, video	Active participating	What are the main principles of the creation of an integral formula?
15 min	Presentation	How to get the formula for calculation area of revolution	Frontal explanation, asking the questions	Active listening, answering questions	
10 min	Example	Solving an example if the curve creates the surface revolving around the x-axis	Frontal explanation, discussion, using GeoGebra	Active listening, answering the questions, solving example	
10 min	Presentation Example	Revolving around the y-axis	Frontal explanation, drawing graphs, solving an example	Active listening, solving example	
15 min	Presentation Example	Surface created by parametric curve	Frontal explanation, drawing graphs, solving an example	Active listening, solving example	What is the differential of an arc?
10 min	Real life example	Applications of integrals	Short discussion, solving example, construction of graphs	Active participation in solving	
20 min	Solving exercises	Solving different exercises	Giving hints for students, commenting solutions, explaining	Solving exercises themselves	

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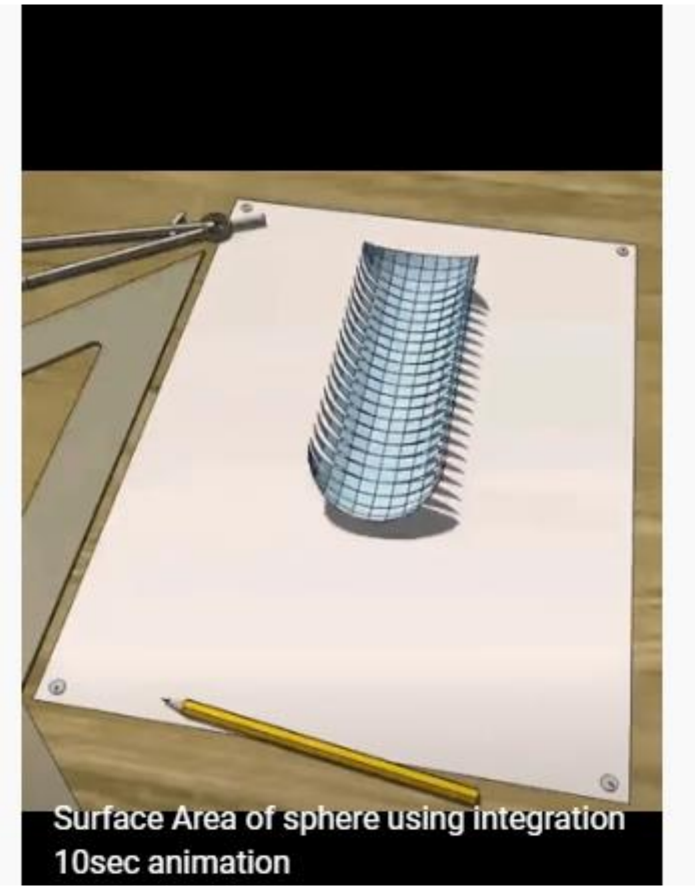
SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• Whiteboard • GeoGebra software • Lesson 13 (unit) https:// https://maremathics.pfst.hr/index.php/2021/07/07/7-integral-calculus/ • Video (Calculate surface area of a sphere) https://www.youtube.com/shorts/98O17lboXRI • GeoGebra applet (creation of a surface) https://www.geogebra.org/m/gxngyyrb
Learning objectives	By the end of the lesson: • ***all*** students know how to calculate the area of a surface of revolution

Lesson 11:

A. The lesson starts with a challenge – lecturer demonstrates 10 seconds long video about the calculation of a surface area of a sphere (https://www.youtube.com/shorts/98O17lboXRI). Lecturer asks how to make the calculation applying integrals.



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B. *"What are the main principles of the creation of an integral formula?"* lecturer asks. With the given object the following operations are performed – cut, replace, calculate, "glue". Next question – *"With what simple geometric solid can every part of surface be replaced?"* has been discussed.

C. Lecturer explains how the integral formula of calculation of the area of a solid of revolution is obtained – every piece is a frustum of a cone. For that it is easy to calculate the lateral surface area.

D. The formula for the surface area is applied to calculate the Example 1.1 (see Unit Calculation of a surface of revolution). Here it is constructed the line segment and appropriate surface by Geogebra tool.

E. Lecturer explains, how changes the formula if the curve is rotating around the y-axis. The example 2.1 can be solved.

F. Lecturer recalls the formula of the differential of an arc if the revolving curve is given parametrically and he/she transforms the formula of calculation of the area of revolution. The examples can be solved.

G. The example of real life application is solved (see Example 11 Unit Application in real situations)

H. At the next part of the class students solve exercises (see the chapter Exercises.) Teacher gives hints and discusses the solving methods. A special method of integration by parts can be demonstrated (see Appendix)

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APPENDIX: Special method for application of partial integration

Example of integration by parts where the evaluation of an integral can be considered as an equation

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t+\sqrt{1+t^2}\right|\right) + C$$

The integral can be evaluated by the method of integration by parts. Let us denote the given integral by Int and apply the method

$$Int = \int \sqrt{1+t^2}\,dt = \left| \begin{matrix} u = \sqrt{1+t^2} & du = \frac{t}{\sqrt{1+t^2}}dt \\ dv = dt & v = t \end{matrix} \right| =$$

$$= t\sqrt{1+t^2} - \int \frac{t \cdot t}{\sqrt{1+t^2}}dt = t\sqrt{1+t^2} - \int \frac{t^2 + 1 - 1}{\sqrt{1+t^2}}dt =$$

$$= t\sqrt{1+t^2} - \int \frac{t^2+1}{\sqrt{1+t^2}}dt + \int \frac{dt}{\sqrt{1+t^2}}dt =$$

$$= t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + ln\left|t+\sqrt{1+t^2}\right|$$

Looking at the given expression, its beginning and end, we have obtained the equation

$$Int = t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + ln\left|t+\sqrt{1+t^2}\right|$$

or

$$Int = t\sqrt{1+t^2} - Int + ln\left|t+\sqrt{1+t^2}\right|$$

We can express unknown Int from the equation

$$2Int = t\sqrt{1+t^2} + ln\left|t+\sqrt{1+t^2}\right|$$

$$Int = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t+\sqrt{1+t^2}\right|\right)$$

or

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t+\sqrt{1+t^2}\right|\right) + C$$

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