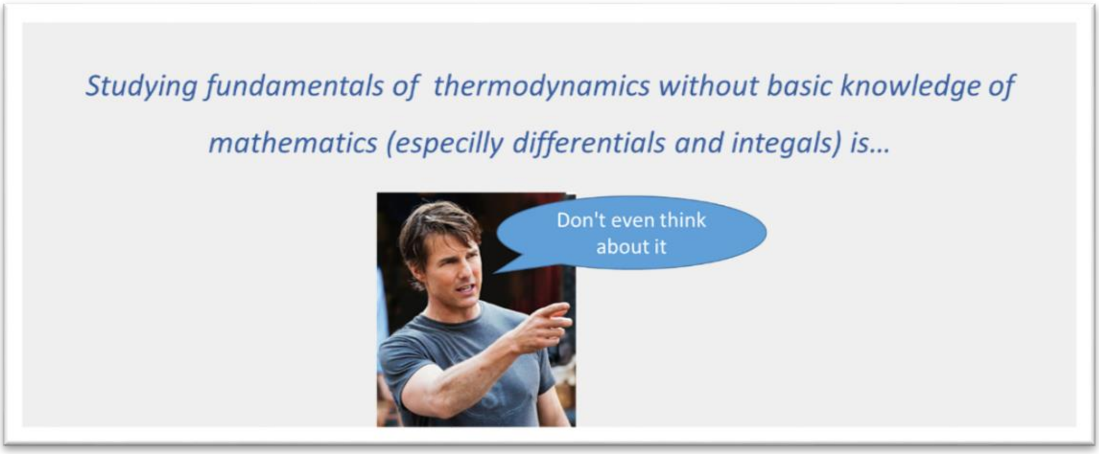
7 INTEGRAL CALCULUS



Thermodynamic is just one of many applications of integration. In fact, integrals are used in a wide variety of mechanical, electrotechnical and physical applications as in nautical navigation. In this chapter, the theory behind integration and some basic integration techniques, definitive and improper integrals. We can use integration to find the area under a curve, an arc length, volume of s solid revolution which has many practical applications in science, business, and maritime fields.

Learning Outcomes:

1. Calculate simple integrals of elementary functions
2. Apply the rules for calculating definite integrals
3. Use initial conditions to determine an integral
4. Calculate the area of plane regions enclosed by curves
5. Apply integration to find volumes of solids
6. Apply integration to solve tasks from different maritime and business applications

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CONTENT

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7.1 Introduction to Indefinite Integrals

DETAILED DESCRIPTION:

The section contains introductory questions about the indefinite integrals. The relation between integrals and derivatives will be considered. The reverse procedure of differentiation of functions will be discussed. The definition of the indefinite integral will be formulated. The section contains examples with geometric interpretation of antiderivatives. The section ends with exercises and their solutions.

For construction of graphs GeoGebra, DESMOS graphing calculator, MS Excel, or other tools can be applied.

Recommended websites:

MareMathics [MareMathics – Innovative Approach in Mathematical Education for Maritime Students (pfst.hr)]

Useful website;

Integral calculus: https://www.mathsisfunitscom/calculus/index.html

Integrals: https://www.khanacademy.org/math/calculus-1/cs1-integrals

Examples for integrals: https://www.wolframalpha.com/examples/mathematics/calculus-and-analysis/integrals/

AIM: **To learn the relationship between indefinite integrals and derivatives of elementary functions. To interpret the family of antiderivatives geometrically.**

Learning Outcomes:

1. Perform the reverse procedure of differentiation for simple elementary functions
2. Express the function with a given rate of change
3. Construct the graph of the specified function

Prior Knowledge: properties of elementary functions; graphs of elementary functions, algebra and trigonometry knowledge; rules of differentiation.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to

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travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

7.1.1 Conceptions of the antiderivative and definition of the indefinite integral

In the previous chapters we learned the differentiation of continuous functions. If function $F(x)$ is given, its derivative is function $f(x)$

$$F'(x) = f(x)$$

or in the form of differentials

$$\frac{d(F(x))}{dx} = f(x).$$

We are interested in the reverse procedure: what function have we differentiated to get the function $f(x)$? For instance, $f(x) = \cos x$.

According to the derivation formulas

$$\sin' x = \cos x$$

Taking into account that the derivative of the constant number is zero

$$C' = 0,$$

we can determine several functions whose derivative is cosine function

$$(\sin x + 1)' = \cos x; \quad (\sin x - 1.5)' = \cos x; \quad (\sin x + 3)' = \cos x$$

We can conclude that all sinus functions plus an arbitrary constant number are the prime functions of the cosine. We determine the family of prime functions of function $\cos x$ for all real numbers C

$$\cos x = (\sin x + C)', C \in R.$$

Definition. For any given function $f(x)$, function $F(x)$ is the ***prime function*** or ***antiderivative*** of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is the reverse procedure of derivation. We call this process integration.

Definition. The ***indefinite integral*** of a given function $f(x)$ is the set of all antiderivatives $F(x) + C$ of the function $f(x)$ and it is denoted

$$\int f(x)dx = F(x) + C,$$

where

the sign ∫ is called the ***integral symbol***,

$f(x)$ is called the ***integrand***,

x is called the ***integration variable***,

C is called the ***integration constant***.

The above-mentioned example can be written

$$\int \cos x\, dx = \sin x + C$$

7.1.2 Geometric interpretation of the indefinite integral

Knowing the geometric meaning of the derivative of a function, the given function $f(x)$ expresses the rate of change of some prime function. Geometric solution of integration of $f(x)$ presents a set of graphs that completely cover the plane. For instance, representatives of the whole family of antiderivatives $F(x) = e^x + C$ are shown in figure 2.1.

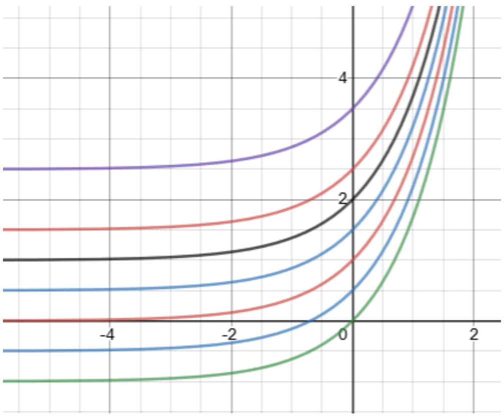

Figure 2.1 The family of antiderivatives $F(x) = e^x + C$

We can get one definite function of the set of answers if some initial condition is given: that is, we have the coordinates of the point belonging to the curve.

Example 2.1 Find function $\omega(x)$ whose rate of change is $\omega'(x) = \cos x$ and the point $(0,2)$ belongs to the graph of the function.

Solution We will solve this problem in two steps.

Step 1. Find antiderivatives of the function $\cos x$

$$\omega(x) = \int \cos x \, dx = \sin x + C$$

Step 2. Calculate the definite value of constant C according to the value of the function at the point $(0,2)$

$$\omega(0) = \sin 0 + C = C$$

$$\omega(0) = 2; \;\; C = 2$$

Answer $\omega(x) = \sin x + 2$.

The graph of this function belongs to the family of functions $\omega(x) = \sin x + C$. The y-intercept is the point (0, 2) where the graph of function $\omega(x) = \sin x + 2$ crosses the y-axis (see Figure 2.2).

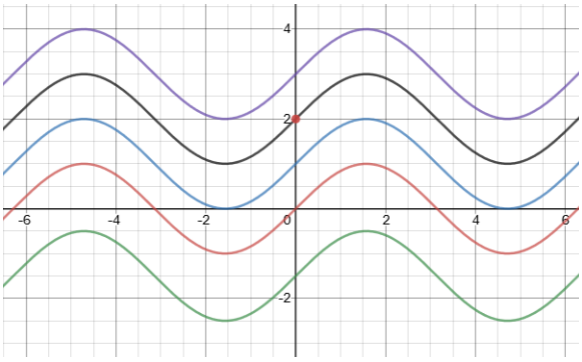

Figure 2.2 The family of functions $\omega(x) = \sin x + C$

An indefinite integral can be used to express the functional relations of physical processes.

Example 2.2 A flare is ejected vertically upwards from the ground at 15 m/s. Find the height of the flare after 2.5 seconds.

Comment: In the solution of this problem we suppose that it is not very hard to apply differentiation to find the function that gives the derivative $-9.8t + 15$

Solution The velocity of a given object can be expressed in terms of time according to gravity

$$v(t) = -9.8t + C$$

At the initial moment the velocity is 15 m/s ($t = 0$). We calculate $C = 15$.

The function of velocity in the given case is

$$v(t) = -9.8t + 15$$

To find the displacement $s(t)$ of the flare we integrate the function of velocity

$$s(t) = \int v(t)dt = \int (-9.8t + 15)dt =$$

$$= -4.9t^2 + 15t + C$$

At the initial position $t = 0,\ s = 0$ therefore $C = 0$. We calculate the height of the flare after 2.5 seconds

$$s(2.5) = -4.92.5^2 + 15 \cdot 2.5 = 6.875\ m$$

7.1.3 Uniqueness of antiderivatives

A question arises when searching for the antiderivatives of the given function $f(x)$. How much these antiderivatives differ from one another? The following theorem states:

Theorem. If functions $F_1(x)$ and $F_2(x)$ are two different antiderivatives of the function $f(x)$ they differ only by a constant number.

It is given $[F_1(x)]' = f(x)$ and $[F_2(x)]' = f(x)$. Then the difference is

$$[F_1(x)]' - [F_2(x)]' = 0 \text{ or } [F_1(x) - F_2(x)]' = 0.$$

We conclude that $F_1(x) - F_2(x) = C$.

7.1.4 Exercises

Using the list of elementary derivatives, find the antiderivatives $f(x)$ of the given functions $f'(x)$ according to the initial conditions. Construct the graph of function (x) .

1. $f'(x) = 3x^2;\quad f(0) = 1$
2. $f'(x) = e^x;\quad f(1) = e$
3. $f'(x) = \dfrac{1}{2x};\quad f(1) = 1.5$
4. $f'(x) = 2\sin x;\quad f\left(\dfrac{\pi}{3}\right) = -0.75$
5. $f'(x) = 4x - 3;\quad f(1) = 1$
6. Car starts from the origin and has acceleration $(t) = 2t - 5\ {}^{m}/_{s^2}$. Find the function of velocity of the car!

7.1.5 Solutions

Solution of exercise 1 We have the formula

$$(x^n)' = nx^{n-1}$$

From the given $f'(x) = 3x^2$ we can decide that $n = 3$ and we find $(x^3)' = 3x^2$.

Using integral we get the set of answers

$$\int f'(x)dx = \int 3x^2 dx = x^3 + C$$

Applying initial condition $x = 0; \ \ y = 1$

$$0 + C = 1; \quad C = 1$$

Answer

$$f(x) = x^3 + 1$$

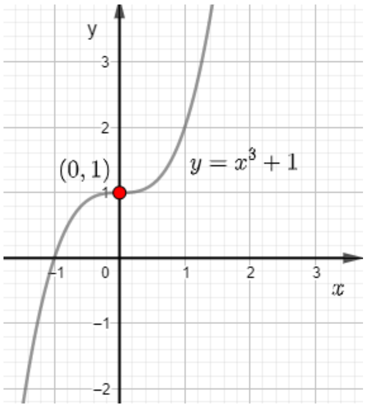

Figure 4.1 Function $f(x) = x^3 + 1$ passing through the point (0; 1).

Solution of exercise 2 We have the formula

$$(e^x)' = e^x$$

Using integral we get the set of answers

$$\int e^x dx = e^x + C$$

Applying initial condition $x = 1; \ \ y = e$

$$e^1 + C = e; \quad C = 0$$

Answer

$$f(x) = e^x$$

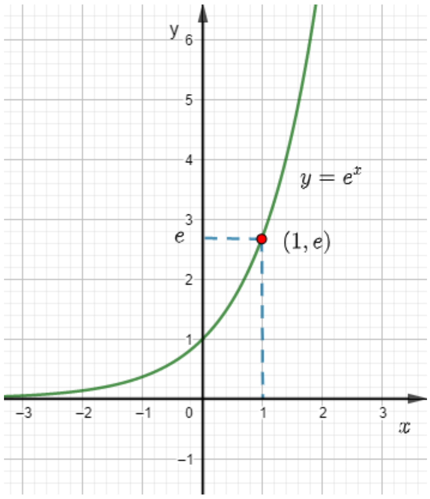

Figure 4.2 Function $f(x) = e^x$ passing through the point (1; e).

Solution of exercise 3 We have the formulas

$(\ln x)' = \frac{1}{x}$ and $(af(x))' = a(f(x))'$, where a is a constant.

Then

$$\left(\frac{1}{2}\ln x\right)' = \frac{1}{2}(\ln x)' = \frac{1}{2} \cdot \frac{1}{x}$$

Using integral

$$\int \frac{1}{2x} dx = \frac{\ln x}{2} + C$$

Applying initial condition $x = 1$; $y = 1.5$

$$\frac{\ln 1}{2} + C = 0 + C = 1.5; \quad C = 1.5$$

Answer

$$f(x) = \frac{\ln x}{2} + 1.5$$

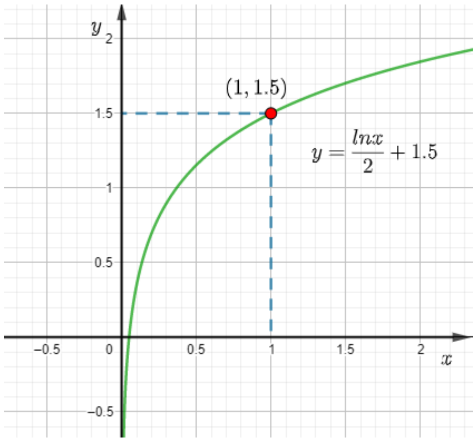

Figure 4.3 Function $f(x) = \frac{\ln x}{2} + 1.5$ passing through the point (1; 1.5).

Solution of exercise 4 We have the formula

$$(\cos x)' = -\sin x$$

Using integral we get

$$\int 2\sin x\, dx = 2\int \sin x\, dx = -\,2\cos x + C$$

Applying initial condition $x = \frac{\pi}{3};\ \ y = -0.75$

$$-2\cos\frac{\pi}{3} + C = -2\cdot\frac{1}{2} + C = -0.75; \quad C = 0.25$$

Answer

$$f(x) = -2\cos x + 0.25$$

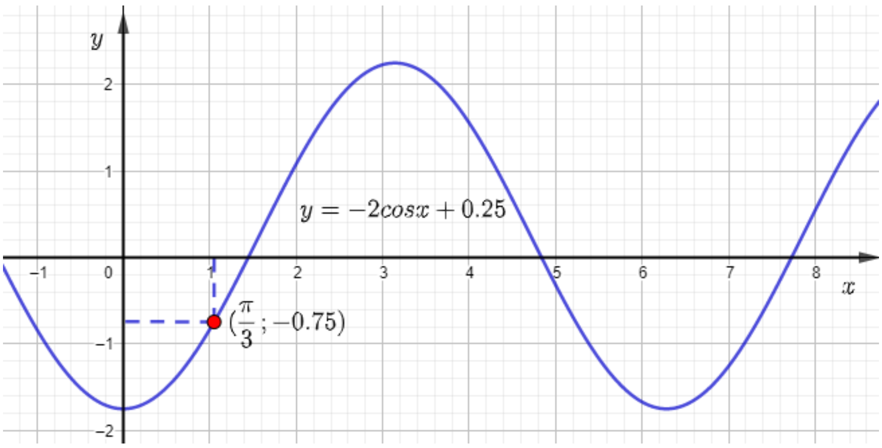

Figure 4.4 Function $f(x) = -2\cos x + 0.25$ passing through the point $\left(\frac{\pi}{3}, -0.75\right)$.

Solution of exercise 5 We know that

$$(x^2)' = 2x; \quad (3x)' = 3$$

and

$$(2x^2 - 3x)' = 2(x^2)' - (3x)' = 4x - 3$$

Using integral we get

$$\int (4x - 3)dx = 2x^2 - 3x + C$$

Applying initial condition $x = 1; \ y = 1$

$$2 - 3 + C = 1; \quad C = 2$$

Answer

$$f(x) = 2x^2 - 3x + 2$$

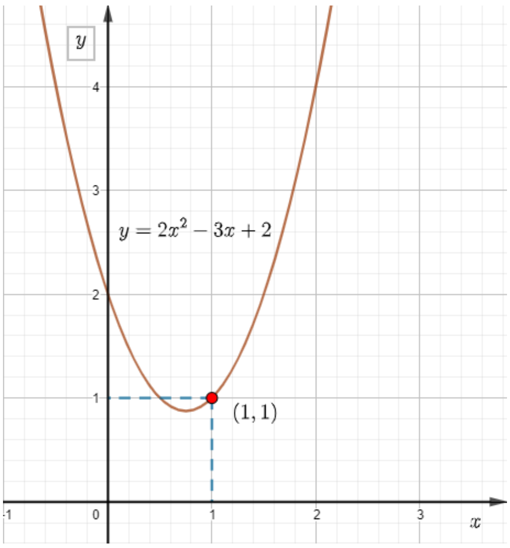

Figure 4.5 Function $f(x) = 2x^2 - 3x + 2$ passing through the point (1; 1).

Solution of exercise 6: Car starts from the origin and has the acceleration $(t) = 2t - 5 \ {}^{m}/_{s^2}$

Find the function of velocity of the car!

Solution

Velocity can be determined

$$v(t) = \int a(t)dt$$

Applying the formula of differentiation of power function, we can detect that expression $2t - 5$ can be derived from the function t^2, and 5 from $5t$. Therefore, the antiderivative should be

$$F(x) = t^2 - 5t$$

Generally, $v(t) = t^2 - 5t + C$.

At the start $t = 0, v(0) = 0$, therefore $C = 0$. Therefore, the function of velocity is

$$v(t) = t^2 - 5t$$

This equation helps to detect the velocity of the car after a time moment, for instance, after 10 seconds

$$v(10) = 100 - 50 = 50\, m/s$$

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7.2 Basic Rules of Integration

DETAILED DESCRIPTION:

This section introduces basic formulas of integration of elementary functions and the main properties of indefinite integrals. The section explains how to derive integration formulas from well-known differentiation rules. Several examples with explanations are discussed. Exercises for individual learning of integration are presented. At the end of the section there is an example on how to check the correctness of the solution of an integral.

AIM: To learn basic formulas and properties of integrals; to introduce methods of integration

Learning Outcomes

7. Learning the basic integration formulas
8. Application of the properties of indefinite integrals
9. Computing simple integrals of elementary functions
10. Transformation of integrands if necessary

Prior Knowledge: rules of differentiation; meaning of the term antiderivative; algebraic and trigonometric formulas to transform the integrands.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

1. Integration formulas
2. List of basic integration formulas
3. Properties of indefinite integrals
4. Alteration of the integrand
5. Exercises
6. Solutions
7. Additional note

7.2.1 Integration formulas

Understanding that integration is the reverse procedure of differentiation, we will write basic formulas of integrals. Any formula

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$$\int f(x)dx = F(x) + C$$

can be proved by differentiation – derivative of the function on the right side of the formula must be equal with the integrand:

$$\frac{d(F(x) + C)}{dx} = f(x)$$

For instance, let us prove the formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C; \quad n \neq -1$$

The derivative of the right side of the formula

$$\left(\frac{x^{n+1}}{n+1} + C\right)' = \left(\frac{x^{n+1}}{n+1}\right)' + C' = \frac{1}{n+1} \cdot (n+1)x^n + 0 = x^n$$

The special case $n = -1$ gives another formula

$$\int x^{-1} dx = \int \frac{dx}{x} = \ln x + C$$

7.2.2 List of basic integration formulas

The list of basic differentiation formulas covers all elementary functions. Therefore, the basic list of integration formulas contain the antiderivatives that are elementary functions – power functions, exponent functions, logarithmic functions, trigonometric functions, and cyclometric functions:

1. $\int dx = x + C$

2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C; \quad n \neq -1$

3. $\int x^{-1} dx = \int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x dx = e^x + C$

5. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

6. $\int \sin x \, dx = -\cos x + C$

7. $\int \cos x \, dx = \sin x + C$

8. $\int \frac{dx}{\sin^2 x} = -\cot x + C$

9. $\int \frac{dx}{\cos^2 x} = \tan x + C$

10. $\int \frac{dx}{1+x^2} = \operatorname{arc}\tan x + C$

11. $\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc}\sin x + C$

Usually this list is been supplemented by additional formulas that relate to hyperbolic functions, cyclometric or inverse trigonometric functions, and similar-looking integrals:

12. $\int \sinh x \, dx = \cosh x + C$

13. $\int \cosh x \, dx = \sinh x + C$

14. $\int \frac{dx}{\sinh^2 x} = -\coth x + C$

15. $\int \frac{dx}{\cosh^2 x} = \tanh x + C$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arc}\tan \frac{x}{a} + C$

17. $\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{arc}\sin \frac{x}{a} + C$

18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right| + C$

19. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left|x + \sqrt{x^2 \pm a^2}\right| + C$

The basic formulas work for any of the mentioned functions whenever the argument of the function is x; t; ω; s, or some other. For instance, the following formulas are true, like formula 6:

$$\int \sin t \, dt = -\cos t + C \quad \text{or} \quad \int \sin\omega \, d\omega = -\cos\omega + C$$

7.2.3 Properties of indefinite integrals

Let us look at the most common properties of integrals.

Property 1. The derivative of integral equals to the integrand

$$\left(\int f(x)dx\right)' = f(x)$$

Property 2. The integral of the sum of two function is equal to the sum of two integrals of given functions

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

Property 3. For any arbitrary constant a

$$\int af(x)dx = a\int f(x)dx$$

Property 4. The integral of the differential of a function is equal to that function plus an arbitrary constant

$$\int d(F(x)) = F(x) + C$$

The first property derives from the definition of the indefinite integral. Other mentioned properties can be proved based on the first property and by the rules of differentials.

Example 3.1 Compute the integral

$$\int (\sin x + e^x)dx$$

Solution

Combining second property and formulas 4 and 6 we get

$$\int (\sin x + e^x)dx = \int \sin x \, dx + \int e^x dx =$$

$$= -\cos x + e^x + C$$

Example 3.2 Compute the integral

$$\int \frac{5}{9 + x^2} dx$$

Solution

Combining third property and formula 16 we get

$$\int \frac{5}{9 + x^2} dx = 5\int \frac{dx}{9 + x^2} =$$

$$= 5 \cdot \frac{1}{3} \operatorname{arc\,tan} \frac{x}{3} + C$$

Example 3.3 Compute the integral

$$\int \left(4x^3 - 2\sqrt{x} + \frac{3\ \ln 7}{x}\right) dx$$

Solution

$$\int \left(4x^3 - 2\sqrt{x} + \frac{3\ \ln 7}{x}\right) dx = \left|\begin{matrix}\text{by property 2} \\ \text{and property 3}\end{matrix}\right| =$$

$$= 4\int x^3 dx - 2\int x^{\frac{1}{2}} dx + 3\ \ln 7 \int \frac{dx}{x} =$$

$$= |\text{applying formulas 2 and 3}| =$$

$$= 4\frac{x^4}{4} - 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3\ \ln 7 \cdot\ \ln|x| + C =$$

$$= x^4 - \frac{4}{3}x\sqrt{x} + 3\ \ln 7 \cdot\ \ln|x| + C$$

7.2.4 Alteration of the integrand

In most cases the integrand function is quite complicated, therefore special techniques of integration are needed. However, there are functions that we can alter to use basic formulas. Let us investigate some examples where we will use algebra and trigonometry formulas.

Example 4.1 Compute the integral

$$\int \frac{dx}{\sqrt{81 - 49x^2}}$$

Solution

We change the expression under the square root to apply basic integration formula 17

$$\int \frac{dx}{\sqrt{81 - 49x^2}} = \int \frac{dx}{\sqrt{49\left(\frac{81}{49} - x^2\right)}} = \frac{1}{7}\int \frac{dx}{\sqrt{\frac{81}{49} - x^2}} = \frac{1}{7}\ \text{arc}\sin\frac{7x}{9} + C$$

Example 4.2 Compute the integral

$$\int (2-x)^2 dx$$

Solution

We will expand the expression, apply the properties 2 and 3, and basic formula 1 and basic formula 2

$$\int (2-x)^2 dx = \int (4 - 4x + x^2) dx = 4x - 2x^2 + \frac{x^3}{3} + C$$

Example 4.3 Compute the integral

$$\int \frac{x^2-5x+6}{x-2}dx$$

Solution

We factorise the numerator and simplify the integrand. We apply the same properties and formulas as in the previous example

$$\int \frac{x^2-5x+6}{x-2}dx = \int \frac{(x-2)(x-3)}{x-2}dx = \int (x-3)dx = \frac{x^2}{2} - 3x + C$$

Example 4.4 Compute the integral

$$\int \frac{x^2 \cdot \sqrt[3]{x}}{x^{-\frac{2}{3}} \cdot \sqrt{x}}dx$$

Solution

Here we use the property of products of power and use the formula 2

$$\int \frac{x^2 \cdot \sqrt[3]{x}}{x^{-\frac{2}{3}} \cdot \sqrt{x}}dx = \int x^{2+\frac{1}{3}+\frac{2}{3}-\frac{1}{2}}dx = \int x^{\frac{5}{2}}dx = \frac{2}{7}x^{\frac{7}{2}} + C = \frac{2}{7}x^3 \cdot \sqrt{x} + C$$

Example 4.5 Compute the integral

$$\int \cos^2\frac{x}{2}dx$$

Solution

Apply the trigonometric formula of the double angle

$$\int \cos^2\frac{x}{2}dx = \int \frac{1+\cos x}{2}dx =$$

$$= \left|\begin{matrix}\text{by properties 2 and 3}\\ \text{by formulas 1 and 2}\end{matrix}\right| =$$

$$= \frac{1}{2}x + \frac{1}{2}\sin x + C$$

For more complex integrals, there are used special integration techniques that we will discuss in the following sections.

7.2.5 Exercises

Compute the following integrals using basic formulas and algebraic transformations if needed.

1. $\int \left(6x^3 + \frac{2}{5x^3} - 12\right) dx$

2. $\int \left(\frac{1}{2\sqrt{x}} - x^{0.5} + \frac{4}{x}\right) dx$

3. $\int \left(3\ \mathrm{sin}x + \frac{2}{\mathrm{sin}^2 x}\right) dx$

4. $\int \left(12^x - \frac{1}{\mathrm{cos}^2 x} + e^x\right) dx$

5. $\int \frac{16}{x^2 + 25} dx$

6. $\int \left(\mathrm{sinh}t - \frac{1}{\sqrt{t^2 - 64}}\right) dt$

7. $\int \sqrt[3]{\frac{x^4 \cdot \sqrt{x^3}}{x^{-\frac{1}{2}}}} dx$

8. $\int \frac{x^2 - 3x + 4}{x + 1} dx$

7.2.6 Solutions

1. $\int \left(6x^3 + \frac{2}{5x^3} - 12\right) dx = 6\int x^3 dx + \frac{2}{5}\int x^{-3} dx - 12\int dx=$

$$= \frac{6x^4}{4} - \frac{2}{5} \cdot \frac{x^{-2}}{2} - 12x + C = 1.5x^4 - \frac{1}{5x^2} - 12x + C$$

2. $\int \left(\frac{1}{2\sqrt{x}} - x^{0.5} + \frac{4}{x}\right) dx = \frac{1}{2}\int x^{-0.5} dx - \int x^{0.5} dx + 4\int \frac{dx}{x} =$

$$= \frac{1}{2}\frac{x^{0.5}}{0.5} - \frac{x^{1.5}}{1.5} + 4\ \ln|x| + C = \sqrt{x} - \frac{2}{3}x\sqrt{x} + 4\ \ln|x| + C$$

3. $\int \left(3\ \mathrm{sin}x + \frac{2}{\mathrm{sin}^2 x}\right) dx = 3\int \mathrm{sin}x\ dx + 2\int \frac{dx}{\mathrm{sin}^2 x} = -3\ \mathrm{cos}x - 2\ \mathrm{cot}x + C$

4. $\int \left(12^x - \frac{1}{\mathrm{cos}^2 x} + e^x\right) dx = \int 12^x dx - \int \frac{dx}{\mathrm{cos}^2 x} + \int e^x dx =$

$$= \frac{12^x}{\ln 12} - \tan x + e^x + C$$

5. $\int \frac{16}{x^2+25}dx = 16\int \frac{dx}{x^2+5^2} = 16 \cdot \frac{1}{5} \text{ arc tan}\frac{x}{5} + C$

6. $\int \left(\sinh t - \frac{1}{\sqrt{t^2-64}}\right)dt = \int \sinh t\ dt - \int \frac{dt}{\sqrt{t^2-64}} =$

$$= \cosh t - \ln\left|t + \sqrt{t^2-64}\right| + C$$

7. $\int \sqrt[3]{\frac{x^4 \cdot \sqrt{x^3}}{x^{-\frac{1}{2}}}}dx = \int \left(x^{4+\frac{3}{2}+\frac{1}{2}}\right)^{\frac{1}{3}} dx = \int x^{6 \cdot \frac{1}{3}}dx = \int x^2 dx = \frac{x^3}{3} + C$

8. $\int \frac{x^2+5x+4}{x+1}dx = \int \frac{(x+1)(x+4)}{x+1}dx = \int (x+4)\,dx =$

$$= \int x\,dx + 4\int dx = \frac{x^2}{2} + 4x + C$$

7.2.7 Additional note

Every integration result can be checked by differentiation of the antiderivative. For instance, let us check the result of example 3.2 (see chapter 7.2.3) by applying the chain rule of differentiation

$$\left(5 \cdot \frac{1}{3} \text{ arc tan}\frac{x}{3} + C\right)' = \frac{5}{3}\left(\text{arc tan}\frac{x}{3}\right)' + C' =$$

$$= \frac{5}{3} \cdot \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \left(\frac{x}{3}\right)' = \frac{5}{3} \cdot \frac{1}{1+\frac{x^2}{9}} \cdot \frac{1}{3} =$$

$$= \frac{5}{3} \cdot \frac{9}{9+x^2} \cdot \frac{1}{3} = \frac{5}{9+x^2}$$

Thus, we get the same function as the integrand. The integral is solved correctly.

7.3 Integration Techniques

7.3.1 Integration Techniques: Substitution

DETAILED DESCRIPTION:

In this chapter we will investigate the methods of integration of composite functions. The Chain Rule of derivation will be presented as an argument for integration of composite functions. The reverse process will be presented that introduces the formula of the Reverse Chain Rule. The method of substitution can help to simplify the notation of an integral. It will be called u-substitution. Several examples are presented in the chapter. There are integrals of composite functions given where the inner function is either linear or non-linear.

AIM: **To master the skills of substitution to solve the integrals of composite functions.**

Learning Outcomes

1. Students will acquire the method of changing the differential to compute integrals
2. Students will be able to carry out integration by making substitution
3. Students will recognize that the method of substitution is useful with integrals of composite functions

Prior Knowledge: rules of differentiation; basic rules of integration; algebraic formulas; knowledge of elementary mathematics.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

1. Integration of composite functions
2. Reverse Chain Rule
3. Application of Reverse Chain Rule
4. The change of differential
5. Method of substitution
6. Examples
7. Exercises

7.3.1.1 Integration of composite functions

In the previous sections we solved integrals where the integrands are elementary simple functions of variable x. How we can compute the integral if the integrand is a composite function?

A ***composite function*** is composed of two functions $f(x)$ and $g(x)$ where one of the given functions is the argument of another function $f(g(x))$.

Function $g(x)$ is the inner function and function $f(x)$ is the outer function.

Examples of composite functions

$$1)\ \sin 3x;\ \ 2)\ \cos(x^2);\ \ 3)\ e^{\tan x};\ \ 4)\ (6x+7)^{13}$$

There are given linear inner functions $3x$ and $6x+7$, and non-linear inner functions x^2 and $\tan x$. Outer functions are $\sin x;\ \cos x;\ e^x;\ x^{13}$.

7.3.1.2 Reverse Chain Rule

Let us remember the ***Chain Rule*** for differentiation of a composite function

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Suppose that we are trying to detect the function whose derivative is $f'(g(x)) \cdot g'(x)$. We will perform the reverse procedure of differentiation to solve this problem. For instance, can we determine the function whose derivative is $\cos(x^2) \cdot 2x$?

Here the composite function is $\cos(x^2)$ whose argument (inner function) is x^2. The derivative of x^2 is

$(x^2)' = 2x$.

Let us check now

$$(\sin(x^2))' = \cos(x^2) \cdot (x^2)' = \cos(x^2) \cdot 2x$$

The performed procedure can be recorded in the notation of integral

$$\int \cos(x^2) \cdot 2x dx = \sin(x^2) + C$$

Let us simplify the notation by substituting the function

$$\text{let } u = x^2 \text{ then } du = d(x^2) = (x^2)'dx = 2xdx$$

$$\int \cos u\, du = \sin u + C, \qquad \text{where } u = x^2.$$

Generalising the case, we write the ***formula of Reverse Chain Rule***:

$$\int \boldsymbol{f'(g(x)) \cdot g'(x) dx = f(g(x)) + C}$$

7.3.1.3 Application of Reverse Chain Rule

Based on the definition of the differential of the function $du = u'dx$ for function $u = u(x)$ we can simplify the Reverse Chain Rule

$$\int \boldsymbol{f(u)du = F(u) + C}$$

Therefore, the basic list of integrals given in the section "Basic Rules of Integration" can also be applicable for composite functions. For instance, formulas

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C$$

can be modified for composite functions whose inner function is the function u

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{\cos^2 u} = \tan u + C$$

Example 3.1 Integrate

$$\int \cos^5 x d(\cos x)$$

Solution

Let us notice

$$\int u^5 du = \frac{u^6}{6} + C$$

We can apply this formula for the given integral

$$\int \cos^5 x d(\cos x) = \frac{\cos^6 x}{6} + C$$

Example 3.2 Integrate

$$\int (8 - 11x)^5 d(8 - 11x)$$

Solution

Notice that we can apply the same formula as in the example 3.1.

$$\int (8 - 11x)^5 d(8 - 11x) = \frac{(8 - 11x)^6}{6} + C$$

Example 3.3 Integrate

$$\int \frac{d(4^x)}{\cos^2(4^x)}$$

Solution

$$\int \frac{d(4^x)}{\cos^2(4^x)} = \tan(4^x) + C$$

7.3.1.4 The change of differential

The examples in the previous chapter demonstrate the integration method where the expression under the integral has the differential of a function as a variable of integration. We can create such differentials in simple cases. Especially if the argument of the composite function is linear

$$\int f(ax+b)dx$$

Let us compute the differential of a linear function

$$d(ax+b) = (ax+b)'dx = adx$$

Calculation shows that differentials dx and $d(ax+b)$ differ only by a constant number a. Therefore, we can easy change the integral

$$\boldsymbol{\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) = \frac{1}{a}F(ax+b)+C}$$

Example 4.1

$$\int \sin 7x dx = \frac{1}{7}\int \sin 7x\, d(7x) = -\frac{1}{7}\cos 7x + C$$

Example 4.2

$$\int \frac{dx}{\sqrt{2x+1}} = \frac{1}{2}\int \frac{d(2x+1)}{\sqrt{2x+1}} = \sqrt{2x+1} + C$$

Example 4.3

$$\int \sin^5 x \cdot \cos x\, dx = \int \sin^5 x\, d(\sin x) = \frac{\sin^6 x}{6} + C$$

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7.3.1.5 Method of substitution

In a more general case, we can try to simplify the integral of a composite function by substitution if we can construct the derivative of the inner function

$$\int \boldsymbol{f(g(x)) \cdot g'(x) dx} = \int \boldsymbol{f(g(x))\, d(g(x))}$$

For example, if the given integral is the following

$$\int x^3 (0.5x^4 + 21)^{10} dx$$

we note the connection between the inner function $0.5x^4 + 21$ and the multiplier x^3. The multiplier is part of derivative of the inner function

$$(0.5x^4 + 21\ )' = 2x^3$$

Therefore, we can change the integral substituting the inner function by u

$$\int x^3 (0.5x^4 + 21)^{10} dx =$$

$$= \left| \begin{matrix} \text{let } u = 0.5x^4 + 21 \\ \text{then } du = 2x^3 dx \end{matrix} \right| =$$

$$= \frac{1}{2} \int 2x^3 (0.5x^4 + 21)^{10} dx \ =$$

$$= \frac{1}{2} \int u^{10} du = \frac{1}{2} \cdot \frac{u^{11}}{11} + C =$$

$$= \frac{(0.5x^4 + 21)^{11}}{22} + C$$

In a more general case we can integrate the composite function $f(u)$ with respect to the function $u = u(x)$ if the integrand contains the derivative of the argument function

$$\int f(u) u' dx = \int f(u) du$$

We can call this method of substitution ***u-substitution***.

7.3.1.6 Examples

Example 6.1 Compute

$$\int \frac{dx}{5x + 2}$$

Solution

$$\int \frac{dx}{5x+2} = \left|\begin{array}{l} \text{let} \quad u = 5x+2 \\ \text{then} \;\; du = (5x+2)'dx = 5dx \end{array}\right| =$$

$$= \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\cdot \ln|u| + C = \frac{1}{5}\cdot \ln|5x+2| + C$$

Example 6.2 Compute

$$\int \frac{dx}{1+16x^2}$$

Solution

$$\int \frac{dx}{1+16x^2} = \left|\begin{array}{l} \text{let} \;\; u = 4x \\ \text{then} \; du = 4dx \end{array}\right| = \frac{1}{4}\int \frac{du}{1+u^2} =$$

$$= \frac{1}{4}\arctan(4x) + C$$

Example 6.3 Compute

$$\int \frac{e^{\tan x}}{\cos^2 x}dx$$

Solution

$$\int \frac{e^{\tan x}}{\cos^2 x}dx = \left|\begin{array}{l} \text{let} \; u = \tan x \\ \text{then} \; du = \dfrac{1}{\cos^2 x}dx \end{array}\right| =$$

$$= \int e^u du = e^u + C = e^{\tan x} + C$$

Example 6.4 Compute

$$\int \frac{tdt}{(t+1)^3}$$

Solution

$$\int \frac{tdt}{(t+1)^3} = \left|\begin{array}{l} \text{let} \; u = t+1; \;\; t = u-1 \\ \text{then} \; du = dt \end{array}\right| =$$

$$= \int \frac{u-1}{u^3}du = \int u^{-2}du - \int u^{-3}du =$$

$$= \frac{u^{-1}}{-1} - \frac{u^{-2}}{-2} + C = -\frac{1}{t+1} + \frac{2}{(t+1)^2} + C$$

7.3.1.7 Exercises

Find the integrals by suitable u-substitution

1. $\int \cos(8\pi x)dx$

2. $\int \frac{dx}{(3-2x)^4}$

3. $\int \frac{dx}{\sqrt{1-(x+6)^2}}$

4. $\int \frac{\ln^5 x}{x} dx$

5. $\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right) dx$

6. $\int e^{\theta}\sqrt{12+e^{\theta}}d\theta$

7. $\int \cot t \, dt$

8. $\int \frac{\arctan^7 x}{1+x^2} dx$

7.3.1.8 Solutions

1. $\int \cos(8\pi x)dx$

Solution

$$\int \cos(8\pi x)dx = \left| \begin{array}{r} \text{let } u = 8\pi x \\ du = 8\pi dx \end{array} \right| =$$

$$= \frac{1}{8\pi}\int \cos u \, du =$$

$$= \frac{1}{8\pi}\sin u + C = \frac{1}{8\pi}\sin(8\pi x) + C$$

2. $\int \frac{dx}{(3-2x)^4}$

Solution

$$\int \frac{dx}{(3-2x)^4} = \left| \begin{array}{r} \text{let } u = 3-2x \\ du = -2dx \end{array} \right| =$$

$$= -\frac{1}{2}\int \frac{du}{u^4} =$$

$$= -\frac{1}{2}\cdot\frac{u^{-3}}{-3} + C = \frac{1}{6(3-2x)^3} + C$$

3. $\int \frac{dx}{\sqrt{1-(x+6)^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-(x+6)^2}} = \left| \begin{array}{r} \text{let } u = x+6 \\ du = dx \end{array} \right| =$$

$$= \int \frac{du}{\sqrt{1-u^2}} =$$

$$= \arcsin u + C = \arcsin(x+6) + C$$

4. $\int \frac{\ln^5 x}{x} dx$

Solution

$$\int \frac{\ln^5 x}{x} dx = \left| \begin{array}{r} \text{let } u = \ln x \\ du = \frac{1}{x} dx \end{array} \right| =$$

$$= \int u^5 du =$$

$$= \frac{u^6}{6} + C = \frac{\ln^6 x}{6} + C$$

5. $\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right) dx$

Solution

$$\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right) dx = \left| \begin{array}{r} \text{let } u = 3+8x^{\frac{3}{2}} \\ du = 12\sqrt{x}dx \end{array} \right| =$$

$$= \frac{1}{12}\int u\, du =$$

$$=\frac{1}{12}\frac{u^2}{2}+C=\frac{\left(3+8x^{\frac{3}{2}}\right)^2}{24}+C$$

6. $\int e^{\theta}\sqrt{12+e^{\theta}}d\theta$

Solution

$$\int e^{\theta}\sqrt{12+e^{\theta}}d\theta=\left|\begin{aligned}\text{let } u&=12+e^{\theta}\\ du&=e^{\theta}d\theta\end{aligned}\right|=$$

$$=\int\sqrt{u}\,du=$$

$$=\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C=\frac{2}{3}\sqrt{12+e^{\theta}}^{3}+C$$

7. $\int \cot t\; dt$

Solution

$$\int \cot t\; dt=\int\frac{\cos t}{\sin t}dt=$$

$$=\left|\begin{aligned}\text{let } u&=\sin t\\ du&=\cos t\; dt\end{aligned}\right|=$$

$$=\int\frac{du}{u}=\ln|u|+C=\ln|\sin t|+C$$

8. $\int\frac{\arctan^7 x}{1+x^2}dx$

Solution

$$\int\frac{\arctan^7 x}{1+x^2}dx=\left|\begin{aligned}\text{let } u&=\arctan x\\ du&=\frac{dx}{1+x^2}\end{aligned}\right|=$$

$$=\int u^7 du=$$

$$=\frac{u^8}{8}+C=\frac{\arctan^8 x}{8}+C$$

7.3.2 Integration Techniques: Integration by Parts

DETAILED DESCRIPTION:

The section starts by recalling the Product rule for differentiation of multiplication of two functions. The integration of this formula produces the method of integration by parts. The application of this method is useful if the integrand is a product of two functions of special type. The most popular cases are discussed and are complemented by examples.

AIM: **to learn the method of integration by parts and to recognize the types of integrals for which the method is useful.**

Learning Outcomes

1. Students recognize the integrals for that the integration by parts is useful.
2. Students can apply the method of partial integration to compute the integrals of different type.

Prior Knowledge: rules of differentiation; rules for integration; the method of substitution; algebra and trigonometry formulas.

Relationship to real maritime problems: a well-known application of the method of integration by parts is the calculation of Fourier coefficients of the Fourier series. Fourier series have broad applications in many disciplines. They are used to describe periodical physical phenomena, for instance, in signal processing, to detect and correct sources of vibration in mechanical devices.

Content

1. Formula for integration by parts
2. Special cases
3. Examples
4. Repeated application of the method
5. Exercises
6. Solutions

7.3.2.1 Formula for integration by parts

We will discuss the method that is often useful to compute the integral if it's integrand is a product of two functions.

Let us have two differentiable functions $u = u(x)$ and $v = v(x)$. We will calculate the differential of the product of these functions according to the ***Product Rule***

$$\boldsymbol{d(uv) = udv + vdu}$$

By integrating both sides of this equation, we obtain

$$\int d(uv) = \int udv + \int vdu.$$

By transposing terms and applying the property of integrals $\int \boldsymbol{d(uv) = uv + C}$, we get

$$\int \boldsymbol{udv = uv - \int vdu.}$$

This formula expresses the method of ***integration by parts*** or ***partial integration***. The constant of integration is not written because it can be considered to be part of the integral of the right side. The method is recommended if the integral on the right side of the formula is not more complicated than the given integral.

Example 1.1 Find the integral

$$\int xe^x dx$$

Solution

$$\int xe^x dx = \left| \begin{matrix} \text{let} & u = x; \;\; dv = e^x dx \\ \text{then} & du = dx; \;\; v = \int e^x dx = e^x \end{matrix} \right| =$$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

Comment. Let us note that here the formula for calculation of the differential of the function $v(x)$ is applied

$$\boldsymbol{dv = v'dx}$$

that we have to integrate to find the function $v(x)$.

7.3.2.2 Special cases

There are some special forms of integrals for which we can apply the method described above. They include polynomials whose degree is decreasing at derivation. On the other hand, there are functions that we cannot simplify by derivation. For instance, the exponential function e^x does not change by derivation. These observations point us to some standard situations for selection of the function $u = u(x)$.

Case 1. The integrand is a product of a polynomial and a trigonometric function $\sin x$ or $\cos x$

$\int P_n(x) \sin x \, dx$; we choose $u = P_n(x)$, then the rest of the expression is the differential $dv = \sin x dx$

Case 2. The integrand is a product of a polynomial and an exponential function e^x or a^x

$\int P_n(x)\, a^x dx$; we choose $u = P_n(x)$, then the differential is $dv = a^x dx$

Case 3. The integrand is a product of a polynomial and a logarithmic function $\ln x$ or $\log_a x$

$\int P_n(x) \log_a x dx$; we choose $u = \log_a x$, then the differential is $dv = P_n(x) dx$

Comment. Instead of the polynomial there can be given an arbitrary power function

$$\int x^k \ln x dx$$

Case 4. The integrand is a cyclometric function $\arcsin x$ or $\arctan x$

$\int \arcsin x\, dx$; we choose $u = \arcsin x$, then the differential is $dv = dx$

7.3.2.3 Examples

Example 3.1 Find the integral

$$\int 2x\cos x\, dx$$

Solution

$$\int 2x\cos x\, dx = \left| \begin{matrix} \text{let} & u = 2x; \;\; dv = \cos x\, dx \\ \text{then} & du = 2dx; \;\; v = \int \cos x\, dx = \sin x \end{matrix} \right| =$$

$$= 2x\sin x - \int 2\sin x\, dx = 2x\sin x + 2\cos x + C$$

Example 3.2 Find the integral

$$\int x^3 \ln x\, dx$$

Solution

$$\int x^3 \ln x\, dx = \left| \begin{matrix} \text{let} & u = \ln x; \;\; dv = x^3 dx \\ \text{then} & du = \frac{1}{x} dx; \;\; v = \int x^3 dx = \frac{x^4}{4} \end{matrix} \right| =$$

$$= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx =$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$$

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Example 3.3 Find the integral

$$\int \arctan x \, dx$$

Solution

$$\int \arctan x \, dx = \left| \begin{matrix} \text{let} & u = \arctan x; & dv = dx \\ \text{then} & du = \dfrac{dx}{1+x^2}; & v = \int dx = x \end{matrix} \right| =$$

$$= x\arctan x - \int \frac{x}{1+x^2} dx = x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2} dx =$$

$$= x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} =$$

$$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

In more general cases the functions can be composite, for instance, $\sin ax$; $\arctan(ax)$.

Example 3.4 Find the integral

$$\int (x+1)\sin 4x \, dx$$

Solution

$$\int (x+1)\sin 4x \, dx = \left| \begin{matrix} \text{let} & u = x+1; & dv = \sin 4x \, dx \\ \text{then} & du = dx; & v = \int \sin 4x \, dx = -\dfrac{1}{4}\cos 4x \end{matrix} \right| =$$

$$= -\frac{1}{4}(x+1)\cos 4x + \frac{1}{4}\int \cos 4x \, dx =$$

$$= -\frac{1}{4}(x+1)\cos 4x + \frac{1}{16}\sin 4x + C$$

7.3.2.4 Repeated application of method

If the polynomial factor of integrand is not linear, we can apply partial integration repeatedly. If the polynomial has degree n, we apply the method n times repeatedly to eliminate the degree of the polynomial.

Example 4.1 Find the integral

$$\int x^3 \sin x \, dx$$

Solution

For the given integral we will use the method of integration by parts three times because the degree of the given polynomial is three.

$$\int x^3 \sin x \, dx = \left| \begin{array}{ll} \text{let} \quad u = x^3; & dv = \sin x \, dx \\ \text{then} \;\; du = 3x^2 dx; & v = \int \sin x \, dx = -\cos x \end{array} \right| =$$

$$= -3x^3 \cos x + 3 \int x^2 \cos x \, dx =$$

$$= \left| \begin{array}{l} \text{let} \quad u = x^2; \;\; dv = \cos x \, dx \\ \text{then} \;\; du = 2x dx; \;\; v = \int \cos x \, dx = \sin x \end{array} \right| =$$

$$= -3x^3 \cos x + 3\left(x^2 \sin x - 2 \int x \sin x \, dx \right) =$$

$$= \left| \begin{array}{l} \text{let} \;\; u = x; \;\; dv = \sin x \, dx \\ \text{then} \;\; du = dx; \;\; v = -\cos x \end{array} \right| =$$

$$= -3x^3 \cos x + 3x^2 \sin x - 6\left(-x \cos x + \int \cos x \, dx \right) =$$

$$= -3x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Example 4.2 Find the integral

$$\int (x^2 - 3x + 7) 2^x dx$$

Solution

$$\int (x^2 - 3x + 7) 2^x dx = \left| \begin{array}{ll} \text{let} \quad u = x^2 - 3x + 7; & dv = 2^x dx \\ \text{then} \;\; du = (2x - 3) dx; & v = \int 2^x dx = \dfrac{2^x}{\ln 2} \end{array} \right| =$$

$$= (x^2 - 3x + 7) \frac{2^x}{\ln 2} - \frac{1}{\ln 2} \int (2x - 3) 2^x dx =$$

$$= \left| \begin{array}{ll} \text{let } u = 2x - 3; & dv = 2^x dx \\ \text{then } u = 2dx; & v = \int 2^x dx = \dfrac{2^x}{\ln 2} \end{array} \right| =$$

$$= (x^2 - 3x + 7) \frac{2^x}{\ln 2} - \frac{1}{\ln 2} \left((2x - 3) \frac{2^x}{\ln 2} - \frac{2}{\ln 2} \int 2^x dx \right) =$$

$$= (x^2 - 3x + 7) \frac{2^x}{\ln 2} - (2x - 3) \frac{2^x}{(\ln 2)^2} + \frac{2 \cdot 2^x}{(\ln 2)^3} + C$$

7.3.2.5 Exercises

Find the integrals

1. $\int \frac{x}{8} \sin x \, dx$

2. $\int 5x \cdot 5^x dx$

3. $\int \frac{\ln x}{x^2} dx$

4. $\int (x^2 + 1) \cos 2x \, dx$

5. $\int \arcsin x \, dx$

7.3.2.6 Solutions

1. $\int \frac{x}{8} \sin x \, dx$

Solution

$$\int \frac{x}{8} \sin x \, dx = \left| \begin{matrix} \text{let } u = \frac{x}{8}, \quad dv = \sin x \, dx \\ du = \frac{1}{8} dx, \quad v = \int \sin x \, dx = -\cos x \end{matrix} \right| =$$

$$= -\frac{x}{8} \cos x + \frac{1}{8} \int \cos x \, dx =$$

$$= -\frac{x}{8} \cos x + \frac{1}{8} \sin x + C$$

2. $\int 5x \cdot 5^x dx$

Solution

$$\int 5x \cdot 5^x dx = \left| \begin{matrix} \text{let } u = 5x, \quad dv = 5^x dx \\ du = 5dx, v = \int 5^x dx = \frac{5^x}{\ln 5} \end{matrix} \right| =$$

$$= 5x \frac{5^x}{\ln 5} - \frac{5}{\ln 5} \int 5^x dx =$$

$$= 5x \frac{5^x}{\ln 5} - \frac{5 \cdot 5^x}{\ln^2 5} + C$$

3. $\int \frac{\ln x}{x^2} dx$

Solution

$$\int \frac{\ln x}{x^2} dx = \left| \begin{matrix} \text{let } u = \ln x, \; dv = \frac{dx}{x^2} \\ du = \frac{1}{x}dx, \; v = \int x^{-2}dx = -x^{-1} \end{matrix} \right| =$$

$$= -\frac{\ln x}{x} + \int x^{-2}dx =$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

4. $\int (x^2+1)\cos 2x \, dx$

Solution

$$\int (x^2+1)\cos 2x \, dx = \left| \begin{matrix} \text{let } u = x^2+1, \; dv = \cos 2x \, dx \\ du = 2xdx; \; v = \int \cos 2x \, dx = \frac{1}{2}\sin 2x \end{matrix} \right| =$$

$$= \frac{x^2+1}{2}\sin 2x - \frac{1}{2}\int \sin 2x \, dx =$$

$$= \frac{x^2+1}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

5. $\int \arcsin x \, dx$

Solution

$$\int \arcsin x \, dx = \left| \begin{matrix} \text{let } u = \arcsin x, \; dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}dx, \; v = x \end{matrix} \right| =$$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx == \left| \begin{matrix} \text{let } u = 1-x^2 \\ du = -2xdx \end{matrix} \right| =$$

$$= x\arcsin x + \frac{1}{2}\int \frac{du}{\sqrt{u}} =$$

$$= x\arcsin x + \sqrt{u} + C =$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

Here we used the substitution for the second integral to simplify the integrand.

7.3.3 Integration Techniques: Integration of Rational Functions

DETAILED DESCRIPTION:

In this chapter we will discuss the problem-solving methods for indefinite integrals of rational functions. We will introduce proper rational functions and improper rational functions, and algebraic methods on how to decompose them into rational fractions or powers and rational fractions. Appropriate integral formulas will be considered. Examples of integration of simple rational functions will be demonstrated.

Students can investigate the examples presented by Symbolab Step-by-Step Calculator; Algebra; Rational Fractions (URL: https://www.symbolab.com/solver/partial-fractions-calculator). With this software it is also possible to check their own solutions by applying the Symbolab calculator for integrals.

AIM: **To acquire the technique of integration of rational functions.**

Learning Outcomes:

1. Perform **the expansion of proper rational function in partial fractions**
2. **Perform the long division of polynomials to get a polynomial plus a proper rational function**
3. **Solve the integrals of rational functions**

Prior Knowledge: algebraic identities; completing the square; factorising of polynomials; roots of polynomials; basic integration and derivation formulas.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

1. Rational functions, proper and improper rational functions
2. Basic integrals for simple cases
3. Decomposition of partial fractions
   3.1. Case 1. Denominator can be factorised in all linear multipliers
   3.2. Case 2. Denominator contains an irreducible quadratic
   3.3. Case 3. Denominator contains the repeated linear factor
4. Computation of improper rational functions

5. Summary
6. Exercises
7. Solutions of the exercises

7.3.3.1 Rational functions, proper and improper rational functions

A ***rational function*** has the form

$$\boldsymbol{f(x) = \frac{P_n(x)}{Q_m(x)}},$$

where $P_n(x)$ and $Q_m(x)$ are polynomials

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

$$Q_m(x) = b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \cdots + b_m$$

If the degree n of polynomial $P_n(x)$ is less than the degree m of polynomial $Q_m(x)$ $(n < m)$ then the rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)}$$

is called a ***proper rational function***, otherwise it is called an ***improper rational function***.

For example,

$$f(x) = \frac{2}{x+3}$$

$$g(x) = \frac{2x-1}{x^2-4}$$

$$t(x) = \frac{x+1}{x-3}$$

$$s(x) = \frac{x^2+3x-4}{x+1}$$

functions $f(x)$ and $g(x)$ are proper rational functions. Functions $t(x)$ and $s(x)$ are improper rational functions.

7.3.3.2 Basic integrals for simple cases

Simple cases of rational functions in a general form are the following

$$f(x) = \frac{A}{x+B}$$

$$g(x) = \frac{A}{(x+B)^m}$$

$$s(x) = \frac{2Ax + B}{Ax^2 + Bx + D}$$

Let us integrate these functions by applying formulas

$$\int \frac{dx}{x + B} = \ln|x + B| + C$$

$$\int \frac{du}{u^m} = \frac{u^{-m+1}}{-m + 1} + C$$

Integral of the first function is

$$\int f(x)dx = \int \frac{A}{x + B}dx = A\int \frac{dx}{x + B} = A \cdot \ln|x + B| + C$$

We will replace the linear argument of integrand $g(x)$ by the function $u = x + B$

$$\int g(x)\,dx = \int \frac{A}{(x + B)^m}dx = \left|\begin{matrix} u = x + B \\ du = dx \end{matrix}\right| = A\int \frac{du}{u^m} =$$

$$= A\frac{u^{-m+1}}{-m + 1} + C = A\frac{(x + B)^{-m+1}}{-m + 1} + C$$

For function $s(x)$ we will use substitution $u = Ax^2 + Bx + D$ and $du = 2Ax + B$

$$\int s(x)dx = \int \frac{2Ax + B}{Ax^2 + Bx + D}dx =$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|Ax^2 + Bx + D| + C$$

Example 2.1

$$\int \frac{2}{x + 3}dx = 2\int \frac{dx}{x + 3} = 2\ln|x + 3| + C$$

Example 2.2

$$\int \frac{dx}{(x - 1)^7} = \frac{(x - 1)^{-7+1}}{-7 + 1} + C = \frac{-1}{6(x - 1)^6} + C$$

Example 2.3

$$\int \frac{2x + 5}{x^2 + 5x + 10}dx =$$

$$= |u = x^2 + 5x + 10 \text{ and } du = (2x + 5)dx| =$$

$$= \ln|x^2 + 5x + 10| + C$$

In other cases, it is necessary to decompose the rational function into partial fractions to simplify the integration.

7.3.3.3 Decomposition of partial fractions

Let us have a proper rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)}; \quad n < m$$

We can apply the ***method of decomposition of partial fractions*** if the denominator can be factorised into fractions. Here we discuss three cases of decomposition of partial fractions:

Case 1: Denominator can be factorised in all linear multipliers;

Case 2: Denominator contains an irreducible quadratic;

Case 3: Denominator contains the repeated linear factor.

1.1. Case 1. Denominator can be factorised in all linear multipliers

The following example shows that we can integrate the function more easily if it is decomposed into partial fractions with linear denominators

Example 3.1

Suppose that we know the decomposition of the given integrand in two terms.

$$\int \frac{x+8}{x^2+x-2}dx = \int \left(\frac{3}{x-1} - \frac{2}{x+2}\right)dx =$$

$$= \int \frac{3}{x-1}dx - \int \frac{2}{x+2}dx =$$

$$= 3\ln|x-1| - 2\ln|x+2| + C$$

If the denominator $Q_2(x) = kx^2 + qx + p$ has real roots $\boldsymbol{x = -a}$ and $\boldsymbol{x = -b}$, it is reducible

$$Q_2(x) = k(x+a)(x+b)$$

We can split the given rational expression into partial fractions

$$\frac{R(x)}{Q_2(x)} = \frac{A}{x+a} + \frac{B}{x+b}$$

We know that the coefficients of polynomials and roots of denominator a and b are definite. To determinate the unknown constants A and B, we will equalize the denominators of partial fractions, equate the numerators, discard them, and get the equation

$$\frac{R(x)}{Q_2(x)} = \frac{kA(x+b)}{(x+a)(x+b)} + \frac{kB(x+a)}{(x+b)(x+a)}$$

$$R(x) = kA(x+b) + kB(x+a)$$

Let us plug the values $x = -a$ and then $x = -b$ into the equation to get

$$R(-a) = kA(b-a)$$

$$R(-b) = kB(a-b)$$

From these equations we can calculate the values of the unknown constants

$$A = \frac{R(-a)}{k(b-a)}$$

$$B = \frac{R(-b)}{k(a-b)}$$

Similarly we split the proper rational part in elementary parts if the denominator has a polynomial with higher degree as two. In the ***general case***, a denominator can have more than two linear multipliers

$$\frac{P_n(x)}{k(x+a_1)(x+a_2)\cdot \ldots \cdot (x+a_m)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \ldots + \frac{A_m}{x+a_m}$$

Example 3.2

Compute the integral

$$\int \frac{4x+7}{x^2+x-6}dx$$

Solution

First part: decomposition of partial fractions

Step 1. Factorise the denominator

$$x^2 + x - 6 = (x-2)(x+3)$$

Step2. Write partial fractions with unknown constants

$$\frac{4x+7}{x^2+x-6} = \frac{4x+7}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

Step 3. Equalize the denominators, equate the numerators, and discard denominators

$$4x + 7 = A(x+3) + B(x-2)$$

Step 4. Plug in $x = 2$ to calculate constant A

$$4 \cdot 2 + 7 = A(2+3) + B(2-2)$$

$$8 + 7 = 5A$$

$$A = 3$$

Step 5. Plug in $x = -3$ to calculate constant B

$$4 \cdot (-3) + 7 = A(-3+3) + B(-3-2)$$

$$-12 + 7 = -5B$$

$$B = 1$$

Second part: integration

$$\int \frac{4x+7}{x^2+x-6}dx = \int \left(\frac{3}{x-2} + \frac{1}{x+3}\right)dx =$$

$$= 3\ln|x-2| + \ln|x+3| + C$$

Answer

$$\int \frac{4x+7}{x^2+x-6}dx = 3\ln|x-2| + \ln|x+3| + C$$

1.2. Case 2. Denominator contains an irreducible quadratic

If the denominator $Q_3(x)$ of a rational function is reducible in the following way

$$Q_3(x) = k(x+a)(x^2+b),$$

we can decompose the given rational expression into partial fractions

$$\frac{R(x)}{Q_3(x)} = \frac{A}{x+a} + \frac{Bx+D}{x^2+b}$$

Similarly, we express

$$R(x) = kA(x^2+b) + kBx(x+a) + kD(x+a)$$

or, considering that the degree of polynomial $R(x)$ does not exceed 2

$$a_0x^2 + a_1x + a_2 = kA(x^2+b) + kBx(x+a) + kD(x+a)$$

Two polynomials are equal by the definition if they have the same degree and all corresponding coefficients are equal.

Therefore, we can write a system of equations and calculate the unknown constants A, B, D

$$\begin{cases} a_0 = kA + kB \\ a_1 = kBa + kD \\ a_2 = kAb + kDa \end{cases}$$

Example 3.3

Compute the integral

$$\int \frac{4x^2+2x-3}{(x-2)(x^2+1)}dx$$

Solution

First part: decomposition of partial fractions

Step1. Write partial fractions with unknown constants

$$\frac{4x^2+2x-3}{(x-2)(x^2+1)}=\frac{A}{x-2}+\frac{Bx+D}{x^2+1}$$

Step 2. Equalize the denominators, equate the numerators, and discard them

$$4x^2+2x-3=A(x^2+1)+Bx(x-2)+D(x-2)$$

Step 3. Create a system of equations

$$\begin{cases} 4=A+B \\ 2=-2B+D \\ -3=A-2D \end{cases}$$

Step 4. Express A from the first equation and plug it into the last equation to get a system of two equations

$$A=4-B$$

$$-3=4-B-2D$$

$$\begin{cases} 2=-2B+D \\ -7=-B-2D \end{cases}$$

Step 5. Multiply the first equation by 2 and add equations

$$-3=-5B;\quad B=0.6$$

Step 6. Calculate A and D

$$A=4-0.6=3.4$$

$$D=2+2B=2+1.2=3.2$$

Second part: integration

$$\int\frac{4x^2+2x-3}{(x-2)(x^2+1)}dx=$$

$$=\int\left(\frac{3.4}{x-2}+\frac{0.6x+3.2}{x^2+1}\right)dx=$$

$$=\int\frac{3.4}{x-2}dx+0.3\int\frac{2xdx}{x^2+1}+3.2\int\frac{dx}{x^2+1}=$$

$$=3.4\ln|x-2|+0.3\ln|x^2+1|+3.2\arctan x+C$$

Answer

$$\int\frac{4x^2+2x-3}{(x-2)(x^2+1)}dx=3.4\ln|x-2|+0.3\ln|x^2+1|+3.2\arctan x+C$$

3.3. Case 3. Denominator contains the repeated linear factor

If the denominator $Q_m(x)$ of a rational function is reducible

$$Q_m(x) = k(x+a)^m = k(x+a)\cdot(x+a)\cdot\ldots\cdot(x+a),$$

we can decompose the given rational expression into partial fractions

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_m}{(x+a)^m}$$

To calculate the unknown constants $A_1, A_2, \ldots, A_m$ we can use the same method as in case 2.

Example 3.4.

Compute

$$\int \frac{x^2-x-4}{(x-1)^3}dx$$

Solution

First part: decomposition of partial fractions

Step1. Write partial fractions with unknown constants

$$\frac{x^2-x-4}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Step 2. Equalize the denominators, equate the numerators, and discard them

$$x^2-x-4 = A(x-1)^2 + B(x-1) + D$$

$$x^2-x-4 = Ax^2 - 2Ax + A + Bx - B + D$$

Step 3. Create a system of equations

$$\begin{cases} 1 = A \\ -1 = -2A + B \\ -4 = A - B + D \end{cases}$$

Step 4. Calculate B and D

$$B = -1 + 2 = 1$$

$$D = -4 - 1 + 1 = -4$$

Second part: integration

$$\int \frac{x^2-x-4}{(x-1)^3}dx =$$

$$= \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{4}{(x-1)^3}\right)dx =$$

$$= \ln|x-1| - (x-1)^{-1} - 4\frac{(x-1)^{-2}}{-2} + C$$

Answer

$$\int \frac{x^2-x-4}{(x-1)^3}dx = \ln|x-1| - \frac{1}{x-1} + \frac{2}{(x-1)^2} + C$$

7.3.3.4 Computation of improper rational functions

An improper rational function can be expressed as a polynomial plus a proper rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)} = T_{n-m}(x) + \frac{R_k(x)}{Q_m(x)},$$

where polynomial $T_{n-m}(x)$ has degree $n-m$ and $n \geq m;\;\; k < m.$ It is necessary to perform *long division of polynomials* to get this result.

Example 4.1.

Compute

$$\int \frac{4x^3-12x+16}{x-2}dx$$

Solution

Step 1. Perform long division of polynomials

$$\begin{array}{r}
4x^2+8x+4 \\
x-2\overline{)4x^3-12x+16} \\
\underline{-(4x^3-8x^2)} \\
8x^2-12x+16 \\
\underline{-(8x^2-16x)} \\
4x+16 \\
\underline{-(4x-8)} \\
24
\end{array}$$

Step 2. Integrate step by step

$$\int \frac{4x^3-12x+16}{x-2}dx =$$

$$= \int \left(4x^2+8x+4+\frac{24}{x-2}\right)dx =$$

$$= \int 4x^2dx + \int 8xdx + \int 4dx + \int \frac{24}{x-2}dx =$$

$$= \frac{4x^3}{3} + \frac{8x^2}{2} + 4x + 24\ln|x-2| + C$$

Answer

$$\int \frac{4x^3 - 12x + 16}{x-2} dx = \frac{4x^3}{3} + 4x^2 + 4x + 24\ln|x-2| + C$$

7.3.3.5 Summary

To evaluate the integral of a rational function, the following steps are recommended

Starting step. Evaluate the given rational function $\int \frac{P_n(x)}{Q_m(x)} dx$

Case 1. The given function is an improper rational function $(n \geq m)$

Step 1.1. Perform long division of polynomials $P_n(x): Q_m(x)$

Step 1.2. Rewrite the integral as an integral of a polynomial plus a proper rational part

Step 1.3. Integrate the polynomial

Step 1.4. For the integral of the proper rational part complete case 2 if necessary or integrate

Case 2. The given function is a proper rational function $f(x) = \frac{P_n(x)}{Q_m(x)}; (n < m)$

Step 2.1. Factorise the denominator if necessary

Step 2.2. Complete the decomposition of the partial fraction

Step 2.3. Integrate simple rational fractions

Comment. There are two methods how to complete Step 2.2. We can apply ***"plug-in" method***, that is, - plug in useful values of x into the polynomial equation (see example 3.2.), or we can apply ***the method of unknown coefficients***, that is, equate the coefficients of two equal polynomials (see examples 3.3 and 3.4), or we can combine both methods.

Example 5.1.

Compute

$$\int \frac{x^4 + 12x - 6}{x^2(x-1)} dx$$

Solution

Let us follow the instructions above. We have **case 1** (given function is an improper rational function) and we complete **step 1.1**

$$\begin{array}{r} x+1 \\ x^3-x^2 \overline{) x^4+12x-6} \\ \underline{-(x^4-x^3)} \\ x^3+12x-6 \\ \underline{-(x^3-x^2)} \\ x^2+12x-6 \end{array}$$

Step 1.2.

$$\int \frac{x^4+12x-6}{x^2(x-1)}dx = \int \left(x+1+\frac{x^2+12x-6}{x^2(x-1)}\right)dx$$

Step 1.3.

$$\int \left(x+1+\frac{x^2+12x-6}{x^2(x-1)}\right)dx = \frac{x^2}{2}+x+\int \frac{x^2+12x-6}{x^2(x-1)}dx$$

Last integral is that of a proper rational function. We complete **step 2.2**.

$$\frac{x^2+12x-6}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

Let us apply the "plug-in" method. Get rid of all the denominators and write the equation

$$x^2+12x-6 = Ax(x-1)+B(x-1)+Cx^2$$

We can use the roots of denominator $x=0$ and $x=1$. We choose the constant $x=-1$ additionally.

If $x=0$ we get the equation

$$-6=-B; \quad B=6$$

If $x=1$ we get the equation

$$1+12-6=C; \quad C=7$$

If $x=-1$ we get the equation

$$1-12-6 = A(-1)(-2)+B(-2)+C$$

$$A=-6$$

We get simple rational fractions and can integrate these

$$\frac{x^2+12x-6}{x^2(x-1)} = \frac{-6}{x}+\frac{6}{x^2}+\frac{7}{x-1}$$

$$\int \left(\frac{-6}{x}+\frac{6}{x^2}+\frac{7}{x-1}\right)dx = -6\ln|x|+6\frac{x^{-1}}{-1}+7\ln|x-1|+C$$

Answer

$$\int \frac{x^4+12x-6}{x^2(x-1)}dx = \frac{x^2}{2} + x - 6\ln|x| - \frac{6}{x} + 7\ln|x-1| + C$$

7.3.3.6 Exercises

Evaluate the following indefinite integrals of rational functions

1. $\int \frac{dx}{1+7x}$
2. $\int \frac{2}{(x-1)(x+2)}dx$
3. $\int \frac{x-1}{x(x-2)(x-3)}dx$
4. $\int \frac{x}{4-x^2}dx$
5. $\int \frac{x+6}{x^2-8x}dx$
6. $\int \frac{x+5}{x^2-4x-12}dx$
7. $\int \frac{2x}{x^2-4x+20}dx$
8. $\int \frac{2x+1}{x^2(x+2)}dx$
9. $\int \frac{3}{x(1+x^2)}dx$
10. $\int \frac{3x}{x+1}dx$
11. $\int \frac{x^3}{x(x+3)}dx$

7.3.3.7 Solution of the exercises

1. $\int \frac{dx}{1+7x}$

Solution

$$\int \frac{dx}{1+7x} = \frac{1}{7}\int \frac{d(7x)}{1+7x} = \frac{1}{7}\ln|1+7x| + C$$

2. $\int \frac{2}{(x-1)(x+2)}dx$

Solution

Let us find elementary partial fractions

$$\frac{2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$2 = A(x+2) + B(x-1)$$

$$x = 1; \quad 2 = A \cdot 3; \quad A = \frac{2}{3}$$

$$x = -2; \;\; 2 = B \cdot (-3); \;\; B = \frac{-2}{3}$$

$$\frac{2}{(x-1)(x+2)} = \frac{2}{3} \cdot \frac{1}{x-1} - \frac{2}{3} \cdot \frac{1}{x+2}$$

Now we can change the integral

$$\int \frac{2}{(x-1)(x+2)}dx =$$

$$= \frac{2}{3}\int \frac{dx}{x-1} - \frac{2}{3}\int \frac{dx}{x+2} =$$

$$= \frac{2}{3}\ln|x-1| - \frac{2}{3}\ln|x+2| + C$$

3. $\int \frac{x-1}{x(x-2)(x-3)}dx$

Solution

Let us find partial fractions

$$\frac{x-1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x - 1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2)$$

$$x = 0; \;\; -1 = A(-2)(-3); \;\; A = -\frac{1}{6}$$

$$x = 2; \;\; 1 = B \cdot 2 \cdot (-1); \;\; B = -\frac{1}{2}$$

$$x = 3; \;\; 2 = C \cdot 3; \;\; C = \frac{2}{3}$$

Let us compute the integral

$$\int \frac{x-1}{x(x-2)(x-3)}dx =$$

$$= -\frac{1}{6}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x-2} + \frac{2}{3}\int\frac{dx}{x-3} =$$

$$= -\frac{1}{6}\ln|x| - \frac{1}{2}\ln|x-2| + \frac{2}{3}\ln|x-3| + C$$

4. $\displaystyle\int\frac{x}{4-x^2}dx$

Solution

Factorise the denominator

$$\frac{x}{4-x^2} = \frac{x}{(2-x)(2+x)}$$

Find partial fractions

$$\frac{x}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$x = A(2+x) + B(2-x)$$

$$x = 2;\;\; 2 = A\cdot 4;\;\; A = \frac{1}{2}$$

$$x = -2;\;\; -2 = B\cdot 4;\;\; B = -\frac{1}{2}$$

Compute the integral

$$\int\frac{x}{4-x^2}dx =$$

$$= \frac{1}{2}\int\frac{dx}{2-x} - \frac{1}{2}\int\frac{dx}{2+x} = =$$

$$= -\frac{1}{2}\ln|2-x| - \frac{1}{2}\ln|2+x| + C$$

Comment

$$\int\frac{dx}{2-x} =$$

$$= -\int\frac{dx}{x-2} = -\ln|x-2| + C =$$

$$= -\ln|2-x| + C$$

5. $\displaystyle\int\frac{x+6}{x^2-8x}dx$

Solution

Factorise the denominator

$$\frac{x+6}{x^2-8x}=\frac{x+6}{x(x-8)}$$

Find partial fractions

$$\frac{x+6}{x(x-8)}=\frac{A}{x}+\frac{B}{x-8}$$

$$x+6=A(x-8)+Bx$$

$$x=0;\;\; 6=A\cdot(-8);\;\; A=-\frac{3}{4}$$

$$x=8;\;\; 14=B\cdot 8;\;\; B=\frac{7}{4}$$

Compute the integral

$$\int\frac{x+6}{x^2-8x}dx=$$

$$=-\frac{3}{4}\int\frac{dx}{x}+\frac{7}{4}\int\frac{dx}{x-8}=$$

$$=-\frac{3}{4}\ln|x|+\frac{7}{4}\ln|x-8|+C$$

6. $\displaystyle\int\frac{x+5}{x^2-4x-12}dx$

Solution

Factorise the denominator

$$\frac{x+5}{x^2-4x-12}=\frac{x+5}{(x-6)(x+2)}$$

Find partial fractions

$$\frac{x+5}{(x-6)(x+2)}=\frac{A}{x-6}+\frac{B}{x+2}$$

$$x+5=A(x+2)+B(x-6)$$

$$x=6;\;\; 11=A\cdot 8;\;\; A=\frac{11}{8}$$

$$x=-2;\;\; 3=B\cdot(-8);\;\; B=-\frac{3}{8}$$

Compute the integral

$$\int\frac{x+5}{x^2-4x-12}dx=$$

$$=\frac{11}{8}\int\frac{dx}{x-6}-\frac{3}{8}\int\frac{dx}{x+2}=$$

$$= \frac{11}{8}\ln|x-6| - \frac{3}{8}\ln|x+2| + C$$

7. $\int \frac{2x}{x^2-4x+20}dx$

Solution

Here we cannot factorise the dominator. We will use another approach: substitution
Let us construct two partial fractions

$$\frac{2x}{x^2-4x+20} = \frac{2x-4+4}{x^2-4x+20} = \frac{2x-4}{x^2-4x+20} + \frac{4}{x^2-4x+20}$$

Now we will integrate two integrals

$$\int \frac{2x}{x^2-4x+20}dx = \int \frac{2x-4}{x^2-4x+20}dx + 4\int \frac{dx}{x^2-4x+20} =$$

$$= \left| \begin{array}{ll} \text{for first integral} & \text{for second integral} \\ \text{let} \quad u = x^2-4x+20; & \text{let } u = x-2; \;\; x^2-4x+4+16 = u^2+16 \\ \text{then } du = (2x-4)dx & \text{then } \; du = dx \end{array} \right| =$$

$$= \int \frac{du}{u} + 4\int \frac{du}{u^2+16} = \ln|u| + 4\cdot\frac{1}{4}\arctan\frac{u}{4} + C =$$

$$= \ln|x^2-4x+20| + \arctan\frac{x-2}{4} + C$$

8. $\int \frac{2x+1}{x^2(x+2)}dx$

Solution

Let us find partial fractions of integrand

$$\frac{2x+1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$2x+1 = Ax(x+2) + B(x+2) + Cx^2$$

$$x = 0; \qquad 1 = 2B; \quad B = 0.5$$

$$x = -2; \quad -4+1 = 4C; \;\; C = -0.75$$

$$x = -1; \quad -2+1 = -A+B+C;$$

$$-1 = -A - 0.25; \quad A = 0.75$$

We can split the integral in separate parts

$$\int \frac{2x+1}{x^2(x+2)}dx =$$

$$= 0.75\int \frac{dx}{x} + 0.5\int \frac{dx}{x^2} - 0.75\int \frac{dx}{x+2} =$$

$$= 0.75\ln|x| - 0.5\frac{1}{x} - 0.75\ln|x+2| + C$$

9. $\int \frac{3}{x(1+x^2)}dx$

Solution

We find partial fractions

$$\frac{3}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$
$$3 = A(1+x^2) + Bx^2 + Cx$$
$$x = 0;\ \ 3 = A$$

For other coefficients we create a system of equations

$$\begin{cases} A + B = 0 \\ C = 0 \end{cases};\quad B = -3$$

We integrate

$$\int \frac{3}{x(1+x^2)}dx =$$
$$= 3\int \frac{dx}{x} - 3\int \frac{xdx}{1+x^2} =$$
$$= 3\ln|x| - 1.5\int \frac{2xdx}{1+x^2} =$$
$$= 3\ln|x| - 1.5\ln(1+x^2) + C$$

10. $\int \frac{3x}{x+1}dx$

Solution

The integrand is an improper rational part. We will transform this rational in the following way

$$\frac{3x}{x+1} = 3\frac{x+1-1}{x+1} = 3 - \frac{3}{x+1}$$

We integrate

$$\int \frac{3x}{x+1}dx =$$
$$= 3\int dx - 3\int \frac{dx}{x+1} =$$
$$= 3x - 3\ln|x+1| + C$$

11. $\int \frac{x^3}{x(x+3)}dx$

Solution

We simplify the expression and then perform a long division

$$\frac{x^3}{x(x+3)} = \frac{x^2}{x+3}$$

$$\begin{array}{r} x-3 \\ x+3\overline{)x^2} \\ \underline{-(x^2+3x)} \\ -3x \\ \underline{-(-3x-9)} \\ 9 \end{array}$$

$$\frac{x^2}{x+3} = x - 3 + \frac{9}{x+3}$$

We integrate

$$\int \frac{x^3}{x(x+3)}dx =$$

$$= \int \frac{x^2}{x+3}dx =$$

$$= \int xdx - 3\int dx + 9\int \frac{dx}{x+3} =$$

$$= \frac{x^2}{2} - 3x + 9\ln|x+3| + C$$

7.3.4 Integration Techniques: Integration of Trigonometric Functions

DETAILED DESCRIPTION:

Various oscillation processes can be described by trigonometric functions. The research of such processes requires the calculation of integrals where integrands are composite functions. Trigonometric identities are useful to modify these integrals. In this chapter we will present the application of trigonometric formulas for more common cases and the appropriate substitution for solving integrals. The method of trigonometric substitution will be introduced additionally.

AIM: **To learn the use of trigonometric identities and special cases of substitution for trigonometric integrands.**

Learning Outcomes:

1. Students will be able integrate the integrals of trigonometric functions applying some trigonometric identities.
2. Students can apply the trigonometric substitution.

Prior Knowledge: rules of integration and differentiation; substitution methods for integrals; algebra and trigonometry formulas.

Relationship to real maritime problems: trigonometric integrals are useful for describing and for solving different problems on sinusoidal processes – for example, to construct an effective shape of a ship propellers' blades, or to calculate wave resistance for steady motion in ship's control equipment.

Content

1. Composite trigonometric functions of a linear argument
2. Product of sines and cosines
3. Powers of trigonometric functions
4. Double-angle trigonometric identity
5. Trigonometric substitution
6. Exercises
7. Solutions

7.3.4.1 Composite trigonometric functions of a linear argument

Some of the simplest cases where integrals involve sine and cosine functions are the following

$$\int \sin ax\,dx\,;\quad \int \cos ax\,dx\,;\quad \int \tan ax\,dx$$

For such cases we can use simple substitution, for instance, as we can see in the example 1.1.

Example 1.1 Compute the integral

$$\int \sin ax\,dx$$

Solution

$$\int \sin ax\,dx = \left|\begin{matrix}\text{let} & u = ax\\ \text{then} & du = a dx\end{matrix}\right| =$$

$$= \frac{1}{a}\int \sin u\,du = -\frac{1}{a}\cos u + C =$$

$$= -\frac{1}{a}\cos ax + C$$

A slightly more complicated case is the integrand that is a tangent function

Example 1.2 Compute the integral

$$\int \tan ax dx$$

Solution

$$\int \tan ax\,dx = \int \frac{\sin ax}{\cos ax}dx =$$

$$= \left|\begin{matrix}\text{let} & u = \cos ax\\ \text{then} & du = -a\sin ax\,dx\end{matrix}\right| =$$

$$= -\frac{1}{a}\int \frac{du}{u} = -\frac{1}{a}\ln|u| + C =$$

$$= -\frac{1}{a}\ln|\cos ax| + C$$

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As additional special cases we can add these formulas to the list of basic integral formulas:

$$\int \mathbf{sin}\boldsymbol{ax}\,\boldsymbol{dx} = -\frac{\mathbf{1}}{\boldsymbol{a}}\mathbf{cos}\boldsymbol{ax} + \boldsymbol{C}$$

$$\int \mathbf{cos}\boldsymbol{ax}\,\boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\mathbf{sin}\boldsymbol{ax} + \boldsymbol{C}$$

$$\int \mathbf{tan}\boldsymbol{ax}\,\boldsymbol{dx} = -\frac{\mathbf{1}}{\boldsymbol{a}}\mathbf{ln}|\mathbf{cos}\boldsymbol{ax}| + \boldsymbol{C}$$

$$\int \mathbf{cot}\boldsymbol{ax}\,\boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\mathbf{ln}|\mathbf{sin}\boldsymbol{ax}| + \boldsymbol{C}$$

7.3.4.2 Product of sines and cosines

Here we will look at the integrals of the product of trigonometric functions that have different arguments, for instance,

$$\int \sin ax \cdot \cos bx\, dx$$

For simplifying integrals of this kind we can apply the product-to-sum identities

$$\mathbf{sin}\boldsymbol{ax}\cdot\mathbf{cos}\boldsymbol{bx} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{sin}(\boldsymbol{ax}+\boldsymbol{bx}) + \mathbf{sin}(\boldsymbol{ax}-\boldsymbol{bx}))$$

$$\mathbf{cos}\boldsymbol{ax}\cdot\mathbf{cos}\boldsymbol{bx} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{cos}(\boldsymbol{ax}+\boldsymbol{bx}) + \mathbf{cos}(\boldsymbol{ax}-\boldsymbol{bx}))$$

$$\mathbf{sin}\boldsymbol{ax}\cdot\mathbf{sin}\boldsymbol{bx} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{cos}(\boldsymbol{ax}-\boldsymbol{bx}) - \mathbf{cos}(\boldsymbol{ax}+\boldsymbol{bx}))$$

The substitution takes place after splitting the integral into two parts.

Example 2.1 Compute the integral

$$\int \sin 5x \cdot \sin 2x\, dx$$

Solution

$$\int \sin 5x \cdot \sin 2x\, dx = \frac{1}{2}\int (\cos 3x - \cos 7x)dx =$$

$$= \frac{1}{2}\int \cos 3x\, dx - \frac{1}{2}\int \cos 7x\, dx =$$

$$= \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$$

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7.3.4.3 Powers of trigonometric functions

In this section we take consider the integrals which include the integer powers of sines and cosines. In general form it is written as

$$\int \sin^n x \cdot \cos^m x \, dx$$

If at least one of the powers is an odd number, we can apply substitution.

Example 3.1 Compute the integral

$$\int \cos^5 x \cdot \sin^3 x \, dx$$

Let us notice that both functions have odd powers and the cosine function has bigger power than the sine function. Therefore, we will substitute the cosine, but first we split the power of the sine into multipliers and then use the trigonometric identity

$$\mathbf{\sin^2 x + \cos^2 x = 1}$$

Solution

$$\int \cos^5 x \cdot \sin^3 x \, dx = \int \cos^5 x \cdot \sin^2 x \cdot \sin x \, dx =$$

$$= \int \cos^5 x \cdot (1 - \cos^2 x) \cdot \sin x \, dx =$$

$$= \left| \begin{matrix} \text{let } u = \cos x \\ du = -\sin x \, dx \end{matrix} \right| =$$

$$= -\int u^5 (1 - u^2) du =$$

$$= -\int u^5 du + \int u^7 du =$$

$$= -\frac{u^6}{6} + \frac{u^8}{8} + C = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$$

Example 3.2 Compute the integral

$$\int \frac{\cot x}{\sin^3 x} dx$$

Solution

$$\int \frac{\cot x}{\sin^3 x} dx = \int \frac{\cos x}{\sin x \cdot \sin^3 x} dx =$$

$$= \int \frac{\cos x}{\sin^4 x} dx = \left| \begin{matrix} \text{let } u = \sin x \\ du = \cos x dx \end{matrix} \right| =$$

$$= \int \frac{du}{u^4} = \int u^{-4} du = \frac{u^{-3}}{-3} + C =$$

$$= \frac{\sin^{-3} x}{-3} + C = -\frac{1}{3\sin^3 x} + C$$

7.3.4.4 Double-angle trigonometric identities

In the previous section we discussed the methods of integration of sine and cosine functions on the integer powers where at least one of the powers is an odd integer. For even cases we can apply double-angle identities to eliminate the powers

$$\mathbf{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$\mathbf{\cos^2 x = \frac{1 + \cos 2x}{2}}$$

Example 4.1 Compute the integral

$$\int \cos^2 x \, dx$$

Solution

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx =$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx =$$

$$= \frac{1}{2} x + \frac{1}{2} \cdot \frac{1}{2} \sin 2x + C$$

Example 4.2 Compute the integral

$$\int \sin^4 x \, dx$$

Solution

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx =$$

$$= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx =$$

$$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx$$

$$=\frac{1}{4}\int dx-\frac{1}{4}\int 2\cos 2x\,dx+\frac{1}{4}\int\frac{1+\cos 4x}{2}dx=$$

$$=\frac{1}{4}\int dx-\frac{1}{4}\int \cos 2x\,d2x+\frac{1}{4}\cdot\frac{1}{2}\int dx+\frac{1}{4}\cdot\frac{1}{2}\int \cos 4x\,dx=$$

$$=\frac{1}{4}x-\frac{1}{4}\sin 2x+\frac{1}{8}x+\frac{1}{8}\cdot\frac{1}{4}\sin 4x+C=$$

$$=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

Notice that in the solution of example 4.2 the double-angle formula is applied repeatedly.

7.3.4.5 Trigonometric substitution

Here we shall look at more complex integrals where an integrand contains the square root of a quadratic expression, for instance,

$$\int x\sqrt{4-x^2}dx;\quad \int\frac{\sqrt{x^2-25}}{x^3}dx$$

Trigonometric substitution is useful to simplify the integrands. The substitution method is based on the trigonometric identity

$$\mathbf{\sin^2 x+\cos^2 x=1}$$

Dividing the identity by $\cos^2 x$ we derive a special case

$$\mathbf{\tan^2 x+1=\frac{1}{\cos^2 x}}$$

We apply the-above mentioned identities for the following cases

Case 1. For $\sqrt{a^2-x^2}$ we substitute $x=a\sin u$ or $x=a\cos u$

Case 2. For $\sqrt{a^2+x^2}$ we substitute $x=a\tan u$

Case 3. For $\sqrt{x^2-a^2}$ we substitute $x=\frac{a}{\cos u}$

Example 5.1 Compute the integral

$$\int x\sqrt{4-x^2}dx$$

Solution

To solve the integral, we use trigonometric substitution and then we change the differential to get the integral of the power function.

$$\int x\sqrt{4-x^2}dx = \left| \begin{matrix} \text{let } x = 2\sin u, \ \text{ then } dx = 2\cos u \, du \\ 4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u \end{matrix} \right| =$$

$$= \int 2\sin u\sqrt{4\cos^2 u}\, 2\cos u \, du =$$

$$= 8\int \cos^2 u \, \sin u \, du =$$

$$= -8\int \cos^2 u \, d(\cos u) = -8\frac{\cos^3 u}{3} + C$$

Now we return to the function with respect to the argument x

$$\cos^3 u = \cos^2 u \cdot \cos u = (1-\sin^2 u)\sqrt{1-\sin^2 u} =$$

$$= \left| \text{as } x = 2\sin u \ \text{ follows } \ 1 - \sin^2 u = 1 - \frac{x^2}{4} \right| =$$

$$= \left(1 - \frac{x^2}{4}\right)\sqrt{1 - \frac{x^2}{4}}$$

The solution of the integral is

$$\int x\sqrt{4-x^2}dx = -8\frac{\cos^3 u}{3} + C =$$

$$= -\frac{8}{3}\left(\sqrt{1-\frac{x^2}{4}}\right)^3 + C =$$

$$= -\frac{\left(\sqrt{4-x^2}\right)^3}{3} + C$$

Example 5.2 Compute the integral

$$\int \frac{\sqrt{x^2-25}}{x^3}dx$$

Solution

$$\int \frac{\sqrt{x^2-25}}{x^3}dx = \left| \begin{matrix} \text{let } x = \dfrac{5}{\cos u} \quad \text{then } \ dx = \dfrac{5\sin u}{\cos^2 u}du \\ x^2 - 25 = 25\tan^2 u \end{matrix} \right| =$$

$$= \int \frac{\sqrt{25\tan^2 u}\cdot \cos^3 u}{125}\frac{5\sin u}{\cos^2 u}du \ =$$

$$= \frac{1}{5}\int \tan u \cdot \cos u \cdot \sin u \, du =$$

$$= \frac{1}{5}\int \sin^2 u \, du = \frac{1}{10}\int (1-\cos 2u)du =$$

$$=\frac{1}{10}\left(u-\frac{1}{2}\sin 2u\right)+C=$$

$$=\left|\text{as }\ \cos u=\frac{5}{x}\quad \text{we get } \sin 2u=2\cdot\frac{5}{x}\sqrt{1-\frac{25}{x^2}}=\frac{10}{x^2}\sqrt{x^2-25}\right|=$$

$$=\frac{1}{10}\left(\arccos\frac{5}{x}-\frac{5}{x^2}\sqrt{x^2-25}\right)+C$$

Comment. The following formula is applied to expand the function $\sin 2u$

$$\mathbf{\sin 2x = 2\sin x \cos x}$$

7.3.4.6 Exercises

Compute the integrals

1. $\int \sin 12x\, dx$

2. $\int \cot\frac{3x}{4}\, dx$

3. $\int \sin 5x \cdot \cos 4.5x\, dx$

4. $\int \sin^{11}x \cdot \cos x\, dx$

5. $\int \frac{\cos^3 x}{\sin^5 x}\, dx$

6. $\int 8(1-\cos^2 x) dx$

2. Apply trigonometric substitution

$$\int \sqrt{1-x^2}\, dx$$

7.3.4.7 Solutions

1. $\int \sin 12x\, dx$

Solution

$$\int \sin 12x\, dx=\frac{1}{12}\int \sin 12x\, d12x=$$

$$=-\frac{1}{12}\cos 12x+C$$

2. $\int \cot\frac{3x}{4}dx$

Solution

$$\int \cot\frac{3x}{4}dx = \int \frac{\cos\frac{3x}{4}}{\sin\frac{3x}{4}}dx =$$

$$= \left| \begin{matrix} u = \sin\frac{3x}{4} \\ du = \frac{3}{4}\cos\frac{3x}{4}dx \end{matrix} \right| =$$

$$= \frac{4}{3}\int\frac{du}{u} = \frac{4}{3}\ln|u| + C =$$

$$= \frac{4}{3}\ln\left|\sin\frac{3x}{4}\right| + C$$

3. $\int \sin 5x \cdot \cos 4.5x\, dx$

Solution

$$\int \sin 5x \cdot \cos 4.5x\, dx = \frac{1}{2}\int(\sin 9.5x + \sin 0.5x)dx =$$

$$= \frac{1}{2}\cdot\frac{2}{19}\int \sin 9.5x\, d9.5x + \frac{1}{2}\cdot 2\int \sin 0.5x\, d0.5x =$$

$$= -\frac{1}{19}\cos 9.5x - \cos 0.5x + C$$

4. $\int \sin^{11}x \cdot \cos x\, dx$

Solution

$$\int \sin^{11}x \cdot \cos x\, dx = \left| \begin{matrix} u = \sin x \\ du = \cos x\, dx \end{matrix} \right| =$$

$$= \int u^{11}du = \frac{u^{12}}{12} + C =$$

$$= \frac{\sin^{12}x}{12} + C$$

5. $\int \frac{\cos^3 x}{\sin^5 x} dx$

Solution

$$\int \frac{\cos^3 x}{\sin^5 x} dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^5 x} dx =$$

$$= \left| \begin{array}{c} u = \sin x \\ du = \cos x \; dx \end{array} \right| = \int \frac{1-u^2}{u^5} du =$$

$$= \int u^{-5} \, du - \int u^{-3} \, du =$$

$$= \frac{u^{-4}}{-4} - \frac{u^{-2}}{-2} + C =$$

$$= -\frac{1}{4\sin^4 x} + \frac{1}{2\sin^2 x} + C$$

6. $\int 8(1-\cos^2 x) dx$

Solution

$$\int 8(1-\cos^2 x) dx = 8 \int \sin^2 x \;\, dx =$$

$$= 4 \int (1-\cos 2x) \, dx =$$

$$= 4 \int dx - 2 \int \cos 2x \; d2x =$$

$$= 4x - 2\sin 2x + C$$

7. Apply trigonometric substitution

$$\int \sqrt{1-x^2} \, dx$$

Solution

$$\int \sqrt{1-x^2} \, dx = \left| \begin{array}{c} \text{let } \; x = \sin u \\ dx = \cos u \; du \end{array} \right| =$$

$$= \int \sqrt{1-\sin^2 u} \cos u \; du =$$

$$= \int \cos u \cdot \cos u \; du = \int \frac{1+\cos 2u}{2} \; du =$$

$$= \frac{1}{2}\int du + \frac{1}{4}\int \cos 2u \, d2u =$$

$$= \frac{1}{2}u + \frac{1}{4}\sin 2u + C =$$

$$= \left| \begin{array}{c} u = \arcsin x \\ \sin 2u = 2\sin u \cos u = 2x\sqrt{1-x^2} \end{array} \right| =$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

7.4 Definite Integrals

DETAILED DESCRIPTION:

This chapter introduces the definite integral. The description starts with the question: how to calculate the area of a region that is bounded by a curve and straight lines? The method of approximate calculation is discussed. Then this method is generalised and the definition of the definite integral is formulated. Basic properties and the Newton-Leibniz formula are presented for calculations of integrals. For students, we can recommend the "Integral Calculator" https://www.integral-calculator.com/. This software shows the solution of an integral step by step and comes with an interactive graph of integrand and antiderivative. With a help of such software students can check their individual solutions of the task. The graphs can be constructed using free software GeoGebra Classic; DESMOS Graphing Calculator, and Microsoft Excel.

AIM: Learn the definition of the definite integral as the limit of a sum.

Learning Outcomes

1. Understand the meaning of the definite integral
2. Understand and apply the rules for calculating definite integrals

Prior Knowledge: basic rules of integration and differentiation; knowledge of the properties of elementary functions and their graphs; algebra and trigonometry formulas.

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculate the area of different shapes, volumes of solids, and to solve other geometric problems. Definite integrals are used for various calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in theory of cargo transport, in economics, in classical signal theory, and in other specialities.

Content

1. Statement of area problem
2. Definition of the definite integral
3. Properties of the definite integral
4. Calculation of a definite integral
5. Exercises
6. Solutions

7.4.1 Statement of area problem

From the results of Euclidean geometry, we know how to calculate the area of rectangle, triangle, circle and other simple plane figures. If the polygon is given, its area can be found by subdividing the polygon into a finite number of non-overlapping triangles. It is a different case we have if we need to calculate the area of a region enclosed by an arbitrary curve.

The example that we will solve is about calculating the area of a region that is placed in the Cartesian coordinate system.

Example 1.1. Find the area of the region enclosed by the parabola $y = x^2$, two vertical straight lines $x = 0$ and $x = 2.4$, and x-axis $y = 0$.

Solution

We can calculate the area of the region approximately. Let us subdivide the interval $[0, 2.4]$ into parts of equal length. For any such subinterval we construct a rectangle whose height is equal to the value of the given function at the endpoint of the chosen subinterval:

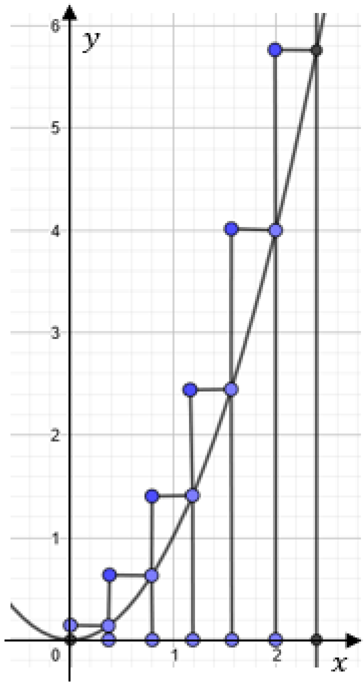

Figure 1.1

The interval $[0;\ 2.4]$ is 2.4 units long (see figure 1.1). Every subinterval is 0.4 units long. We can calculate the area of all rectangles

$$0.4 \cdot 0.4^2 + 0.4 \cdot 0.8^2 + 0.4 \cdot 1.2^2 + 0.4 \cdot 1.6^2 + 0.4 \cdot 2^2 + 0.4 \cdot 2.4^2 = 5.824\ \mathrm{sq.\,units}$$

Figure 1.1 shows that the calculated area of rectangles is larger than the area of the region that we need to find. Subdividing the interval $[0, 2.4]$ into shorter subintervals will make the error of calculations smaller. If the length of the subinterval is 0.01, we get

$$0.01(0.01^2 + 0.02^2 + 0.03^2 + \cdots + 2.4^2) \approx 4.63684 \text{ square units}$$

It is useful to do this very long summation of 261 addends by a software program, for instance, by Microsoft Excel. Thus, we can subdivide the interval $[0, 2.4]$ into more and more detail, thus reducing the error. Anyway, we need to find the correct answer.

7.4.2 Definition of the definite integral

An arbitrary continuous function $y = f(x)$ is defined on the closed interval $[a, b]$. The task is to calculate area *S* of the region between the curve determined by the function, x-axis, and two vertical straight lines $x = a$, $x = b$ (see figure 2.1).

We choose a set of points inside the interval $[a, b]$, say

$$\{x_1 = a, x_2, x_3, \dots, x_n = b\}, \text{ where}$$

$$x_1 < x_2 < x_3 < \cdots < x_n$$

We call this set of points a ***partition of interval*** $[\boldsymbol{a}, \boldsymbol{b}]$ into *n* subintervals $[x_{i-1}, x_i]$, where $i = 1,2,3, \dots, n$. The length of any subinterval is

$$\Delta x_i = x_i - x_{i-1}$$

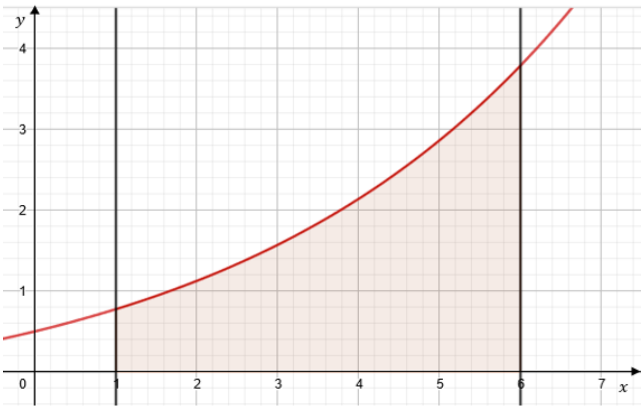

Figure 2.1

We chose an inner point c_i in every interval $c_i \in [x_{i-1}, x_i]$, where $i = 1, 2, 3, \dots, n$ and construct the rectangle with side length Δx_i and $f(c_i)$ (see figure 2.2).

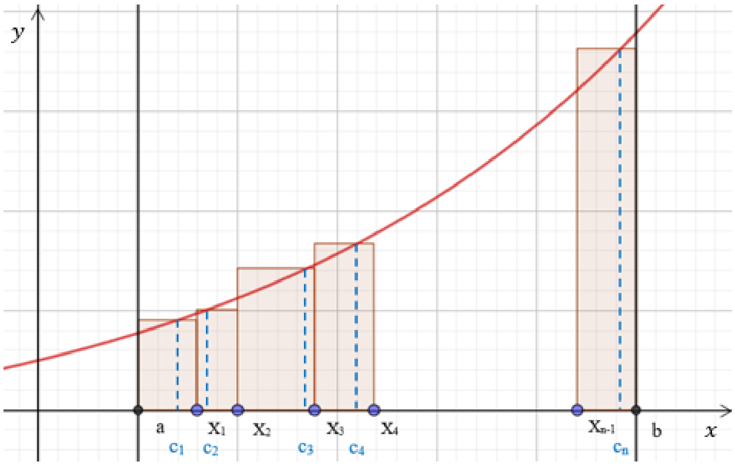

Figure 2.2

Now we compute the area of every rectangle and calculate their sum. The sum of all areas can be written using a ***sigma notation***

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

The sum expresses the approximate value of the area under the curve determined by the function $f(x)$ over the interval $[a, b]$. If we make the partition of this interval into smaller and smaller parts, so that the length of the longest subinterval tends to zero

$$\max\Delta x_i \to 0,$$

then the difference between the sum and the area S of the given region will decrease. Taking the limit, we can calculate the precise value of area S

$$S = \lim_{\max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Definition. If the limit S exists at any partition of the interval $[a, b]$ and any selection of inner points c_i, then we call the limit the ***definite integral*** of the function $f(x)$ on the interval $[a, b]$. The definite integral is denoted by the symbol

$$\lim_{\max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

If the limit exists, we say that the function $f(x)$ is ***integrable*** on interval $[a, b]$.

The sign ∫ is called the ***integral sign***, it resembles the letter S since it represents the limit of the sum.

Numbers a and b are called the ***limits of integration***, a is the ***lower limit***, and b is the ***upper limit***.

The function $f(x)$ is called ***integrand***; x is the variable of integration.

dx is the ***differential of*** $\boldsymbol{x}$.

The variable x can be replaced with any other variable without changing the value of the integral

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Example 2.1 Find the area of the region bounded by $f(x) = 1, x = a, x = b, y = 0.$

Solution

The area of the given region can be calculated by the integral

$$\int_a^b 1 \cdot dx$$

Described region is bounded by straight lines that define a rectangle (see figure 2.3).

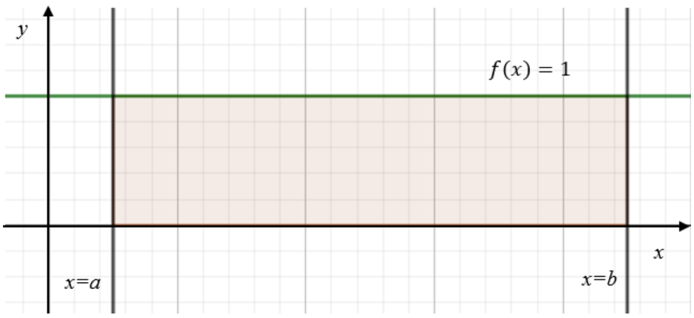

Figure 2.3

The area of this rectangle is equal to the value of the integral

$$\int_a^b dx = (b-a) \cdot 1 = b - a \quad \text{square units}$$

7.4.3 Properties of the definite integral

Some of the most important properties of definite integrals are included in the following list. Most of the properties listed below can be directly deduced from the definition of the definite integral.

Let functions $f(x)$ and $g(x)$ be continuous and differentiable on the interval $[a, b]$, then

1. $\displaystyle\int_a^a f(x)dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c is a particular constant.

4. $\int_a^b (cf(x) + kg(x))\,dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx$, where c and k are particular constants.

5. If $f(x) \le g(x)$ for all arguments $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

6. For all arguments $x \in [a,b]$ is true $\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$

7. If the integral has symmetric limits and the function $f(x)$ is an odd function $(f(-x) = -f(x))$, then $\int_{-a}^a f(x)\,dx = 0$

8. If the integral has symmetric limits and the function $f(x)$ is an even function $(f(-x) = f(x))$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

Other properties present methods of evaluation of the definite integral.

9. $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$, where m is the minimum value of the function $f(x)$ in the interval $[a,b]$, M is the maximum value in the interval $[a,b]$.

The 10th property is called ***the Mean value theorem***.

10. $\int_a^b f(x)dx = f(c)(b-a)$, where $c \in [a,b]$.

By applying the Mean value theorem, we can calculate the ***average value*** of an integrable function $f(x)$ on the interval $[a,b]$:

$$\boldsymbol{\frac{1}{b-a}\int_a^b f(x)dx}$$

7.4.4 Calculation of the definite integral

The Newton–Leibniz formula. If the function $f(x)$ is continuous on the interval $[a,b]$ and the function $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b \boldsymbol{f(x)dx = F(b) - F(a)}$$

How to apply this formula? First, we need to compute the corresponding indefinite integral

$$\int f(x)dx = F(x) + C,$$

and then calculate the values of the antiderivative at the upper limit of integral and at its lower limit, and subtract them

$$F(b) + C - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$$

Calculation shows that we can omit the constant C of integration. Therefore, we will expand the Newton – Leibniz formula with an evaluation symbol (vertical line segment)

$$\int_a^b \boldsymbol{f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$$

Now we can precisely calculate the value of the region defined in example 1.1. We calculate the integral

$$\int_0^{2.4} x^2 dx = \frac{x^3}{3}\Big|_0^{2.4} = \frac{2.4^3}{3} - 0 = 4.608 \quad \text{sq. units}$$

Example 4.1

Application of the Newton – Leibniz formula

$$\int_1^3 2x dx = x^2\Big|_1^3 = 3^2 - 1 = 8$$

Example 4.2 Compute the integral

$$\int_1^4 (3 + \sqrt{x})\, dx$$

Solution

Here we use property 4 of definite integrals in the solution

$$\int_1^4 (3 + \sqrt{x})\, dx = \int_1^4 3\, dx + \int_1^4 \sqrt{x}\, dx =$$

$$= \left(3x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_{1}^{4} =$$

$$= 3(4-1) + \frac{2}{3}\left(\sqrt{4^3} - 1\right) =$$

$$= 9 + \frac{2}{3} \cdot 7 = 13\frac{2}{3}$$

Example 4.3 Compute the integral

$$\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\varphi}{2} d\varphi$$

Solution

To solve this integral we change the differential

$$\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\varphi}{2} d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi - \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos 2\varphi \; d(2\varphi) = \frac{\varphi}{2}\Bigg|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin 2\varphi\Bigg|_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{4} - \frac{1}{4}\sin\pi - (0 - \sin 0) = \frac{\pi}{4}$$

Example 4.4 Compute the integrals

$$\int_{-1}^{1} x^2 \, dx + \int_{1}^{2} \sqrt[3]{x^4} \cdot x^{\frac{2}{3}} \, dx$$

Solution

We apply property 3 to simplify the task

$$\int_{-1}^{1} x^2 \, dx + \int_{1}^{2} \sqrt[3]{x^4} \cdot x^{\frac{2}{3}} \, dx = \int_{-1}^{1} x^2 \, dx + \int_{1}^{2} x^2 \, dx =$$

$$= \int_{-1}^{2} x^2 \, dx = \frac{x^3}{3}\Bigg|_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3$$

Example 4.5

$$\int_{-\pi}^{\pi} \sin^5 2x \, dx = 0,$$

while the integrand is an odd function (see property 7).

Example 4.6

Calculate the average value of the function $f(x) = \tan^2 x$ over the interval $\left[0, \frac{\pi}{4}\right]$.

Solution

We apply the mean value theorem

$$\frac{1}{b-a} \int_a^b f(x)dx$$

The length of interval is

$$b - a = \frac{\pi}{4}$$

We calculate the average value of the function

$$\frac{1}{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \tan^2 x \, dx = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x} dx =$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{1 - cos^2 x}{\cos^2 x} dx =$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx - \frac{4}{\pi} \int_0^{\frac{\pi}{4}} 1 \, dx = \frac{4}{\pi} (\tan x - x) \Bigg|_0^{\frac{\pi}{4}} =$$

$$= \frac{4}{\pi} \left(1 - \frac{\pi}{4}\right) = \frac{4}{\pi} - 1$$

Example 4.7. *Application of integrals in real situations*

The bananas storage temperature in cargo holds must be between 13.3 and 13.6 degrees Celsius. The temperature recorded during a half of a day followed the curve

$$f(t) = 0.001t^3 - 0.01t^2 + 13.4$$

where t is a number of hours ($0 \leq t \leq 12$). What is the average temperature in the cargo holds during this time period?

Solution. The average temperature can be calculated using the mean value theorem for integrals. If the function $y = f(x)$ is integrable on the interval $[a, b]$ then the mean value of this function on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

The given interval for variable t is here [0, 12]. Then the mean value of the function we will calculate in the following way

$$f_{mean} = \frac{1}{12-1}\int_1^{12}(0.001t^3 - 0.01t^2 + 13.4)\,dx =$$

$$= \frac{1}{11}\left(0.001 \cdot \frac{t^4}{4} - 0.01 \cdot \frac{t^3}{3} + 13.4t\right)\Bigg|_1^{12} =$$

$$= \frac{1}{11}(5.184 - 5.76 + 160.8 - 0.000025 + 0.0333 - 13.4) \approx$$

$$\approx 13.35\ (℃)$$

Answer. The average temperature in the cargo holds is 13.35 ℃, it is within acceptable limits.

7.4.5 Exercises

Compute the integrals

1. $\displaystyle\int_0^2 (x^4 - x^3)\,dx$

2. $\displaystyle\int_{-\frac{\pi}{4}}^0 \sin x\,dx$

3. $\displaystyle\int_{-1}^1 \frac{dt}{\sqrt{4-t^2}}$

4. $\displaystyle\int_0^{\frac{1}{3}} \frac{2^{3x}}{7}\,dx$

5. $\int_{-2}^{7} \frac{2dx}{x+3}$

7.4.6 Solutions

1. $\int_{0}^{2} (x^4 - x^3)\, dx$

Solution

$$\int_{0}^{2} (x^4 - x^3)\, dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)\Bigg|_0^2$$

$$= \frac{32}{5} - \frac{16}{4} = 2.4$$

2. $\int_{-\frac{\pi}{4}}^{0} \sin x\, dx$

Solution

$$\int_{-\frac{\pi}{4}}^{0} \sin x dx = -\cos x\Bigg|_{-\frac{\pi}{4}}^{0} =$$

$$= -\cos 0 + \cos\left(-\frac{\pi}{4}\right) =$$

$$= -1 + \frac{\sqrt{2}}{2}$$

3. $\int_{-1}^{1} \frac{dt}{\sqrt{4 - t^2}}$

Solution

$$\int_{-1}^{1} \frac{dt}{\sqrt{4 - t^2}} = \arcsin \frac{t}{2}\Bigg|_{-1}^{1} =$$

$$= \arcsin\frac{1}{2} - \arcsin\left(-\frac{1}{2}\right) =$$

$$= \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

We calculate the value of the integral according to the odd property of the function $\arcsin x$.

4. $\displaystyle\int_0^{\frac{1}{3}} \frac{2^{3x}}{7} dx$

Solution

$$\int_0^{\frac{1}{3}} \frac{2^{3x}}{7} dx = \frac{1}{7} \cdot \frac{1}{3} \int_0^{\frac{1}{3}} 2^{3x} d(3x) = \frac{1}{27} \cdot \frac{2^{3x}}{\ln 2} \Bigg|_0^{\frac{1}{3}} =$$

$$= \frac{2}{27\ln 2} - \frac{1}{27\ln 2} = \frac{1}{27\ln 2}$$

Here is another calculation method using the properties of powers and logarithms

$$\int_0^{\frac{1}{3}} \frac{2^{3x}}{7} dx = \frac{1}{7} \int_0^{\frac{1}{3}} 8^x dx = \frac{1}{7} \cdot \frac{8^x}{\ln 8} \Bigg|_0^{\frac{1}{3}} =$$

$$= \frac{1}{7} \cdot \frac{\sqrt[3]{8}}{\ln 8} - \frac{1}{7} \cdot \frac{1}{\ln 8} =$$

$$= \frac{2}{7 \cdot 3\ln 2} - \frac{1}{7 \cdot 3\ln 2} = \frac{1}{27\ln 2}$$

5. $\displaystyle\int_{-2}^{7} \frac{2dx}{x+3}$

Solution

$$\int_{-2}^{7} \frac{2dx}{x+3} = 2 \int_{-2}^{7} \frac{d(x+3)}{x+3} = 2\ln|x+3| \Bigg|_{-2}^{7} =$$

$$= 2\ln 10 - 2\ln 1 = 2\ln 10$$

7.5 Some Methods for Calculation of the Definite Integral

DETAILED DESCRIPTION:

The basic rules for calculation of the definite integral were discussed in the previous section. In this section we present methods of integration of composite functions and the method of integration by parts for the definite integral.

AIM: **to introduce certain methods of calculation of the definite integral if the integrand is non-trivial.**

Learning Outcomes:

Students can evaluate different definite integrals using various integration methods

Prior Knowledge: basic rules of integration and differentiation; methods of integration of indefinite integrals; the Newton-Leibniz formula.

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculated the area of various shapes; the volumes of solids, and to solve other geometric problems. Definite integrals are used for different calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in the theory of cargo transport, in economics, in classical signal theory, and in other specialities.

Content

1. Integration by parts
2. Substitution method for the definite integral
3. Exercises
4. Solutions

7.5.1 Integration by parts

Suppose that we have an integral that can be integrated by parts. We can find the antiderivative part by part according the formula for indefinite integrals. Let us recall it

$$\int u dv = uv - \int v du$$

Following the method for integration of the definite integral, we need to find the antiderivative and to apply the Newton-Leibniz formula as follows

$$\int_a^b u dv = \left(uv - \int v du\right)\Bigg|_a^b$$

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or

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Example 1.1 Find the integral

$$\int_0^2 x e^x dx$$

Solution

$$\int_0^2 x e^x dx = \left| \begin{matrix} \text{let } u = x, & dv = e^x dx \\ du = dx, & v = e^x \end{matrix} \right| =$$

$$= x e^x \Big|_0^2 - \int_0^2 e^x dx =$$

$$= 2 \cdot e^2 - 0 - e^x \Big|_0^2 =$$

$$= 2e^2 - e^2 + e^0 = e^2 + 1$$

Example 1.2 Find the integral

$$\int_4^{e^2} \frac{\ln x}{\sqrt{x}} dx$$

Solution

$$\int_4^{e^2} \frac{\ln x}{\sqrt{x}} dx = \left| \begin{matrix} \text{let } u = \ln x, & dv = \frac{dx}{\sqrt{x}} \\ du = \frac{1}{x} dx, & v = 2\sqrt{x} \end{matrix} \right| =$$

$$= 2\sqrt{x} \ln x \Big|_4^{e^2} - 2 \int_4^{e^2} \frac{\sqrt{x}}{x} dx =$$

$$= 2\sqrt{e^2} \ln(e^2) - 2 \cdot 2\ln 4 \Big|_4^{e^2} - 2 \cdot 2\sqrt{x} \Big|_4^{e^2} =$$

$$= 4e - 4\ln 4 - 4\sqrt{e^2} + 8 = 8 - 4\ln 4$$

7.5.2 Substitution method for the definite integral

Let us recall the substitution method for indefinite integrals. If the integrand contains a composite function multiplied by the derivative of its argument, we can simplify the notation of the integral by introducing a new argument

$$\int f(g(x))g'(x)dx = \int f(u)du$$

The indefinite integral represents a set of antiderivatives $F(x) + C$. The value of a definite integral is numerical. Since the definite integral has limits of integration defined by the interval $[a, b]$ with respect to the argument x, the introduced argument u belongs to a new interval $[\alpha, \beta]$. The case is expressed precisely by the following theorem:

Theorem. Suppose that the function $g(x)$ is a differentiable function on the interval $[a, b]$, and satisfies $g(a) = \alpha$ and $g(b) = \beta$. Suppose that the function $f(x)$ is continuous on the range of $g(x)$. Then by performing the substitution $u = g(x)$ it follows

$$\boldsymbol{\int_a^b f(g(x))g'(x)dx = \int_\alpha^\beta f(u)du}$$

Example 2.1 Find the integral

$$\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\, dx$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\, dx = \left| \begin{array}{cc} \text{let } u = \sin x, & du = \cos x\, dx \\ u_1 = \sin 0 = 0, & u_2 = \sin\frac{\pi}{2} = 1 \end{array} \right| =$$

$$= \int_0^1 e^u du = e^u \Big|_0^1 = e - 1$$

Example 2.2 Find the integral

$$\int_{\frac{1}{2}}^{1} (2 - 8x^5)^2 x^4 dx$$

Solution

$$\int_{\frac{1}{2}}^{1}(2-8x^5)^2x^4dx = \begin{vmatrix}\text{let } u = 2-8x^5, & du = -40x^4dx \\ u_1 = 1.75, & u_2 = -6\end{vmatrix} =$$

$$= -\frac{1}{40}\int_{1.75}^{-6}u^2du = \frac{1}{40}\int_{-6}^{1.75}u^2du =$$

$$= \frac{1}{40}\cdot\frac{u^3}{3}\Bigg|_{-6}^{1.75} = \frac{1}{120}\left(\frac{343}{64}-\left(\frac{-1}{216}\right)\right) \approx 0.045$$

Solving this integral we reversed the limits of integration.

Example 2.3 Find the integral

$$\int_1^2 \frac{2t^3}{t^2+1}dt$$

Solution

$$\int_1^2 \frac{2t^3}{t^2+1}dt = \int_1^2 \frac{2t\cdot t^2}{t^2+1}dt =$$

$$= \begin{vmatrix}\text{let } u = t^2+1, \ du = 2tdt \\ u_1 = 1+1 = 2, \ u_2 = 2^2+1 = 5 \\ t^2 = u-1\end{vmatrix} =$$

$$= \int_2^5 \frac{u-1}{u}du = \int_2^5 du - \int_2^5 \frac{du}{u} = (u-\ln u)\Bigg|_2^5 =$$

$$= 5-\ln 5-2+\ln 2 = 3-\ln 2.5$$

7.5.3 Exercises

Find the integrals

1. $\int_0^2 (x^2+1)x\,dx$

2. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\varphi \, d\varphi$

3. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5^{\arcsin x} dx}{\sqrt{1-x^2}}$

4. $\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75+2t^4} dx$

5. $\int_{-1}^{3} \frac{4dx}{\sqrt{x}(x+1)}$

7.5.4 Solutions

1. $\int_{0}^{2} (x^2+1)x \, dx$

Solution

$$\int_{0}^{2} (x^2+1)x \, dx = \left| \begin{matrix} \text{let } u = x^2+1, \;\; du = 2xdx \\ u_1 = 1, \;\; u_2 = 5 \end{matrix} \right| =$$

$$= \frac{1}{2} \int_{1}^{5} u du = \left. \frac{u^2}{4} \right|_1^5 = \frac{25}{4} - \frac{1}{4} = 6$$

2. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\varphi \, d\varphi$

Solution

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\varphi \, d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\varphi}{\sin\varphi} d\varphi =$$

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$$= \left| \begin{matrix} \text{let } u = \sin\varphi, \ \ du = \cos\varphi\, d\varphi \\ u_1 = \frac{\sqrt{2}}{2}, \ \ u_2 = \frac{\sqrt{3}}{2} \end{matrix} \right| =$$

$$= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{du}{u} = \ln u \Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} =$$

$$= \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2} = 0.5\ln 1.5$$

3. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5^{\arcsin x} dx}{\sqrt{1-x^2}}$

Solution

$$\int_{-\frac{1}{2}}^{0} \frac{5^{\arcsin x} dx}{\sqrt{1-x^2}} = \left| \begin{matrix} \text{let } u = \arcsin x, \ \ du = \frac{dx}{\sqrt{1-x^2}} \\ u_1 = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \ \ u_2 = 0 \end{matrix} \right| =$$

$$= \int_{-\frac{\pi}{6}}^{0} 5^u du = \frac{5^u}{\ln 5}\Big|_{-\frac{\pi}{6}}^{0} = \frac{1}{\ln 5} - \frac{5^{-\frac{\pi}{6}}}{\ln 5}$$

4. $\displaystyle\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75 + 2t^4} dx$

Solution

$$\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75 - 16\,t^4} dt = \left| \begin{matrix} \text{let } u = 0.75 - 16t^4, \ \ du = -64t^3 dt \\ u_1 = 0.75, \ \ u_2 = -0.25 \end{matrix} \right| =$$

$$= -\frac{1}{64} \int_{0.75}^{-0.25} \frac{du}{u} = \frac{1}{64} \int_{-0.25}^{0.75} \frac{du}{u} =$$

$$= \frac{1}{64} \ln u \Big|_{-0.25}^{0.75} = \frac{1}{64}(\ln 0.75 - \ln|-0.25|) =$$

$$= \frac{1}{64}(\ln 0.75 - \ln 0.25) = \frac{1}{64}\ln\frac{0.75}{0.25} = \frac{\ln 3}{64}$$

5. $\displaystyle\int_{-1}^{3} \frac{4dx}{\sqrt{x}(x+1)}$

Solution

$$\int_{1}^{3} \frac{4dx}{\sqrt{x}(x+1)} = \left| \begin{array}{c} \text{let} \quad u = \sqrt{x}, \quad du = \dfrac{dx}{2\sqrt{x}} \\ u_1 = 1, \; u_2 = \sqrt{3} \\ x = u^2 \end{array} \right| =$$

$$= 4 \cdot 2 \int_{1}^{\sqrt{3}} \frac{du}{1+u^2} = 8\arctan\sqrt{x} \Big|_{1}^{\sqrt{3}} =$$

$$= 8\left(\arctan\sqrt{3} - \arctan 1\right) = 8\left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right) = \tfrac{2\pi}{3}$$

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7.6 Improper Integrals

DETAILED DESCRIPTION:

In this section, we will extend the concept of the definite integral. Special integrals are investigated over infinite intervals. We will introduce two types of improper integrals: integrals with infinite limits and integrals with an infinite discontinuity in the region of integration. Such integrals are defined using the notion of the limit. The ways of calculation of improper integrals are presented. Examples of convergent and divergent improper integrals are discussed.

The software GeoGebra, DESMOS, Excel are recommended for construction of graphs. To check their solutions and for deeper understanding, students can use *Definite and Improper Integral Calculator* (https://www.emathhelp.net/calculators/calculus-2/definite-integral-calculator/)

AIM: to learn about improper integrals and methods of their evaluation, to understand the concepts of convergence and divergence of integrals.

Learning Outcomes:

1. Acquire the methods of evaluation of improper integrals of type I.
2. Distinguish the improper integrals of type II and acquire the methods of their calculation.

Prior Knowledge: definite integrals; limits; detection of the domain of function; elementary functions and their graphs.

Relationship to real maritime problems: By describing the shape of the hull of a ship mathematically, it is possible to research the ship's wave resistance that can be presented by an improper integral. Improper integrals are used to express the electrical potential of a given field. A probability density function for a continuous random variable can be described by an improper integral.

Content

1. Improper integrals with infinite upper limit
2. Improper integrals with an infinite discontinuity in the region of integration
3. Exercises
4. Solutions

7.6.1 Improper integrals with infinite upper limit

In the previous sections we got acquainted with the definite integral where integrand is defined on the closed interval. Let us investigate a continuous function over a left-bounded interval $[a, \infty)$ (see figure 1.1).

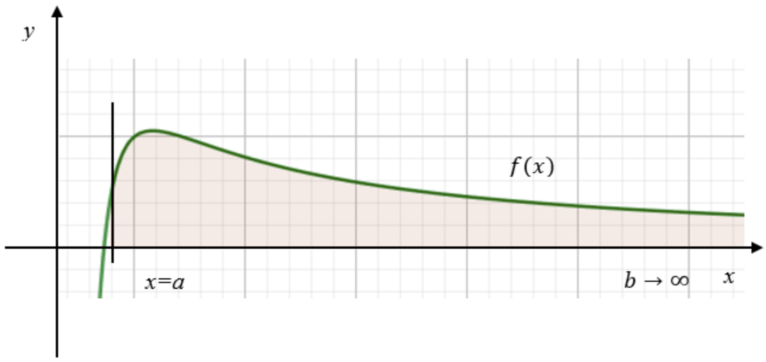

Figure 1.1

Suppose that function $f(x)$ is integrable on the whole interval, so we can calculate the value of any integral with upper limit B∈ $[a, \infty)$

$$\int_a^B f(x)dx$$

By choosing different values of the number B

$$a \le B_1 < B_2 < B_3 < \cdots,$$

we get a sequence of numbers

$$\int_a^{B_1} f(x)dx; \int_a^{B_2} f(x)dx; \int_a^{B_3} f(x)dx; \ \ldots$$

It can be a convergent or a divergent sequence.

Definition. ***The improper integral of type I*** (or an integral with an infinite upper limit) is a definite integral with infinite limits of integration evaluated by the limit

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

If the limit is a finite number, we say that the improper integral ***converges***. If the limit does not exist or is infinity (positive or negative), the improper integral ***diverges***.

The evaluation of an improper integral follows the known rules. We find the corresponding antiderivative, apply the Newton-Leibniz formula, and calculate the limit as b tends to infinity

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx = \lim_{b\to\infty} F(x)\Big|_a^b = \lim_{b\to\infty} F(b) - F(a)$$

We will write symbolically

$$\int_{a}^{\infty} \boldsymbol{f(x)dx = F(\infty) - F(a)},$$

Remembering that $F(\infty)$ means the calculation of a limit.

Similarly, the improper integral can have an infinite lower limit, or both

$$\int_{-\infty}^{b} f(x)dx\,; \quad \int_{-\infty}^{\infty} f(x)dx$$

Example 1.1 Evaluate the integral

$$\int_{0.5}^{\infty} \frac{dx}{x^2}$$

Solution

Let us construct the graph

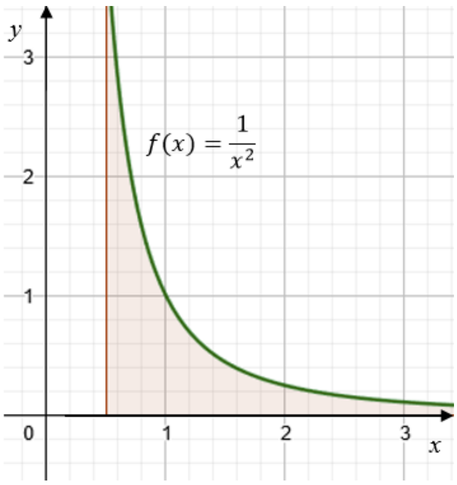

Figure 1.2

Figure 1.2 shows that the graph of the given integrand approaches the x-axis asymptotically. Let us evaluate the integral

$$\int_{0.5}^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_{0.5}^{b} x^{-2}\,dx =$$

$$= \lim_{b\to\infty} \frac{x^{-1}}{-1}\bigg|_{0.5}^{b} = -\lim_{b\to\infty} \left(\frac{1}{x}\right)\bigg|_{0.5}^{b} =$$

$$= -\left(\lim_{b\to\infty} \frac{1}{b} - 2\right) = 2$$

Answer The given integral converges to 2.

Example 1.2 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Solution

The graph of the integrand is

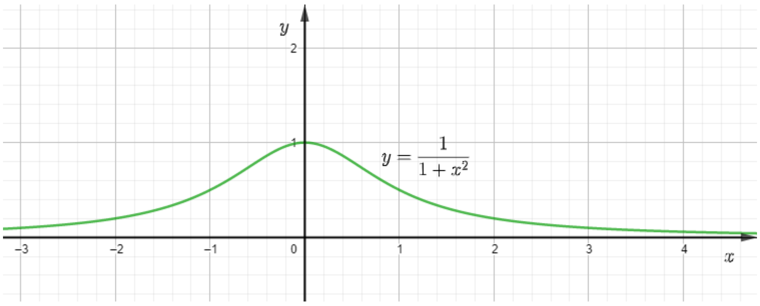

Figure 1.3

Figure 1.3 demonstrates the graph of an even function; its graph is symmetric with respect to the y-axis. To evaluate this integral we will split it into two integrals with half-bounded integration limits taking the intermediate value $x = 0$. We will also follow the principle of symmetry.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} =$$

$$= 2 \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = 2 \lim_{b\to\infty} \operatorname{arctan} x \Big|_{0}^{b} =$$

$$= 2 \lim_{b\to\infty} \operatorname{arctan} b - \operatorname{arctan} 0 = 2\frac{\pi}{2} = \pi$$

The graph of function $f(x) = \operatorname{arctan} x$ has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ (see figure 1.4)

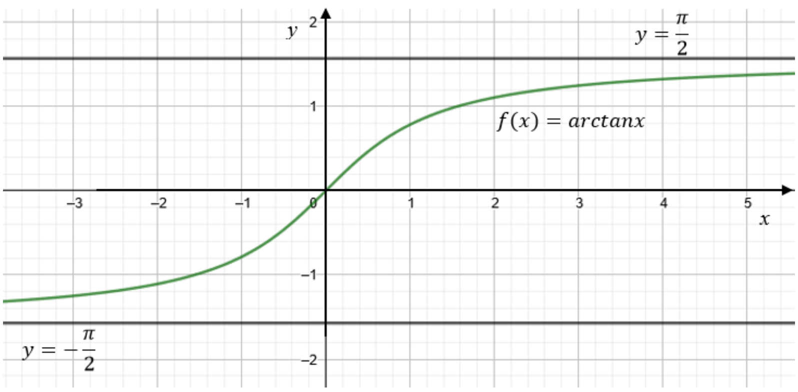

Figure 1.4

Answer The given integral converges to the number π.

Example 1.3 Evaluate the integral

$$\int_{0.2}^{\infty} \frac{dx}{x}$$

Solution

$$\int_{0.2}^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \ln x \Big|_{0.2}^{b} =$$

$$= \lim_{b\to\infty} \ln b - \ln 0.2 = \infty$$

The graphs of integrand $f(x)$ and of antiderivative $F(x)$ are presented in figure 1.5. Function $F(x) = \ln x$ increases indefinitely.

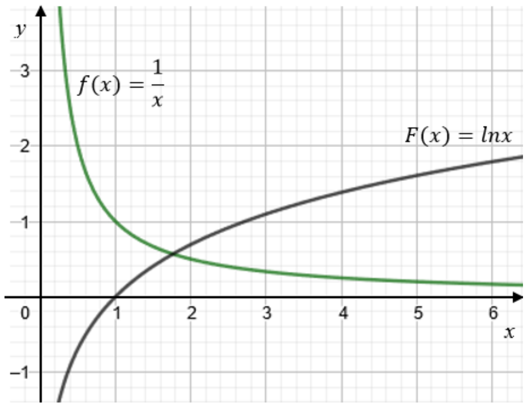

Figure 1.5

Answer The given improper integral diverges.

Example 1.4 We would like to escape from the ground of the Earth. What must be the minimum velocity to overcome the gravitational force?

Solution An object with a mass m must be moved from the ground of the Earth. The object must have enough energy to move arbitrarily far away from Earth into universe. It is necessary to calculate the amount of work needed to move it.

Here are some constants useful for calculation:

$M \approx 5.97 \cdot 10^{24}\ kg$ – the mass of the earth

$G \approx 6.67 \cdot 10^{-11} Nm^2/kg^2$ – the constant of gravity

$R \approx 6.37 \cdot 10^{6}\ m$ – the distance between the centre of the Earth and the centre of object

We will calculate the energy needed to overcome the force of gravity

$$\int_R^{\infty} \frac{mMG}{x^2} dx = -\frac{mMG}{x}\bigg|_R^{\infty} = mMG\left(-\frac{1}{\infty} + \frac{1}{R}\right) = \frac{mMG}{R}$$

The energy needed to move an object is kinetic energy that can be calculated

$$E_{kin} = \frac{mv^2}{2}$$

So we have the equation

$$\frac{mv^2}{2} = \frac{mMG}{R}$$

Let us calculate the velocity by inserting well known constants in the equation

$$v = \sqrt{\frac{2MG}{R}} \approx \sqrt{\frac{2 \cdot 5.97 \cdot 10^{24} \cdot 6.67 \cdot 10^{-11}}{6.37 \cdot 10^6}} \approx 11.2\ m/s$$

The result is minimum velocity to overcome the Earth's pull. It is called ***escape velocity***.

7.6.2 Improper integrals with an infinite discontinuity in the region of integration

Let us investigate the case when the function $f(x)$ becomes unbounded as its argument x approaches one or both endpoints of the interval $[a, b]$. Figure 2.1 presents the function whose value tends to infinity when the argument x approaches endpoint b of the interval. To integrate the function over such an interval we will evaluate the one-sided limit.

Definition. Let function $f(x)$ be continuous on the interval $[a, b)$ and be discontinuous at endpoint b of the interval, ***the improper integral of type II*** is defined in the following way

$$\boldsymbol{\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx}$$

If the limit exists, we say that the improper integral ***converges***. If the limit does not exist or it is infinity, the improper integral ***diverges***.

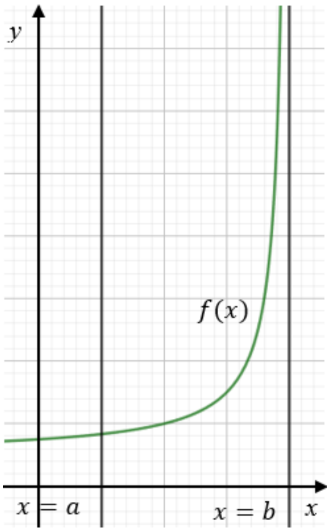

Figure 2.1

Similarly, if the function is discontinuous at the left endpoint of the interval (see figure 2.2), we have

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)dx$$

It is necessary to separate the integral into two parts if the function is discontinuous at the inner point of the interval (see figure 2.3). The improper integral converges only in the case if both its parts converge.

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon\to 0+} \int_{c+\varepsilon}^b f(x)dx$$

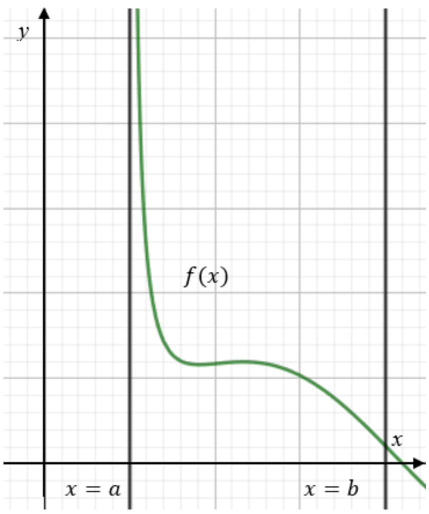

Figure 2.2

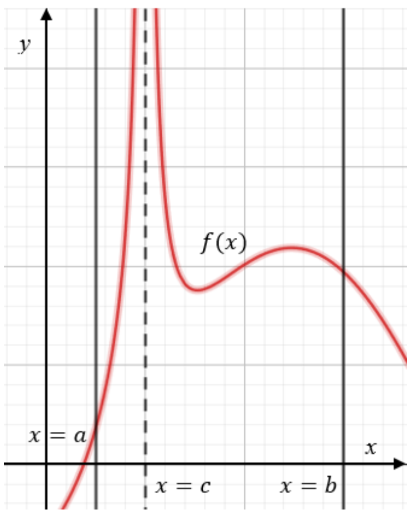

Figure 2.3

Example 2.1 Evaluate the integral

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x}}$$

Solution

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x}} = \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} (1-x)^{-1/3} dx =$$

$$= \begin{vmatrix} \text{let } u = 1-x, & du = -dx \\ u_1 = 1, & u_2 = 0 \end{vmatrix} =$$

$$= \lim_{\varepsilon\to 0+} \left(-\int_1^{0+\varepsilon} u^{-1/3} du \right) =$$

$$= -\lim_{\varepsilon\to 0+} \frac{3}{2} u^{\frac{2}{3}} \Big|_1^{0+\varepsilon} = -\frac{3}{2}(0-1) = \frac{3}{2}$$

Answer This integral converges to $\frac{3}{2}$.

Example 2.2 Evaluate the integral

$$\int_0^3 \frac{2dx}{(x-1)^2}$$

Solution The integrand is not defined at the inner point $x = 1$ of the integration interval. It is necessary to split the interval into two parts.

$$\int_0^3 \frac{2dx}{(x-1)^2} = \int_0^1 \frac{2dx}{(x-1)^2} + \int_1^3 \frac{2dx}{(x-1)^2} =$$

$$= \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{2dx}{(x-1)^2} + \lim_{\varepsilon\to 0+} \int_{1+\varepsilon}^{3} \frac{2dx}{(x-1)^2}$$

Let us solve these limits separately

$$\lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{2dx}{(x-1)^2} = 2 \lim_{\varepsilon\to 0+} \frac{-1}{x-1} \Big|_0^{1-\varepsilon} =$$

$$= -2 \lim_{\varepsilon\to 0+} \frac{1}{-\varepsilon} + 1 = \infty$$

Similarly, the second improper integral tends to infinity. It is not necessary to calculate it, as if at least one of the addends tends to infinity, the given integral diverges.

7.6.3 Exercises

Evaluate the integral and draw the graph of its integrand and of its antiderivative.

1. $\displaystyle\int_{2}^{\infty}\frac{dx}{x\ln x}$
2. $\displaystyle\int_{-\infty}^{0} xe^{-x^2}dx$
3. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$
4. $\displaystyle\int_{-1}^{2}\frac{x}{\sqrt{x+1}}dx$
5. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\tan x\,dx$

7.6.4 Solutions

1. $\displaystyle\int_{2}^{\infty}\frac{dx}{x\ln x}$

Solution The graph of the integrand in figure 4.1 demonstrates a decreasing function

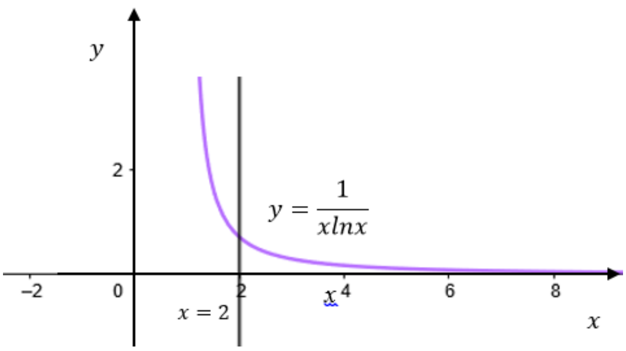

Figure 4.1

The upper limit of the given integral is infinity. Therefore, the given integral is an improper integral of type 1. After the primitive function or antiderivative $\ln u$ is found, we calculate the limit. While the logarithmic function is an increasing function, its limit is infinite as x tends to infinity (see figure 4.2).

$$\int_2^{\infty} \frac{dx}{x\ln x} = \left| \begin{matrix} \text{let } u = \ln x \quad \text{then } du = \dfrac{dx}{x} \\ u_1 = \ln 2, \ \ u_2 = \infty \end{matrix} \right| =$$

$$= \int_{ln2}^{\infty} \frac{du}{u} = \ln|u| \Big|_{ln2}^{\infty} =$$

$$= \lim_{u\to\infty} \ln u - \ln \ln 2 = \infty$$

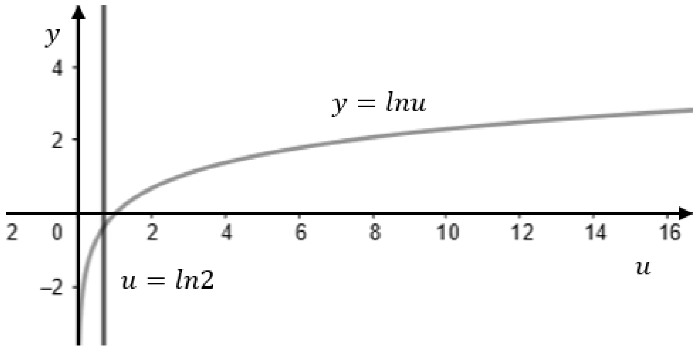

Figure 4.2

The given improper integral diverges.

2. $\displaystyle\int_{-\infty}^{0} xe^{-x^2}dx$

Solution

Figure 4.3 shows the graph of the integrand (red curve) and of the antiderivative (green curve). Both graphs are symmetric. The integrand function is odd, because its graph is symmetric with respect to the origin of coordinate system. Antiderivative is even because its graph is symmetric with respect to the y-axis.

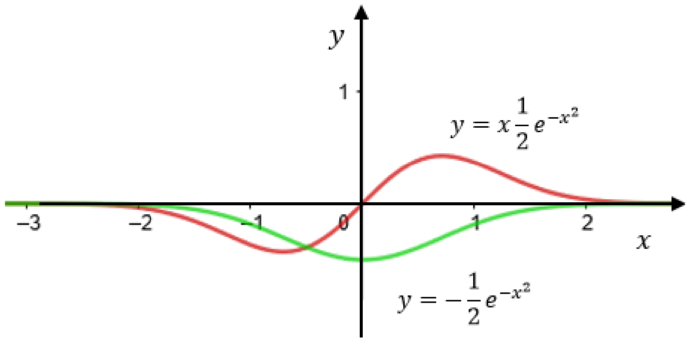

Figure 4.3

Since there is infinity in the lower bound, this is an improper integral of type 1. We evaluate the integral by changing the differential, and find the limit.

$$\int_{-\infty}^{0} xe^{-x^2}dx = -\frac{1}{2}\int_{-\infty}^{0} e^{-x^2}d(x^2) = -\frac{1}{2}e^{-x^2}\Bigg|_{-\infty}^{0} =$$

$$= -\frac{1}{2}\left(e^0 - \lim_{x\to-\infty} e^{-x^2}\right) =$$

$$= -\frac{1}{2}(1-0) = -\frac{1}{2}$$

The given integral converges.

3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Solution

The integral with both infinite limits is an improper integral of type I. We split it in two parts

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2}$$

Both integrals have the same integrand. Therefore, we compute the corresponding indefinite integral by completing the full square

$$\int \frac{dx}{x^2+2x+2} = \left| \begin{array}{r} x^2+2x+2 = x^2+2x+1+1 = \\ = (x+1)^2+1 \end{array} \right| =$$

$$= \int \frac{d(x+1)}{(x+1)^2+1} =$$

$$= \arctan(x+1) + C$$

We use this antiderivative for evaluation of improper integrals, and we apply the odd property of $\arctan x$ function

$$\int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2} =$$

$$= \arctan(x+1)\Big|_{-\infty}^{0} + \arctan(x+1)\Big|_{0}^{\infty} =$$

$$= \arctan 0 - \arctan(-\infty) + \arctan\infty - \arctan 0 =$$

$$= 2\lim_{x\to\infty} \arctan x = 2\frac{\pi}{2} = \pi$$

The integral converges. The graphs of both functions are presented in figure 4.4. The graph of integrand is in red colour, the graph of antiderivative is green.

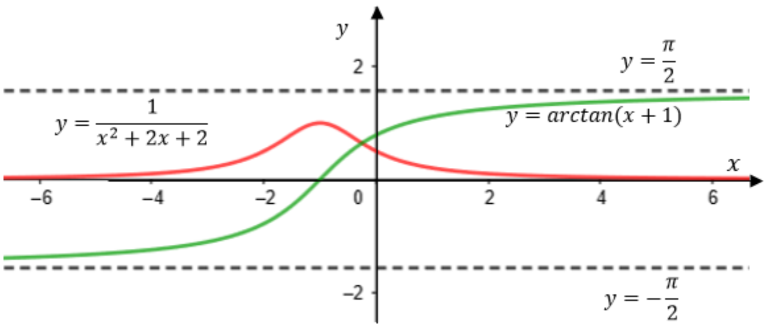

Figure 4.4

4. $\displaystyle\int_{-1}^{2} \frac{x}{\sqrt{x+1}}dx$

Solution

We detect the domain of the integrand

$$x > -1$$

The function is not defined at the lower point of the integration region; therefore, the given integral is an improper integral of type II. The graph of this function is sketched in figure 4.5

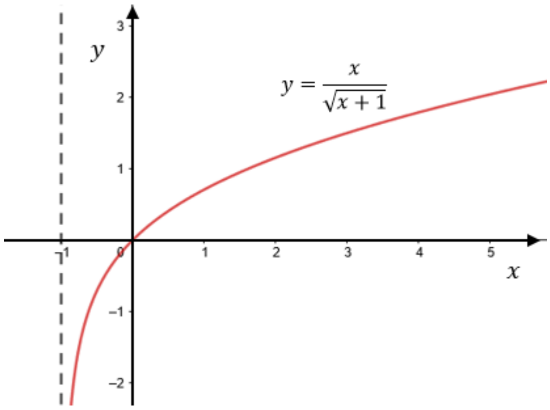

Figure 4.5

We compute the corresponding indefinite integral by applying substitution

$$\int \frac{x}{\sqrt{x+1}}dx = \begin{vmatrix} \text{let } x+1 = u^2 \\ dx = 2udu \end{vmatrix} =$$

$$= \int \frac{u^2-1}{u} 2udu =$$

$$= 2\int (u^2-1)du = 2\left(\frac{u^3}{3} - u\right) + C =$$

$$= 2\left(\frac{\sqrt{x+1}^3}{3} - \sqrt{x+1}\right) + C$$

Return to the improper integral

$$\int_{-1}^{2} \frac{x}{\sqrt{x+1}}dx = \lim_{\varepsilon\to 0+} \int_{-1+\varepsilon}^{2} \frac{x}{\sqrt{x+1}}dx =$$

$$= \lim_{\varepsilon\to 0+} 2\left(\frac{\sqrt{x+1}^3}{3} - \sqrt{x+1}\right)\Bigg|_{-1+\varepsilon}^{2} =$$

$$= 2\frac{\sqrt{3}^3}{3} - 2\sqrt{3} - 2\lim_{\varepsilon\to 0+}\left(\frac{\sqrt{-1+\varepsilon+1}^3}{3} - \sqrt{-1+\varepsilon+1}\right) =$$

$$= 2\frac{\sqrt{3}^3}{3} - 2\sqrt{3} = 0$$

The given improper integral converges as the limit is finite.

The graph of the antiderivative is presented in figure 4.6. The value of this function at the point $x = 2$ is 0, the graph crosses the x-axis at this point. The extreme value of the antiderivative in the given interval is at point $x = 0$.

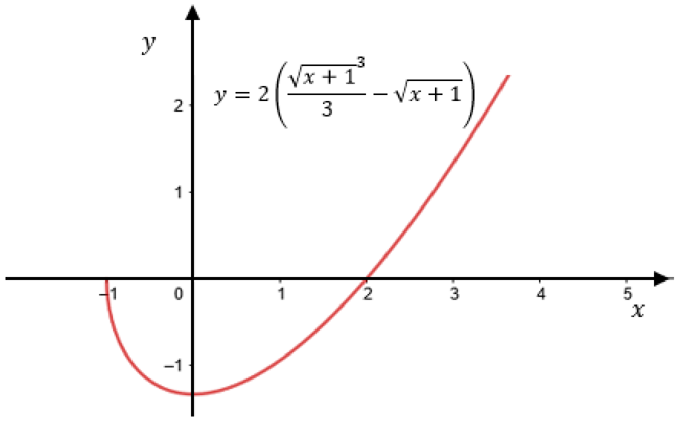

Figure 4.6

5. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \tan x \, dx$

Solution

The tangent function is not defined at the point $x = \frac{\pi}{2}$. We solve the improper integral of type II. The graphs of the integrand and the corresponding antiderivative are given in figure 4.7.

We solve the integral by the change of differential

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \tan x dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{\cos x} dx =$$

$$= -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d(\cos x)}{\cos x} =$$

$$= -\ln|\cos x| \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= -\lim_{\varepsilon \to 0+} \ln\left(\cos\left(\frac{\pi}{2} - \varepsilon\right)\right) + \ln\left(\cos\frac{\pi}{3}\right) = \infty$$

Since the value of the integral is not finite, it is divergent.

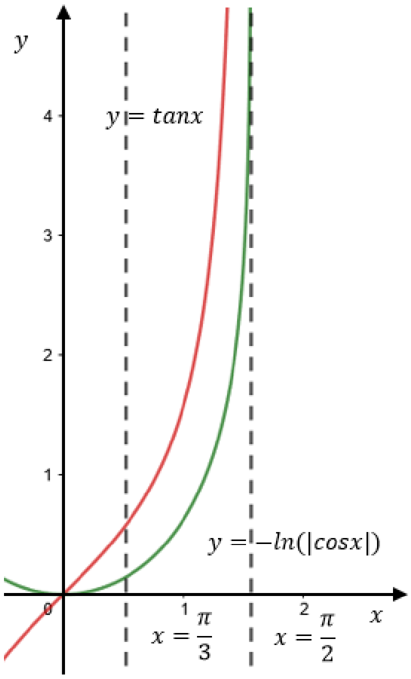

Figure 4.7

7.7 Application of Definite Integrals: Areas of Plane Regions

DETAILED DESCRIPTION:

Definite integrals are used to solve various problems. One of the usual applications is the calculation of the area of a plane region bounded by curves. This chapter presents different types of regions and gives the methods to calculate their areas. Formulas of definite integrals are given for curves expressed analytically, expressed by parametrical equations, as well as for curves given in the polar coordinate system. To construct the curves, the software programs GeoGebra Classic or Desmos Graphing Calculator, or others, can be used. Students can check their solutions with the integral calculator (https://www.integral-calculator.com/) that also constructs graphs of the integrand and the antiderivative.

AIM: to explore the methods of calculation of the area of plane regions of different types.

Learning Outcomes:

1. Students understand the geometrical meaning of the definite integral.
2. Students can calculate the area of plane regions enclosed by curves.
3. Students distinguish the cases if a region must be divided into two or more parts.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Relationship to real maritime problems: Calculation of the area of various specific construction parts is one of the core questions in shipbuilding. However, the shapes are so complex that mostly numerical calculations are used. Calculation of the area of a region is part of solving physics problems: for instance, to detect the pressure that is applied to an object it is necessary to calculate the area of the object’s surface.

Content

1. Area under the graph of a function
2. Area between two curves
3. The problem of the compound region
4. Area under a parametric curve
5. Curve in a polar coordinate system
6. Exercises
7. Solutions

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7.7.1 Area under the graph of a function

The definite integral was introduced as a tool for calculation of the area of a given region. If the region is bounded by the graph of a continuous function $f(x)$ on the interval $[a, b]$, two vertical lines $x = a$ and $x = b$, and x-axis, we can calculate the area S under the graph of the given function in square units

$$\boldsymbol{S = \int_a^b f(x)dx}$$

Example 1.1

Calculate the area of the region bounded by the function $y = \cos x$, vertical straight lines $x = -\frac{\pi}{3}$, $x = \frac{\pi}{3}$, and x-axis.

Solution Let us construct the graph (see figure 1.1) and let us express the integral

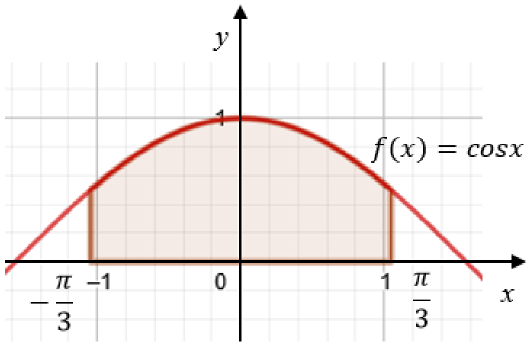

Figure 1.1

$$S = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x\, dx = 2\int_0^{\frac{\pi}{3}} \cos x\, dx = 2\sin x \Big|_0^{\frac{\pi}{3}} = 2\sin\frac{\pi}{3} - 0 = \sqrt{3}\ \text{sq. units}$$

We notice that this integral has symmetric integration boundaries and cosine function is an even function, so the interval was halved.

Example 1.2 Find the area of the plane region bounded by $y = (x-1)^3 + 1$, $x = 0.5$, $x = 2$, and

$y = 0$.

Solution The region is bounded by two vertical lines $= 0.5$, $x = 2$, x-axis, and the cubic parabola (see figure 1.2).

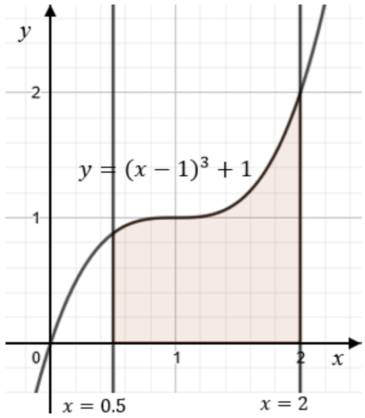

Figure 1.2

Thus, the area of the region is

$$S = \int_{0.5}^{2} ((x-1)^3 + 1)dx = \int_{0.5}^{2} (x-1)^3 d(x-1) + \int_{0.5}^{2} dx =$$

$$= \left(\frac{(x-1)^4}{4} + x\right)\Bigg|_{0.5}^{2} = \frac{1}{4} + 2 - \frac{1}{64} - 0.5 \approx 1.73 \ \text{sq. units}$$

If the function has a break point that separates the interval of integration into subintervals where the function has only positive and only negative values, we need to integrate the function separately on every such subinterval, taking the absolute value of the result.

Example 1.3 Calculate the area of the region enclosed by $f(x) = \log_2 x,\ x = 0.5,\ x = 2,\ y = 0$.

Solution The graph shows that the function has negative values in the interval $[0.5, 1]$ and positive values in the interval $[1, 2]$ (see figure 1.3). Therefore, we will separate these intervals.

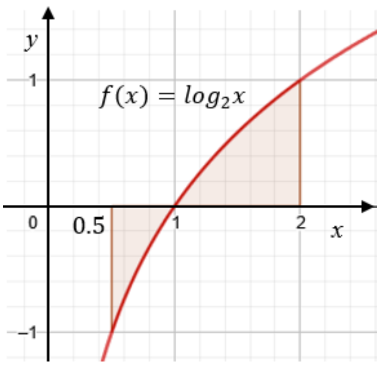

Figure 1.3

We split the integral into two parts to calculate the area of the region

$$S=\int_{0.5}^{2}|\log_2 x|dx=\left|\int_{0.5}^{1}\log_2 xdx\right|+\int_{1}^{2}\log_2 xdx$$

Let us evaluate the corresponding indefinite integral by applying the method of integration by parts

$$\int \log_2 xdx=\left|\begin{matrix}\text{let } u=\log_2 x, & dv=dx\\ du=\dfrac{dx}{x\ln 2}, & v=x\end{matrix}\right|=x\log_2 x-\frac{1}{\ln 2}\int\frac{xdx}{x}=$$

$$=x\log_2 x-\frac{x}{\ln 2}+C$$

Now we use this antiderivative for the calculation of area S according to the Newton-Leibniz formula

$$S=\left|x\log_2 x-\frac{x}{\ln 2}\right|\Big|_{0.5}^{1}+\left(x\log_2 x-\frac{x}{\ln 2}\right)\Big|_{1}^{2}=$$

$$=\left|\log_2 1-\frac{1}{\ln 2}-0.5\log_2 0.5+\frac{1}{2\ln 2}\right|+2\log_2 2-\frac{2}{\ln 2}-\log_2 1+\frac{1}{\ln 2}=$$

$$=\left|\frac{1}{2}-\frac{1}{2\ln 2}\right|+2-\frac{1}{\ln 2}\approx|-0.22|+0.56\approx 0.78\ \text{sq. units}$$

Example 1.4 At what value of the upper limit b is the integral equal to 4?

$$\int_{1}^{b}\frac{dx}{4\sqrt{x}}=4$$

Solution We calculate the integral

$$\int_{1}^{b}\frac{dx}{4\sqrt{x}}=\frac{1}{4}\cdot 2\sqrt{x}\Big|_{1}^{b}=\frac{1}{2}\left(\sqrt{b}-1\right)$$

We solve the equation

$$\frac{1}{2}\left(\sqrt{b}-1\right)=4$$

$$\sqrt{b}-1=8$$

$$\sqrt{b}=9;\quad b=81$$

Answer The upper limit of the integral should be $b=81$.

7.7.2 Area between two curves

If functions $f(x)$ and $g(x)$ are continuous functions over the interval $[a, b]$ and $f(x) \geq g(x)$ for all arguments $x \in [a, b]$ then area S of the region between the curves $f(x)$ and $g(x)$ in this interval is expressed by the integral

$$\boldsymbol{S = \int_a^b (f(x) - g(x))dx}$$

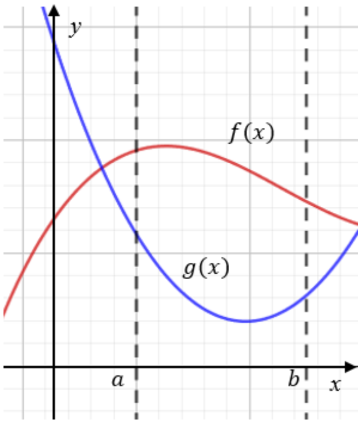

Figure 2.1

Example 2.2 Find the area of a plane region bounded by two curves $y = x^2$, $y = x + 2$

Solution We construct the graphs of given functions (see figure 2.2). To detect the integration interval, we need to calculate the coordinates of the projection of the region on the x-axis.

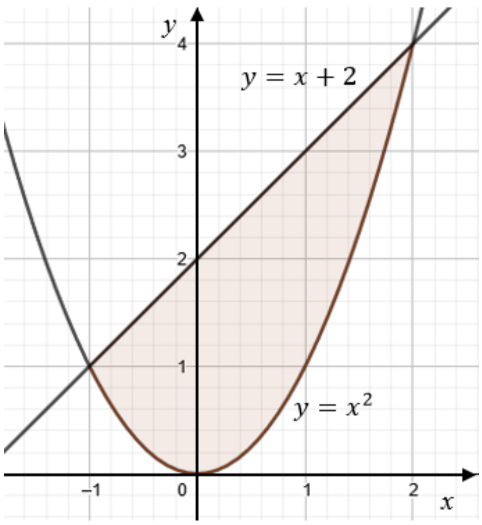

Figure 2.2

$$\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$$

$$x^2 - x - 2 = 0$$

The equation has two roots

$$x_1 = -1, \;\; x_2 = 2$$

Considering that the parabola is the lower curve, the area is

$$S = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} =$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5 \;\; \text{sq. units}$$

7.7.3 The problem of the compound region

We will investigate the case of the region bounded by more than two curves. Let it be bounded by curves $f(x),\;\; g(x),\;\; z(x)$ (see figure 3.1).

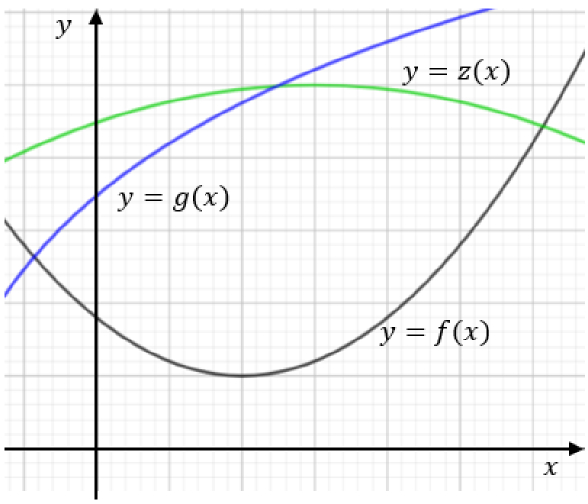

Figure 3.1

There are given two upper functions $g(x)$ and $z(x)$ and one lower function $f(x)$. We determine the intersection points of graphs that define two separate regions with different intervals of projection $[a, b]$ and $[b, c]$ (see figure 3.2).

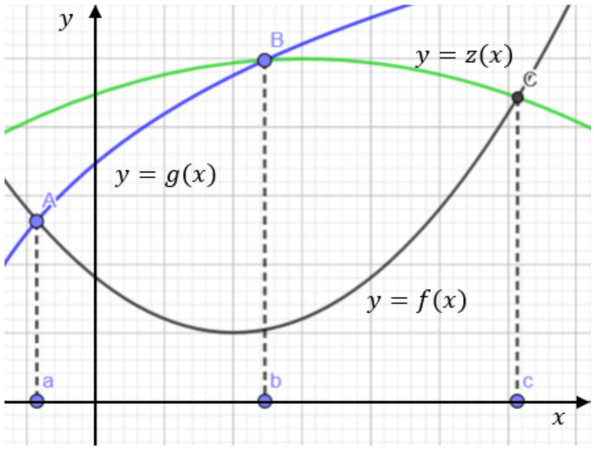

Figure 3.2

We compose two integrals to solve the problem of the area

$$\boldsymbol{S = \int_a^b (g(x) - f(x))\,dx + \int_b^c (z(x) - f(x))\,dx}$$

Example 3.1 Calculate the area of a region enclosed by the curve $y = x^2 - 2x + 1$ and two lines $x + y = 3,\ y = 0$.

Solution

The figure 3.3 shows several closed regions. We find the region that is enclosed by the curve and exactly two lines, where one of the lines is the x-axis (see the coloured region).

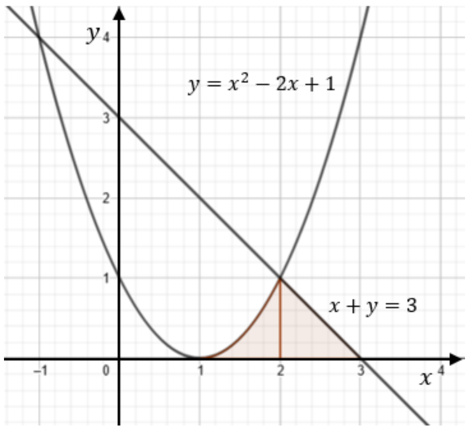

Figure 3.3

Now our solution has the following steps:

Step 1. Determine the boundaries of integration

$$x^2 - 2x + 1 = 0; \quad x = 1$$

$$x + y - 3 = 0; \quad x = 3$$

The boundaries are given by the interval $[1,3]$.

Step 2.

Calculate the point of intersection of the curve and the line $x+y=3$.

$$\begin{cases} y = x^2 - 2x + 1 \\ y = 3 - x \end{cases}$$

$$3 - x = x^2 - 2x + 1$$

$$x^2 - x - 2 = 0$$

The equation has two roots $x=-1;\quad x=2$. The point $x=2$ belongs to the interval $[1,3]$.

Step 3. To calculate the area of the region it is necessary to break up the interval of boundaries into two parts

$$[1,3] = [1,2] + [2,3]$$

and set up two integrals of two different upper functions

$$S = \int_1^2 (x^2 - 2x + 1)\,dx + \int_2^3 (3 - x)\,dx$$

Step 4. Calculate the area

$$S = \int_1^2 (x^2 - 2x + 1)\,dx + \int_2^3 (3 - x)\,dx =$$

$$= \left(\frac{x^3}{3} - x^2 + x\right)\Bigg|_1^2 + \left(3x - \frac{x^2}{2}\right)\Bigg|_2^3 =$$

$$= \frac{8}{3} - 4 + 2 - \frac{1}{3} + 1 - 1 + 9 - \frac{9}{2} - 6 + 2 = \frac{5}{6}\ \text{sq. units}$$

Example 3.2 Calculate the area of a region enclosed by $y=\sqrt{x},\ \ 3x-5y-12=0,\ \ y=0$.

We will calculate the area of the given region in two different ways.

Solution 1

Step 1. Construct the given region.

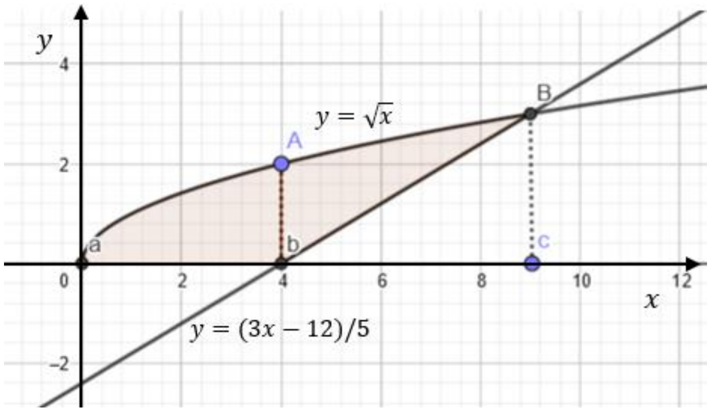

Figure 3.4

Step 2. Detect the boundaries of the integrals. Figure 3.4 presents the compound region whose area will be calculated as the sum of two integrals. The first integral is defined in the interval $[0,4]$ because the point $x=4$ is the x-intercept of the straight line. The boundaries of the second integral are $[4,9]$. We can find the upper bound x=9 by solving the system of equations

$$\begin{cases} y = \sqrt{x} \\ y = \dfrac{3x-12}{5} \end{cases}$$

$$5\sqrt{x} = 3x - 12$$

$$25x = 9x^2 - 72x + 144$$

$$9x^2 - 97x + 144 = 0$$

$$x = 9; \;\; x = \frac{16}{9}$$

Point B has coordinates $B(9,3)$

Step 3. Set up integrals

$$S = \int_0^4 \sqrt{x}\,dx + \int_4^9 \left(\sqrt{x} - \frac{3x-12}{5}\right) dx =$$

$$= \left.\frac{x^{3/2}}{3/2}\right|_0^4 + \left.\frac{x^{3/2}}{3/2}\right|_4^9 - \left.\left(\frac{3x^2}{10} - \frac{12x}{5}\right)\right|_4^9 =$$

$$= \frac{2}{3}\cdot 27 - \frac{3}{10}\cdot 65 + \frac{12}{5}\cdot 5 = 10.5 \;\text{ square units}$$

Solution 2

We solved the problem by the calculation of two integrals. If we turn the construction with x-axis up, we can express the given functions as functions with respect to the argument y (see figure 3.5)

$$x = y^2; \quad x = \frac{5y + 12}{3}$$

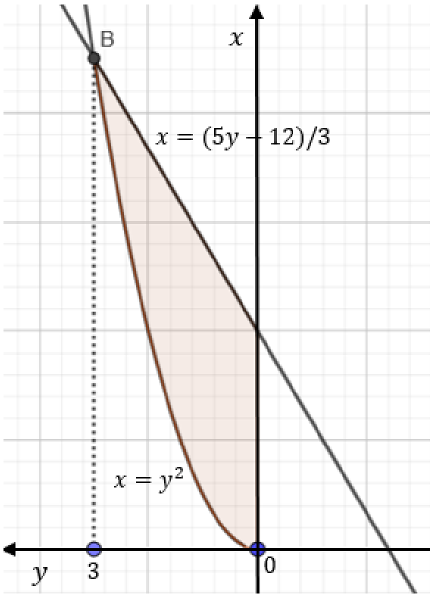

Figure 3.5

Point B has coordinates $(9, 3)$ (see figure 3.5). The boundaries on the y-axis are $[0, 3]$. Therefore, we can set up a simpler integral

$$S = \int_0^3 \left(\frac{5y + 12}{3} - y^2\right) dy = \left(\frac{5y^2}{6} + 4y - \frac{y^3}{3}\right)\Bigg|_0^3 =$$

$$= 5 \cdot \frac{9}{6} + 12 - 9 = 10.5 \ \text{sq. units}$$

7.7.4 Area under a parametric curve

Parametric equations are used to describe many different types of curves. Circle, ellipse, cycloid, and hypocycloid are some of the best-known curves that can be expressed parametrically. To calculate the area under the curve, we modify the area formula by substitution.

The area of a region S enclosed by function $f(x)$, two vertical lines $x = a$ and $x = b$ and x-axis can be calculated by the formula

$$\boldsymbol{S = \int_a^b f(x)dx}$$

If the function is described by $x = x(t)$ and $y = y(t)$ and the parameter t runs between t_1 and t_2 where

$$a = x(t_1); \ \ b = x(t_2)$$

We substitute

$$\boldsymbol{S} = \int_{t_1}^{t_2} \boldsymbol{y(t)d(x(t))} = \int_{t_1}^{t_2} \boldsymbol{y(t)x'(t)dt}$$

Example 4.1 Calculate the area of an ellipse.

Solution

The ellipse is symmetric with respect to its axes. Therefore, we calculate the area of the fourth part of the ellipse (see figure 4.1)

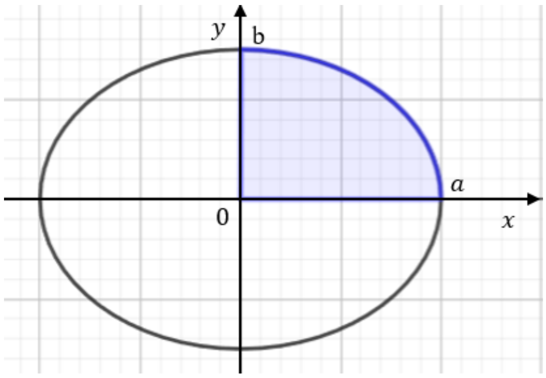

Figure 4.1

Parametric equations of the ellipse are

$$\begin{cases} x = a\,\mathrm{cost} \\ y = b\,\mathrm{sint} \end{cases}$$

We calculate

$$S = 4\int_0^a f(x)dx = \left| \begin{matrix} \text{let } x = a\,\mathrm{cost}, & \text{then } dx = -a\,\mathrm{sint}\,dt \\ x_1 = 0, \text{ then } t_1 = \frac{\pi}{2}; & x_2 = a, \; t_2 = 0 \end{matrix} \right| =$$

$$= -4\int_{\frac{\pi}{2}}^{0} b\,\mathrm{sint}\,a\,\mathrm{sint}\,dt = -4ab\int_{\frac{\pi}{2}}^{0} \sin^2 t\,dt =$$

$$= 4ab\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2t}{2}\,dt =$$

$$= 2ab\int_0^{\frac{\pi}{2}} dt - ab\int_0^{\frac{\pi}{2}} \cos 2t\,d2t = 2ab\left|\begin{matrix}\frac{\pi}{2}\\0\end{matrix}\right. - ab\sin 2t\left|\begin{matrix}\frac{\pi}{2}\\0\end{matrix}\right. =$$

$$= 2ab\frac{\pi}{2} = \pi ab \quad \mathrm{sq.units}$$

7.7.5 Curve in a polar coordinate system

The curvilinear sector is given by the function $r = r(\varphi)$ and two rays $\varphi = \alpha; \ \ \varphi = \beta$. To calculate the area of this sector we apply the formula (see figure 5.1)

$$\boldsymbol{S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)d\varphi}$$

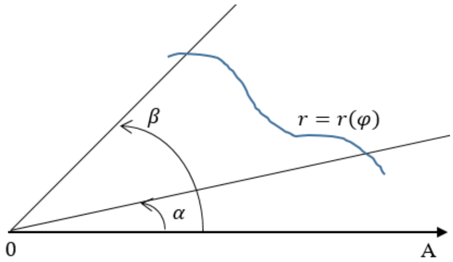

Figure 5.1

Example 5.1 Find the area inside the cardioid $r = 2 + 2\cos\varphi$.

Solution

The shape of the given cardioid is represented in figure 5.2

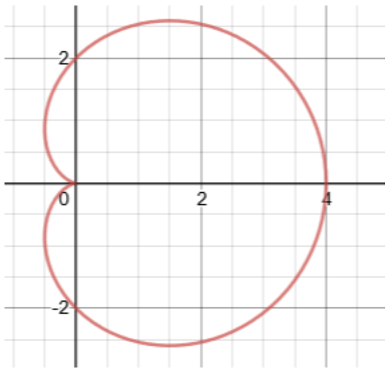

Figure 5.2

Polar axis is the symmetry line of the cardioid. We create the integral for half of the region where the angle changes from 0 to 180°. The area of this region is

$$S = 2 \cdot \frac{1}{2}\int_0^{\pi} (2 + 2\cos\varphi)^2 d\varphi = \int_0^{\pi} (4 + 4\cos\varphi + \cos^2\varphi)d\varphi$$

$$=$$

$$= 4\int_0^{\pi} d\varphi + 4\int_0^{\pi} \cos\varphi \, d\varphi + \frac{1}{2}\int_0^{\pi} (1 + \cos 2\varphi)d\varphi =$$

$$= \left(4\varphi + 4\,\sin\varphi + \frac{1}{2}\varphi + \frac{1}{4}\,\sin 2\varphi\right)\Big|_{0}^{\pi} = 4.5\pi \quad \text{sq. units}$$

7.7.6 Exercises

1. Calculate the area of the region between the curve $y = \sin x$ and x-axis in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{4}\right]$.
2. Calculate the area of a region enclosed by straight lines $y = x$ and $x + 2y - 6 = 0$, and x-axis.
3. Calculate the area between two curves $y = (x+2)^2$ and $y = 4 - x^2$.
4. Calculate the area enclosed by $y = 0.5^x$, $y = 0.5x\sqrt{1+x^2}$, $x = -2$ and y-axis.
5. Calculate the area under one arc of the cycloid

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

6. Calculate the area of one petal of the polar rose $r = 4\cos 3\varphi$.

7.7.7 Solutions

1. Calculate the area of the region between the curve $y = \sin x$ and x-axis in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{4}\right]$.

   Solution

   We construct the curve and vertical lines (see figure 7.1).

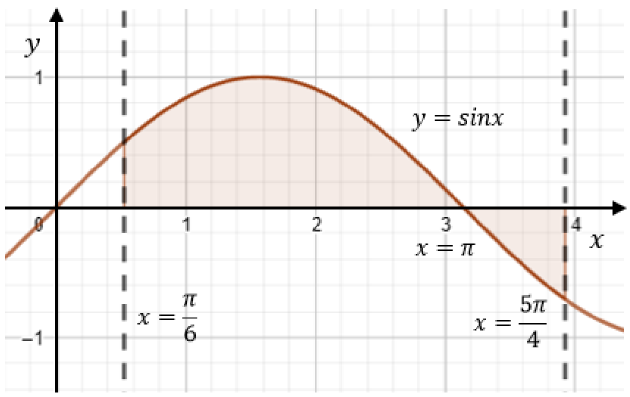

Figure 7.1

The function $y = \sin x$ has positive and negative values over the given interval. To calculate the area of the region it is necessary to divide the interval into two parts.

The x-intercept of the function is $x = \pi$. We compose two integrals to calculate the area

$$S = S_1 + S_2$$

$$= \int_{\frac{\pi}{6}}^{\pi} \sin x \, dx + \left| \int_{\pi}^{\frac{5\pi}{4}} \sin x \, dx \right| = -\cos x \Big|_{\frac{\pi}{6}}^{\pi} + |-\cos x| \Big|_{\pi}^{\frac{5\pi}{4}} =$$

$$= -\left(\cos\pi - \cos\frac{\pi}{6}\right) + \left|\cos\frac{5\pi}{4} - \cos\pi\right| =$$

$$= 1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} + 1 = \frac{4 + \sqrt{3} - \sqrt{2}}{2} \approx 2.16 \quad \text{sq. units}$$

2. Calculate the area of a region enclosed by straight lines $y = x$ and $x + 2y - 6 = 0$, and x-axis.

Solution

We construct the straight lines and choose the projection of the region to the y-axis.

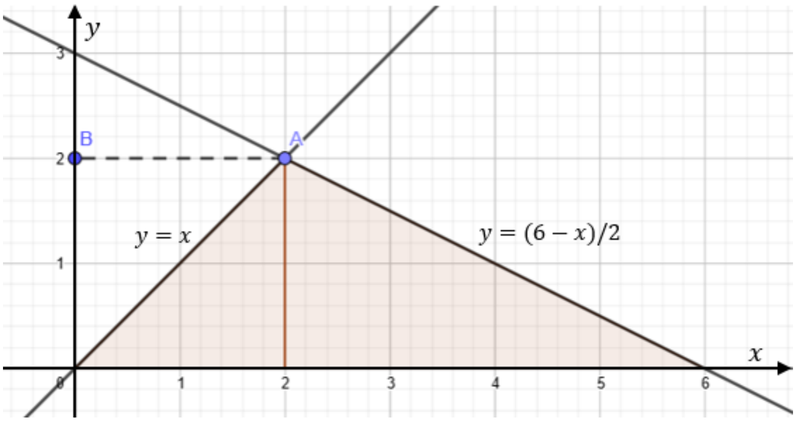

Figure 7.2

Let us calculate the coordinates of the intersection point A

$$\begin{cases} y = x \\ y = (6 - x)/2 \end{cases}$$

$$x = (6 - x)/2$$

$$2x = 6 - x$$

$$3x = 6; \quad x = 2$$

The intersection point A has the coordinates A (2,2). The boundaries of the integral are [0,2] with respect to variable y. The integral is

$$S=\int_0^2 (6-2y-y)\,dy=\left(6y-\frac{3y^2}{2}\right)\Bigg|_0^2=12-3\cdot 2=6\quad \text{sq. units}$$

3. Calculate the area between two curves $y=(x+2)^2$ and $y=4-x^2$.

Solution

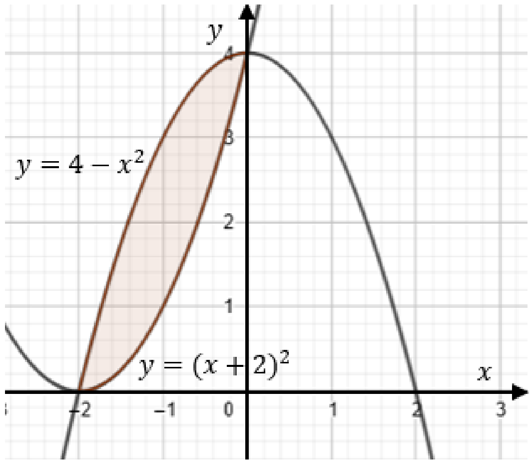

Figure 7.3

The region is defined in the interval $[-2,0]$. We find its area

$$S=\int_{-2}^{0}\left(4-x^2-(x+2)^2\right)dx=\left(4x-\frac{x^3}{3}-\frac{(x+2)^3}{3}\right)\Bigg|_{-2}^{0}=$$

$$=0-\frac{8}{3}+8-\frac{8}{3}-0=\frac{8}{3}\quad \text{sq. units}$$

4. Calculate the area enclosed by $y=0.5^x$, $y=0.5x\sqrt{1+x^2}$, $x=-2$ and y-axis.

Solution

Construct the curves and the vertical line (see figure 7.4)

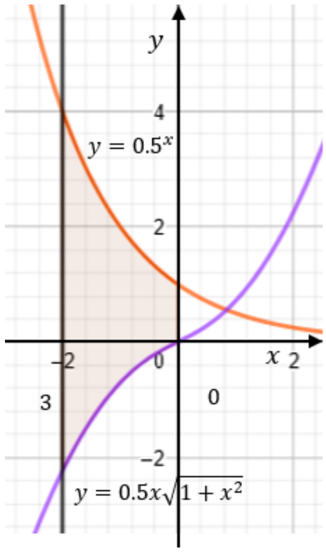

Figure 7.4

Set up the integral

$$S = \int_{-2}^{0} \left(0.5^x - 0.5x\sqrt{1+x^2}\right) dx = \int_{-2}^{0} 0.5^x\, dx - \int_{-2}^{0} 0.5x\sqrt{1+x^2}\, dx$$

Let us solve the second integral separately

$$\int_{-2}^{0} 0.5x\sqrt{1+x^2}\, dx =$$

$$= \left| \begin{matrix} \text{let } u = 1 + x^2, \text{then } \quad du = 2xdx \\ u_1 = 5, \ \ u_2 = 1 \end{matrix} \right| =$$

$$= \frac{1}{4}\int_{5}^{1} \sqrt{u}\, du = \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{5}^{1} = \frac{1}{6}\left(1 - 5\sqrt{5}\right)$$

Now

$$S = \frac{0.5^x}{\ln 0.5}\Bigg|_{-2}^{0} - \frac{1 - 5\sqrt{5}}{6} =$$

$$= \frac{1}{\ln 0.5}\left(1 - 0.5^{-2}\right) + \frac{5\sqrt{5} - 1}{6} \approx 6.03 \quad \text{sq. units}$$

5. Calculate the area under one arc of the cycloid

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

Solution

Comment. The ***cycloid*** is the locus of a point on the rim of a circle of radius R rolling along a straight line. We can see the way of construction of the cycloid on the webpage:

Weisstein, Eric W. "Cycloid." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/Cycloid.html

The radius of the given cycloid is $R = 2$. Then the area under the first arc is over the interval $[0, 2\pi R] = [0, 4\pi]$ (see figure 7.5).

Figure 7.5

We have to calculate the integral of the function given in parametric form. We calculate the boundaries with respect to the argument t in the following way:

We have $0 \leq x \leq 4\pi$.

For the lower bound $x = 0$ then $0 = 2(t - \sin t)$. We calculate $t_1 = 0$

For the upper bound $x = 4\pi$ then $4\pi = 2(t - \sin t)$; $2\pi = t - \sin t$. We calculate $t_2 = 2\pi$.

According to the formula given in chapter 4, we differentiate the function x with respect to variable t

$$x' = 2(1 - \cos t)$$

The area is

$$S = \int_0^{2\pi} 2(1 - \cos t)2(1 - \cos t)\, dt = 4\int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt =$$

$$= 4\int_0^{2\pi} (1 - 2\cos t) dt + 2\int_0^{2\pi} (1 + \cos 2t)\, dt =$$

$$= (4t - 8\sin t + 2t + \sin 2t)\Big|_0^{2\pi} =$$

$$= 8\pi + 4\pi = 12\pi \quad \text{sq. units}$$

6. Calculate the area of one petal of the polar rose $r = 4\cos 3\varphi$.

Solution

The given polar rose has three petals:

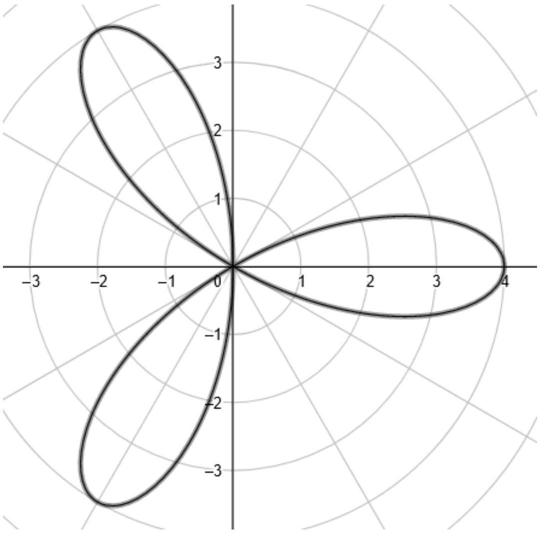

Figure 7.6

We calculate the area of the petal whose line of symmetry is the polar axis. Therefore, we can calculate half of the petal's area and double the integral. First, we need to detect the upper bound of the integral. It appears when the distance of a point on the ray is zero

$$0 = 4\cos 3\varphi;\ 3\varphi = \frac{\pi}{2};\ \ \varphi_2 = \frac{\pi}{6}$$

We set up the integral to calculate the area of one petal

$$S = 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{6}} (4\cos 3\varphi)^2\, d\varphi =$$

$$= \frac{16}{2}\int_0^{\frac{\pi}{6}} (1 + \cos 6\varphi)\, d\varphi = 8\left(\varphi + \frac{1}{6}\sin 6\varphi\right)\Bigg|_0^{\frac{\pi}{6}} = \frac{4\pi}{3}\quad \text{sq. units}$$

7.8 Application of the Definite Integral: Arc Length

DETAILED DESCRIPTION:

Definite integrals can be applied to calculate the length of various curves. This chapter explains the creation of the formula for calculation of arc length. The formula can be transformed for curves that are given as parametric equations or in polar form. The content is supplemented with examples of graphs constructed with GeoGebra and Desmos.

AIM: to demonstrate the calculation of the arc length for curves given in the Cartesian coordinate system and for curves given in the polar coordinate system.

Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can calculate the arc length of given curves.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of a functions; the construction of the graph of a function; algebra and trigonometry formulas.

Relationship to real maritime problems: With the help of definite integrals it is possible to calculate the lengths of different objects that can be described by functions. For instance, it is possible to calculate the length of a rope hanging between two supports by integration.

Content

1. The formula for calculation of the length of an arc
2. The length of an arc given by parametric equations
3. The arc length of a polar curve
4. Exercises
5. Solutions

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Arc Length

7.8.1 The formula for calculation of the length of an arc

Let the function $y = f(x)$ be given over the interval $[a, b]$. We will calculate the length of the arc $\overset{\frown}{AB}$ of this curve (see figure 1.1).

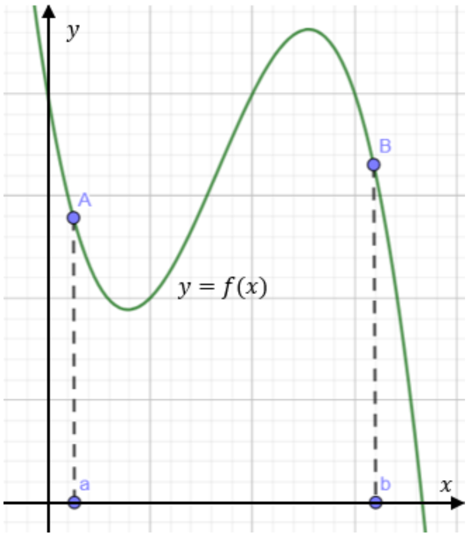

Figure 1.1

We will form a polygonal line by choosing points $A = P_0, P_1, P_2, P_3, \dots,$ and $P_n = B$ (see figure 1.2).

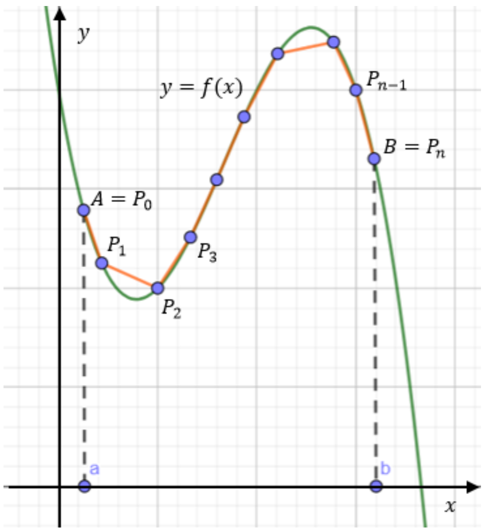

Figure 1.2

Now we can calculate the length of every segment $[P_{i-1}, P_i]$ for every index i (see figure 1.3) applying the theorem of Pythagoras:

$$\Delta l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

where

$$\Delta x_i = x_i - x_{i-1};\ \ \Delta y_i = f(x_i) - f(x_{i-1})$$

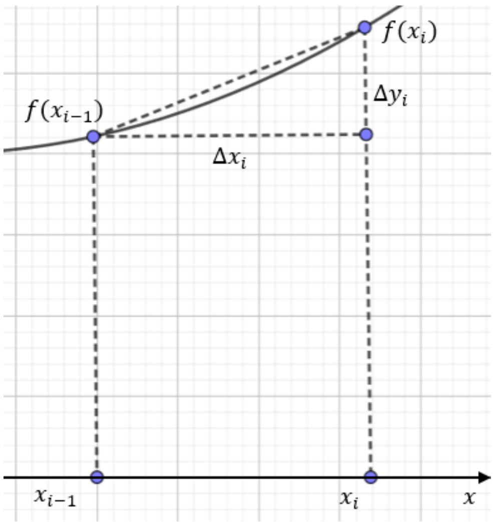

Figure 1.3

We change the expression

$$\Delta l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

The approximate value of the arc length is the sum of the lengths of all segments Δl_i

$$\overset{\frown}{AB} \approx \sum_{i=1}^{n} \Delta l_i = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

The result will be better if we divide the arc into smaller and smaller parts. Taking the limit when the maximum length of the interval Δx_i tends to zero, we get the real length of the arc in the given units

$$\overset{\frown}{AB} = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

Here we can recall the definition of derivative of the function $y = f(x)$ with respect to the argument x

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'$$

Thus, we express the limit as an integral in the following way getting the formula for calculation of the arc length

$$\overset{\frown}{AB} = \int_a^b \sqrt{1 + (y')^2}\, dx$$

Example 1.1

Calculate the length of the line segment given by equation $y = 3x - 2$ from $a = -2$ and $b = 3$.

Solution

It is necessary to differentiate the given function to apply the formula

$$y' = (3x - 2)' = 3$$

Now

$$L = \int_{-2}^{3} \sqrt{1 + (3)^2}\, dx = \sqrt{10} \int_{-2}^{3} dx = \sqrt{10}x \bigg|_{-2}^{3} =$$

$$= \sqrt{10}(3 + 2) = 5\sqrt{10} \approx 15.8 \text{ units}$$

7.8.2 The length of an arc given by parametric equations

If the arc $\overset{\frown}{AB} = L$ is described by parametric equations on the interval $[a, b]$ with respect to the argument t

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

we can apply substitution with respect to the argument t

$$L = \int_{a}^{b} \sqrt{1 + (y')^2}\, dx = \left| \begin{matrix} x = x(t), \ \ dx = d(x(t)) = (x'(t))_t dt = \dot{x} dt \\ y' = \dfrac{\dot{y}}{\dot{x}}, \ \ x(t_1) = a, \ \ x(t_2) = b \end{matrix} \right| =$$

$$= \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2} \cdot \dot{x}\, dt = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

We can calculate the arc length for a parametrically given function by the formula

$$\boldsymbol{L = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt}$$

Example 2.1

Calculate the arc length of an astroid

$$\begin{cases} x = 4\cos^3 t \\ y = 4\sin^3 t \end{cases}$$

Solution

Comment. An ***astroid*** is the locus of a point on a circle as it rolls inside a fixed circle with four times the radius (see figure 2.1). Parameter t expresses the angle. The curve can be constructed if the parameter t changes from 0 to 2π.

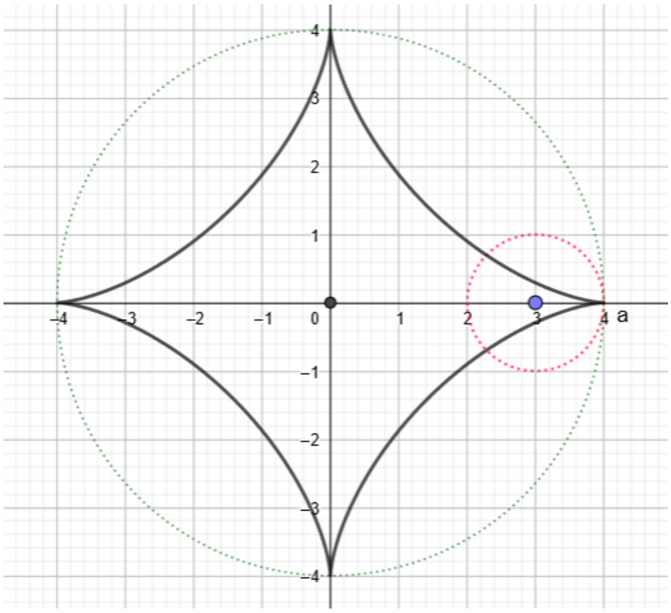

Figure 2.1

The curve is centrally symmetric with respect to the origin. Therefore, we can calculate one-fourth of the astroid where parameter t changes from 0 to $\pi/2$.

To create an integral, it is necessary to differentiate the parametric functions

$$\begin{cases} \dot{x} = 4 \cdot 3\cos^2 t(-\sin t) \\ \dot{y} = 4 \cdot 3\sin^2 t \cdot \cos t \end{cases}$$

The length of the given astroid is

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{(4 \cdot 3\cos^2 t(-\sin t))^2 + (4 \cdot 3\sin^2 t \cdot \cos t)^2} dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{144\cos^4 t\sin^2 t + 144\sin^4 t\cos^2 t}\, dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{144\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)}\, dt =$$

$$= 4\int_0^{\frac{\pi}{2}} 12\sin t \cdot \cos t\, dt = 48\int_0^{\frac{\pi}{2}} \sin t\, d(\sin t) = 48\frac{\sin^2 t}{2}\Bigg|_0^{\frac{\pi}{2}} =$$

$$= 24(1-0) = 24 \text{ units}$$

7.8.3 The arc length of a polar curve

The arc length $\overset{\frown}{MN}$ of a polar curve $r = r(\varphi)$ between the rays α and β (see figure 3.1) is given by the integral

$$\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\varphi}$$

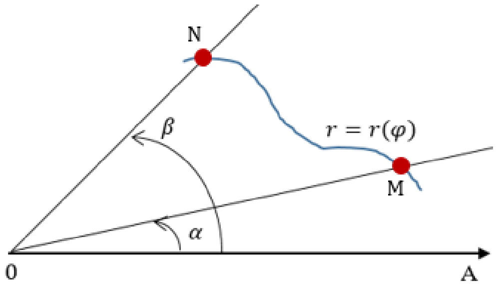

Figure 3.1

Example 3.1

Calculate the length of the arc of the circle $r = 6$ between $0 \leq \varphi \leq \frac{2\pi}{3}$

Solution

We will calculate the length of the arc that is one-third of the circle with radius 6 (see figure 3.2)

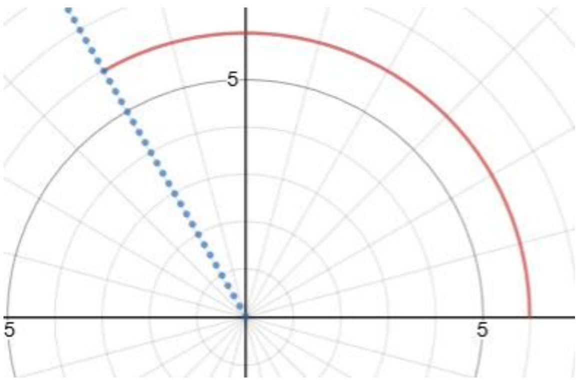

Figure 3.2

Provided that the value of derivative $r' = 0$, the length of the arc is

$$L = \int_{0}^{\frac{2\pi}{3}} \sqrt{36 + 0}\, d\varphi = \int_{0}^{\frac{2\pi}{3}} 6\, d\varphi = 6\varphi \Big|_{0}^{\frac{2\pi}{3}} = 4\pi \quad \text{units}$$

7.8.4 Exercises

1. Calculate the arc length of the curve $y = \frac{4}{3}\sqrt{x^3}$ from $x = 1$ to $x = 2$.

2. Calculate the arc length of the curve $y = \ln x$ from $x = 1$ to $x = \sqrt{2}$.

3. Find the length of one arc of the cycloid given by parametric equations

$$\begin{cases} x = 2(1 - \sin t) \\ y = 2(t - \cos t) \end{cases}$$

4. Calculate the arc length of the circle $r = 4\cos\varphi$ included between the polar rays $\varphi = -\frac{\pi}{3}$ and $\varphi = \frac{\pi}{5}$.

7.8.5 Solutions

1. Calculate the arc length of the curve $y = \frac{4}{3}\sqrt{x^3}$ from $x = 1$ to $x = 2$.

Solution

Construct the graph

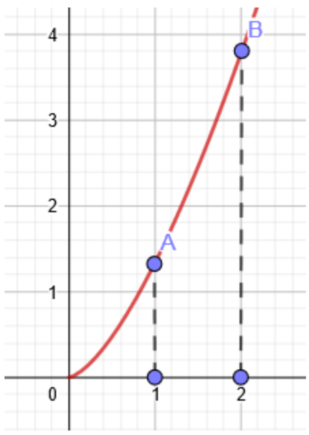

Figure 5.1

We calculate the arc length between the points A and B. The derivative of the function

$$y' = \left(\frac{4}{3}x^{\frac{3}{2}}\right)' = 2\sqrt{x}$$

We use the arc length formula

$$\overset{\frown}{AB} = \int_a^b \sqrt{1+(y')^2}\, dx$$

$$L = \int_1^2 \sqrt{1+\left(2\sqrt{x}\right)^2}\, dx = \int_1^2 \sqrt{1+4x}\, dx = \left| \begin{array}{c} u = 1+4x, \;\; du = 4dx \\ x_1 = 1, \;\; u_1 = 5; \;\; x_2 = 2, \;\; u_2 = 9 \end{array} \right| =$$

$$= \frac{1}{4}\int_5^9 u^{\frac{1}{2}}\, dx = \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_5^9 = \frac{1}{6}\left(3^3 - \sqrt{5^3}\right) = \frac{1}{6}\left(27 - 5\sqrt{5}\right) \approx 2.64 \;\; \text{units}$$

2. Calculate the arc length of the curve $y = \ln x$ from $x = 1$ to $x = \sqrt{2}$.

Solution

Construct the graph of the given function

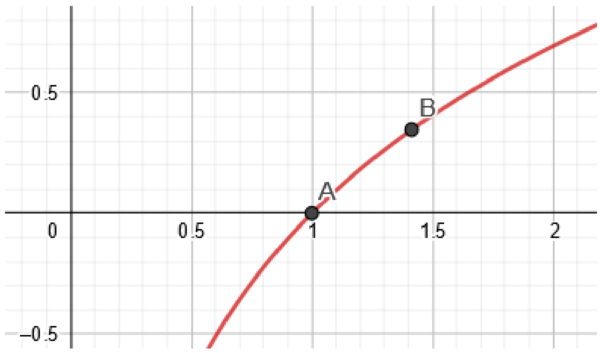

Figure 5.2

The derivative of the function $y = \ln x$ is

$$y' = (\ln x)' = \frac{1}{x}$$

To calculate the integral, we use an algebraic transformation of the integrand and apply substitution

$$L = \int_1^{\sqrt{2}} \sqrt{1+\frac{1}{x^2}}\, dx = \int_1^{\sqrt{2}} \sqrt{\frac{x^2+1}{x^2}}\, dx =$$

$$= \int_1^{\sqrt{2}} \frac{\sqrt{x^2+1}}{x}\, dx = \int_1^{\sqrt{2}} \frac{\sqrt{x^2+1}}{x^2}\, xdx = \left| \begin{array}{c} u^2 = x^2+1, \;\; 2udu = 2xdx \\ u_1 = \sqrt{2}, \;\; u_2 = \sqrt{3} \end{array} \right|$$

$$=$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} \frac{u}{u^2-1} u du = \int_{\sqrt{2}}^{\sqrt{3}} \frac{u^2 - 1 + 1}{u^2-1} du =$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} du + \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{u^2-1} du = \left(u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \right) \Bigg|_{\sqrt{2}}^{\sqrt{3}} =$$

$$= \sqrt{3} - \sqrt{2} + \frac{1}{2} \left(\ln \left| \frac{\sqrt{3}-1}{\sqrt{3}+1} \right| - \ln \left| \frac{\sqrt{2}-1}{\sqrt{2}+1} \right| \right) \approx 0.54 \quad \text{units}$$

3. Find the length of one arc of the cycloid given by parametric equations

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

Solution

We can calculate the length of one arc of the cycloid (see figure 5.3) if the range of the parameter $t \in [0,2\pi]$.

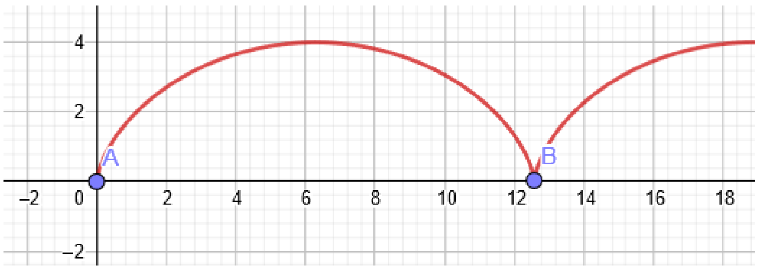

Figure 5.3

First we will calculate the derivatives

$$\dot{x} = 2(1 - \cos t)$$
$$\dot{y} = 2\sin t$$

Now we will simplify the expression by applying algebra and trigonometry formulas

$$\dot{x}^2 + \dot{y}^2 = 4(1 - \cos t)^2 + 4\sin^2 t =$$

$$= 4(1 - 2\cos t + \cos^2 t + \sin^2 t) =$$

$$= 4(2 - 2\cos t) = 8 \cdot 2\sin^2 \frac{t}{2} = 16\sin^2 \frac{t}{2}$$

We used trigonometry formulas

$$\cos^2 t + \sin^2 t = 1$$

$$1 - \cos t = 2\sin^2\frac{t}{2}$$

The length of the first arc of cycloid is

$$L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^{2\pi} \sqrt{16\sin^2\frac{t}{2}}\, dt =$$

$$= \int_0^{2\pi} 4\sin\frac{t}{2}\, dt = -8\cos\frac{t}{2}\Big|_0^{2\pi} = -8(\cos\pi - \cos 0) =$$

$$= -8 \cdot (-2) = 16 \quad \text{units}$$

4. Calculate the arc length of the circle $r = 4\cos\varphi$ included between the polar rays $\varphi = -\frac{\pi}{3}$ and $\varphi = \frac{\pi}{5}$.

Solution

Construct the curve in the polar coordinate system (see figure 5.4)

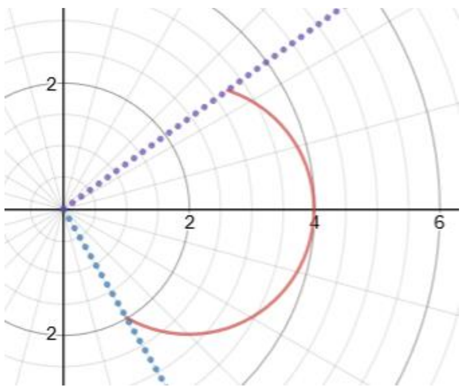

Figure 5.4

Calculate the derivative and transform the trigonometric expression

$$r' = (4\cos\varphi)' = -4\sin\varphi$$

$$r^2 + r'^2 = 16\cos^2\varphi + 16\sin^2\varphi = 16$$

Create an integral

$$L = \int_{-\frac{\pi}{3}}^{\frac{\pi}{5}} \sqrt{r^2 + r'^2}\, d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{5}} \sqrt{16}\, d\varphi = 4\varphi \Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{5}} =$$

$$= 4\left(\frac{\pi}{5} + \frac{\pi}{3}\right) = \frac{32\pi}{15} \quad \text{units}$$

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7.9 Application of the Definite Integral: Volume of a Solid of Revolution

DETAILED DESCRIPTION:

This chapter introduces the main principles for calculation of the volume of solids of revolution. Different examples are discussed where solids are generated by elementary curves and lines. Some composite constructions are explained. The case of the parametrically given curve is included to describe the solid of revolution. The content is supplemented with examples of graphs and surfaces constructed with GeoGebra tools.

AIM: to show the methods of calculation of the volume of solids of revolution.

Learning Outcomes:

1. Students understand the application of the definite integral in solving geometry tasks.
2. Students can construct regions of a revolution and understand what surfaces they form.
3. Students can calculate the volume of solids of revolution.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Relationship to real maritime problems: Volume is a very important concept if we are speaking about the capacity of cargo holds, the capacity of fuel oil tanks or ballast water tanks, tanks of lubricating oil, or others. It is important to know the amount of material required for producing a specific part with a definite volume. Calculations of the volume of containers, cauldrons, and tanks are among the necessary premises for designing a ship's engineering equipment.

Content

1. Volume of a solid of revolution obtained by rotating an area about x-axis
2. Volume of a solid of revolution generated by two curves
3. Rotation about the y-axis
4. Revolution of parametrically given curves
5. Exercises
6. Solutions

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Application of the Definite Integral. Volume of a Solid of Revolution

7.9.1 Volume of a solid of revolution obtained by rotating an area about x-axis

Let us recall the concept of the solid of revolution.

Definition. The ***solid of revolution*** is a solid figure obtained by rotating a plane curve around a straight line (the axis of revolution) that lies in the same plane.

Let the function $y = f(x)$ be a continuous non-negative function on the interval $[a, b]$.

Consider the solid formed by rotating (revolving) the region bounded by the curve $f(x)$, straight lines $x = a$, $x = b$, and x- axis about the $\boldsymbol{x}$-axes (see figure 1.1). This solid is called a solid of revolution.

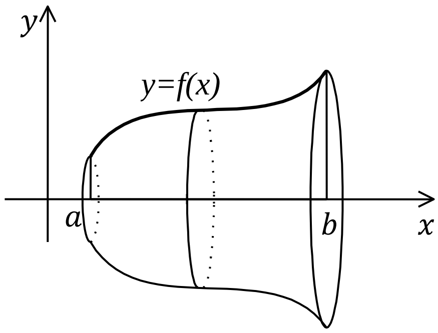

Figure 1.1

The volume of this solid can be calculated by the formula:

$$\boldsymbol{V = \pi \int_a^b (f(x))^2 dx}$$

Example 1.1

Let us find the volume of solid of revolution obtained by revolving the area bounded by the curve $y = x^3$ and x-axis between $x = 0$ and $x = 2$ about x-axis.

Solution

The region is given in figure 1.2. Figure 1.3 presents the solid of revolution.

The Volume is

$$V = \pi \int_0^2 (x^3)^2 dx = \pi \int_0^2 x^6 dx = \pi \left.\frac{x^7}{7}\right|_0^2 =$$

$$= \frac{2^7 \cdot \pi}{7} = \frac{128\pi}{7} \approx 57.45 \text{ cubic units}$$

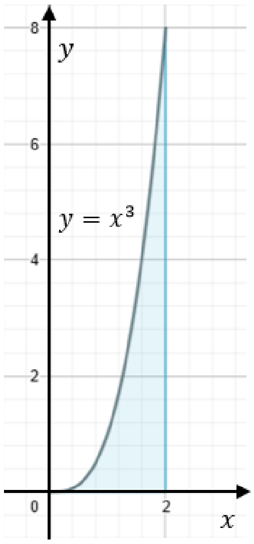

Figure 1.2

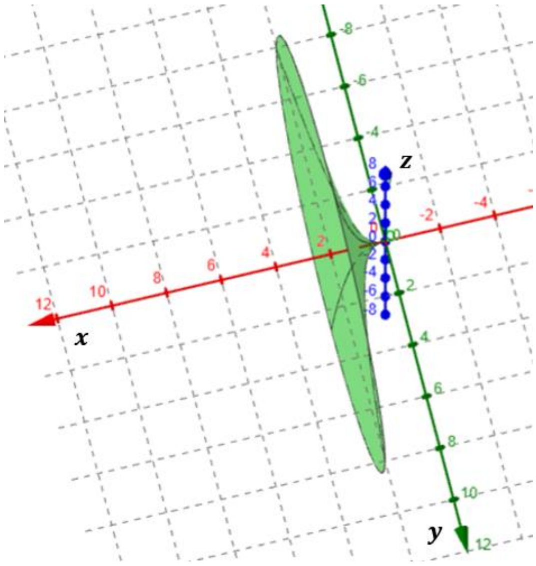

Figure 1.3

7.9.2 Volume of a solid of revolution generated by two curves

Let us consider two functions $f_1(x)$ and $f_2(x)$ that are continuous and non-negative on the interval $[a,b]$ and $f_1(x) \leq f_2(x)$ (see figure 2.1).

The volume of the solid formed by rotating the area bounded by two curves $y = f_2(x)$ and $y = f_1(x)$ between $x = a$ and $x = b$ about the x-axis (see figure 2.2) is defined as:

$$\boldsymbol{V = \pi \int_a^b [(f_2(x))^2 - (f_1(x))^2]dx}$$

Note that due to $f_1(x) \leq f_2(x)$ the curve $y = f_2(x)$ bounds the area on the top and curve $y = f_1(x)$ bounds the area on the bottom.

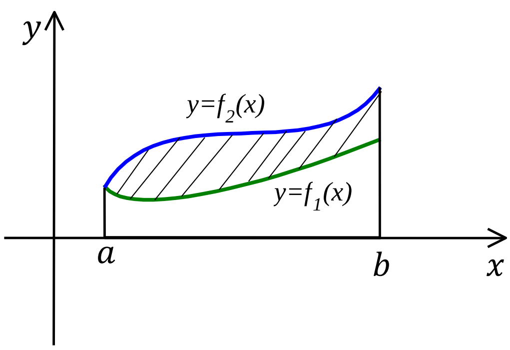

Figure 2.1

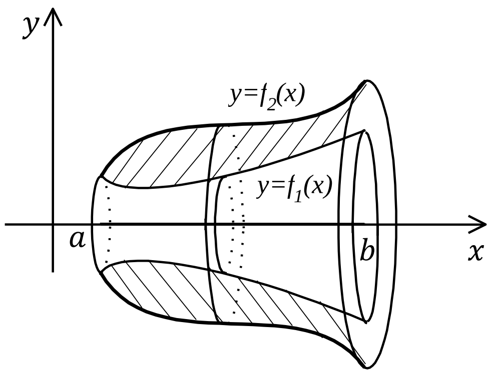

Figure 2.2

Example 2.1

Find the volume of a solid of revolution obtained by rotating the area bounded by the curve $y = x^3$ and the straight line $y = 4x$ about the x-axis.

Solution

We sketch the region of the revolution (see figure 2.3) and the solid of revolution (see figure 2.4).

The intersection points of the two lines are $(0,0)$ and $(2,8)$, therefore the area is between $x = 0$, $x = 2$. In this region $x^3 < 4x$. It means that the region is bounded by the line $y = 4x$ on the top and by the curve $y = x^3$ on the bottom.

We use the formula to calculate the volume of a solid:

$$\boldsymbol{V = \pi \int_a^b [(f_2(x))^2 - (f_1(x))^2]dx}$$

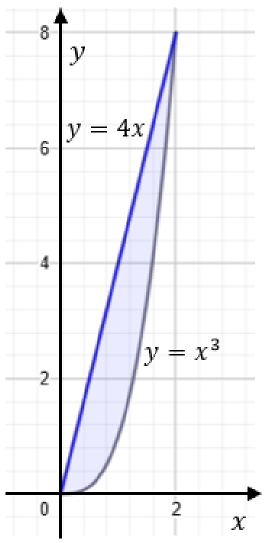

Figure 2.3

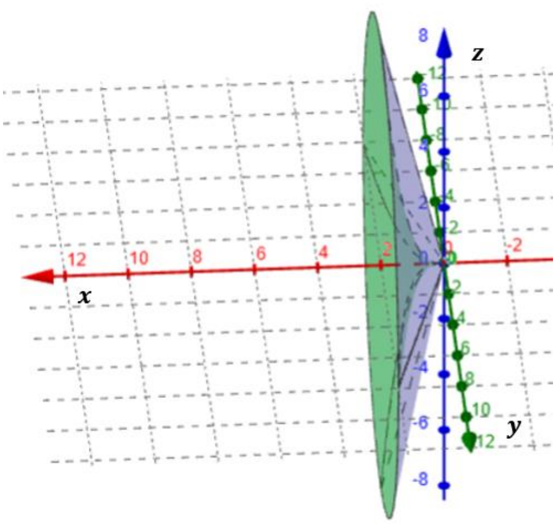

Figure 2.4

For the given case

$$V = \pi \int_0^2 [(4x)^2 - (x^3)^2]dx = \pi \int_0^2 [16x^2 - x^6]dx =$$

$$= \pi \left(\frac{16x^3}{3} - \frac{x^7}{7} \right) \Bigg|_0^2 =$$

$$= \pi \left(\frac{128}{3} - \frac{128}{7} \right) = \frac{512\pi}{21} \approx 76.59 \text{ cub. units}$$

7.9.3 Rotation about the y-axis

The volume of the solid, when the region bounded by the curve $x = x(y)$ and y-axis between $y = c$ and $y = d$, revolves **about the y-axis** (see figure 3.1) can be found by using the formula:

$$\boldsymbol{V = \pi \int_c^d \left(x(y)\right)^2 dy}$$

Here the function $x = x(y)$ is continuous on the interval $[c, d]$.

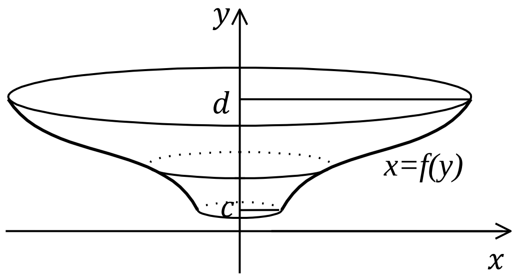

Figure 3.1

In the case when the area located between $y = c$ and $y = d$ and bounded by the curve $x = x_2(y)$ on the right side and by the curve $x = x_1(y)$ on the left side of the area, revolves **about the y-axis**, the volume of the obtained solid of revolution is calculated by the formula

$$\boldsymbol{V = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2\right] dy}$$

The functions $x = x_1(y)$ and $x = x_2(y)$ are continuous on the interval $[c, d]$ and $x_1(y) \leq x_2(y)$ on the interval $y \in [c, d]$.

Example 3.1

We find the volume of the solid obtained by revolving about the y-axis the area bounded by the curve $y = x^3$, the line $y = 8$, and the y-axis.

Solution

The region is sketched in figure 3.2. The solid is sketched in figure 3.3.

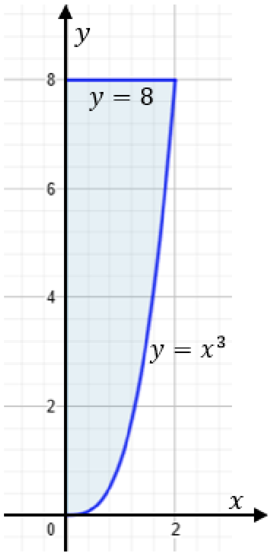

Figure 3.2

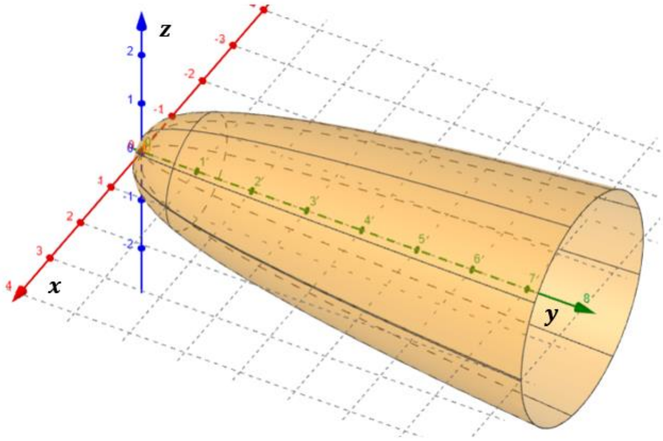

Figure 3.3

We use the formula

$$\boldsymbol{V = \pi \int_{c}^{d} (x(y))^{2} dy}$$

From the equation $y = x^3$ we express $x = \sqrt[3]{y}$.

Then

$$V = \pi \int_{0}^{8} \left(\sqrt[3]{y}\right)^{2} dy = \pi \int_{0}^{8} y^{\frac{2}{3}} dy =$$

$$= \pi \left. \frac{3y^{\frac{5}{3}}}{5} \right|_{0}^{8} = \pi \frac{3\left(\sqrt[3]{8}\right)^{5}}{5} = \frac{96\pi}{5} \approx 60.32 \text{ cub. units}$$

Example 3.2

Let us consider the givens from Example 1.1 (see figure 1.2). Let the region bounded by the functions $y = x^3$ and $x = 2$ revolve around the y-axis (see figure 3.4). We will find the volume of such solid.

Solution

The region of the revolution is bounded by the line $x = 2$ on the right side and by the curve $y = x^3$ on the right side, therefore we use the formula

$$\boldsymbol{V = \pi \int_{c}^{d} \left[(x_2(y))^{2} - (x_1(y))^{2} \right] dy}$$

To find the interval of integration, we find the value of the function $y = x^3$ at $x = 2$:

$$y(2) = 8$$

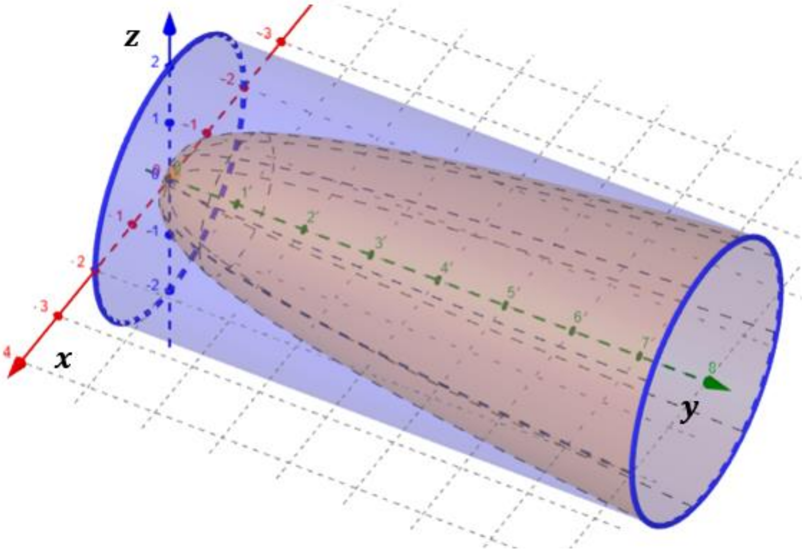

Figure 3.4

From the equation of the curve $y = x^3$ we find $x = \sqrt[3]{y}$.

Then

$$V = \pi \int_0^8 \left[(2)^2 - \left(\sqrt[3]{y}\right)^2\right] dy = \pi \int_0^8 \left[4 - y^{\frac{2}{3}}\right] dy =$$

$$= \pi \left(4y - \frac{3y^{\frac{5}{3}}}{5}\right)\Bigg|_0^8 = \pi \left(32 - \frac{3\left(\sqrt[3]{8}\right)^5}{5}\right) =$$

$$= \pi \left(32 - \frac{96}{5}\right) = \frac{64\pi}{5} \approx 40.2 \text{ cub. units}$$

Example 3.3

Let us consider the region from example 2.1 (see figure 2.3) now rotating about the y-axis. The area is bounded by the curve $y = x^3$ and the straight line $y = 4x$. To find the volume of the solid of revolution obtained by rotating this region about the y-axis (see figure 3.5), we will use the same formula as in the previous example. The region is bounded by $x = y/4$ and $x = \sqrt[3]{y}$. The variable y belongs to the interval $[0,8]$. The volume is

$$V = \pi \int_0^8 \left[\left(\sqrt[3]{y}\right)^2 - \left(\frac{y}{4}\right)^2\right] dy =$$

$$= \pi \int_0^8 \left[y^{\frac{2}{3}} - \frac{y^2}{16} \right] dy = \pi \left(\frac{3y^{\frac{5}{3}}}{5} - \frac{y^3}{16 \cdot 3} \right) \Bigg|_0^8 =$$

$$= \pi \left(\frac{3\left(\sqrt[3]{8}\right)^5}{5} - \frac{8^3}{48} \right) = \pi \left(\frac{96}{5} - \frac{32}{3} \right) = \frac{128\pi}{15} \approx 26{,}8 \ \text{cub. units}$$

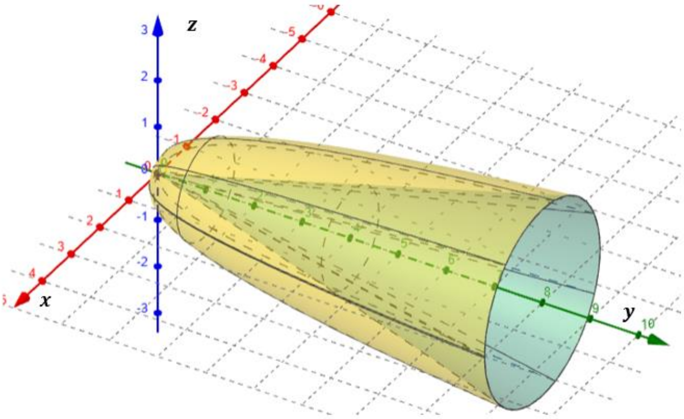

Figure 3.5

7.9.4 Revolution of parametrically given curves

Let the plane area be bounded by the line defined in parametric form $\boldsymbol{x = x(t)}$, $\boldsymbol{y = y(t)}$ and by the lines $\boldsymbol{x = a, x = b, y = 0}$.

1) If the corresponding values of the parameter t to the variable $x \in [a, b]$ belong to the interval $[t_1, t_2]$, then the volume of the solid of revolution around the $\boldsymbol{x}$-axis is calculated by the formula:

$$\boldsymbol{V = \pi \int_{t_1}^{t_2} \left(y(t)\right)^2 x'(t) dt}$$

2) In the case when the plane area, bounded by the line given in parametric form $\boldsymbol{x = x(t)}$, $\boldsymbol{y = y(t)}$ and by the lines $\boldsymbol{y = c}$, $\boldsymbol{y = d}$, $\boldsymbol{x = 0}$, revolves about the $\boldsymbol{y}$-axis, the volume of the solid of revolution is calculated by using the following formula:

$$\boldsymbol{V = \pi \int_{t_1}^{t_2} \left(x(t)\right)^2 y'(t) dt}$$

Example 4.1

The plane area is bounded by quarter of an ellipse presented in parametric form

$$\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

and by lines $x = 0$, $y = 0$ (see figure 4.1). Find the volume of the solid of revolution obtained by rotating this region a) about the x-axis; b) about the y-axis.

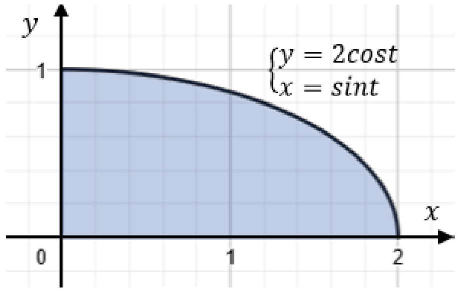

Figure 4.1

Solution

Case a)

The region revolves about the x-axis (see figure 4.2):

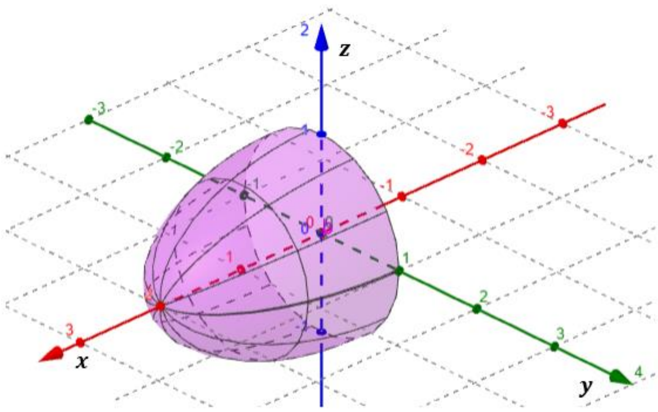

Figure 4.2

We will find the values of parameter t corresponding to the endpoints of a projection of quarter of ellipse on the x-axis, that is, $x = 0$ and $x = 2$:

If $x = 0$ then $\cos t = 0$ and $t = \pi/2$.

If $x = 2$ then $2\cos t = 2$ and $t = 0$.

We find the volume of the solid of revolution obtained when the area revolves about the x-axis by using the formula:

$$\boldsymbol{V = \pi \int_{t_1}^{t_2} \left(y(t)\right)^2 x'(t)dt}$$

We differentiate the function $x = 2\cos t$ with respect to the argument t

$$x' = (2\cos t)' = -2\sin t$$

The volume of the solid is

$$V=\pi\int_{\frac{\pi}{2}}^{0}(\sin t)^2(-2\sin t)dt=-2\pi\int_{\frac{\pi}{2}}^{0}\sin^3 t dt=2\pi\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\sin t dt=$$

$$=-2\pi\int_0^{\frac{\pi}{2}}(1-\cos^2 t)d(\cos t)=-2\pi\left(\cos t-\frac{\cos^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}}=$$

$$=-2\pi\left(\cos\frac{\pi}{2}-\frac{\cos^3\frac{\pi}{2}}{3}\right)+2\pi\left(\cos 0-\frac{\cos^3 0}{3}\right)=2\pi\left(1-\frac{1}{3}\right)=$$

$$=\frac{4\pi}{3}\ \text{cub. units}$$

Case b)

The region revolves about the y-axis (see figure 4.3)

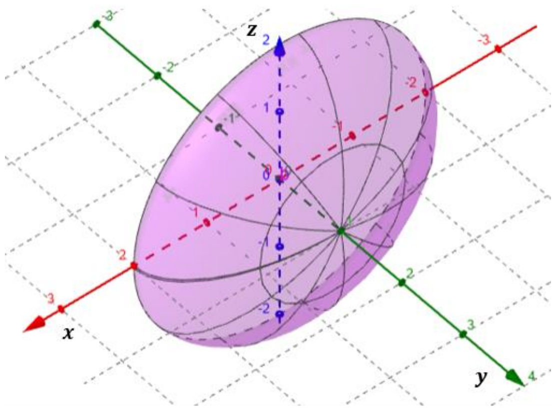

Figure 4.3

We find the values of parameter t that correspond to $y=0$ and $y=1$.

If $y=0$ then $\sin t=0$ and $t=0$.

If $y=3$ then $\sin t=1$ and $t=\pi/2$.

We find the volume of the solid of revolution obtained when the area revolves about the y-axis by using the formula:

$$\boldsymbol{V=\pi\int_{t_1}^{t_2}\left(x(t)\right)^2 y'(t)dt}$$

We differentiate $y'=(\sin t)'=\cos t$.

$$V=\pi\int_0^{\frac{\pi}{2}}(2\cos t)^2(\cos t)dt=4\pi\int_0^{\frac{\pi}{2}}\cos^2 t\cdot\cos t\,dt=$$

$$= 4\pi\int_0^{\frac{\pi}{2}}(1-\sin^2 t)d(\sin t) = 4\pi\left(\sin t - \frac{\sin^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= 4\pi\left(1-\frac{1}{3}\right) - 0 = \frac{8\pi}{3} \quad \text{cub. units}$$

7.9.5 Exercises

1. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2 + 1$ and the straight lines $y = -1, x = 1, y = 0$ about the x-axis.

2. Find the volume of the solid obtained by rotating the region bounded by two sine functions $y = 3\sin x$ and $y = \sin x$ between $x = 1$ and $x = \pi$ about the x-axis.

3. Calculate the volume of the solid obtained by revolving about y-axis the area bounded by the hyperbola $xy = 4$ and the lines $y = 1$, $y = 4$, and $x = 0$.

4. Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = \ln x$ and the lines $y = 0$, $x = e$ about the y-axis.

5. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = \mathrm{x}^2$, the line $x + y = 2$, and $y = 0$ a) about the x-axis; b) about the y-axis.

6. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the part of asteroid given in parametric form $x = \cos^3 t$ and $y = 2\sin^3 t$ on the interval $t \in [-\frac{\pi}{2}; \frac{\pi}{2}]$.

7. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the part of line given in parametric form $x = 2\tan t$ and $y = 2\cos^2 t$, $x = -2$, $x = 2$, $y = 0$.

7.9.6 Solutions

1. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2 + 1$, straight lines $y = -1, x = 1, y = 0$ about the x-axis.

Solution

The region is bounded by a curve and straight lines (see figure 6.1)

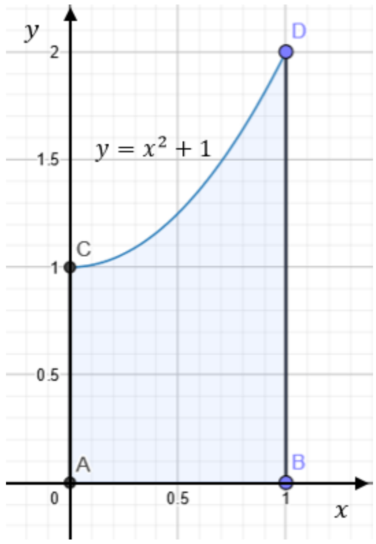

Figure 6.1

As the region is rotating about the x-axis we use the following formula to calculate the volume of the solid of revolution (see figure 6.2):

$$\boldsymbol{V = \pi \int_a^b (f(x))^2 dx}$$

Then

$$V = \pi \int_{-1}^{1} (x^2+1)^2 dx = \pi \int_{-1}^{1} (x^4+2x^2+1) dx =$$

$$= \pi \int_{-1}^{1} (x^4+2x^2+1) dx = \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_{-1}^{1} =$$

$$= \pi \left(\tfrac{1}{5} + \tfrac{2}{3} + 1 \right) - \pi \left(-\tfrac{1}{5} - \tfrac{2}{3} - 1 \right) = \tfrac{56\pi}{15} \approx 11.73 \quad \text{cub. units}$$

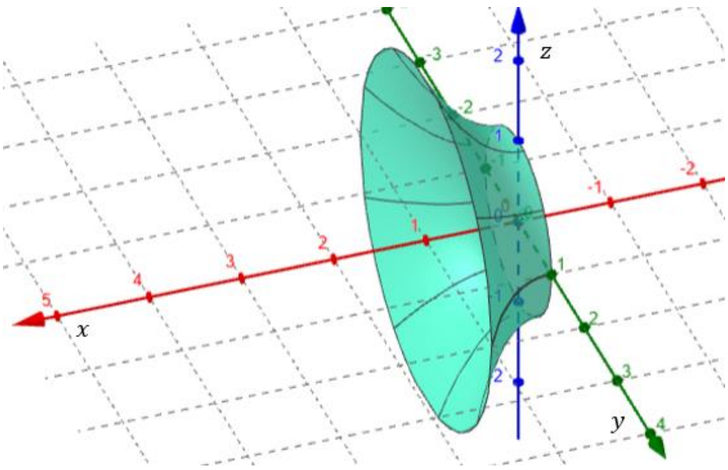

Figure 6.2

2. Find the volume of the solid obtained by rotating the region bounded by two sine functions $y = 3\sin x$ and $y = \sin x$ between $x = 1$ and $x = \pi$ about the x-axis.

Solution

Let us construct the given region (see figure 6.3):

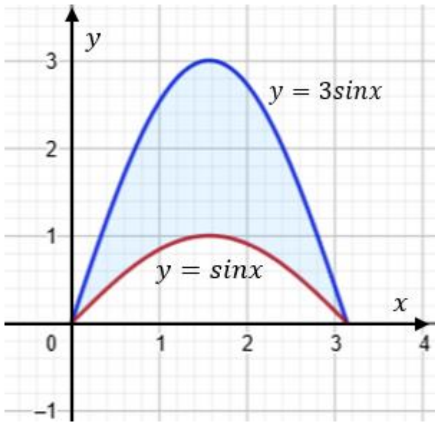

Figure 6.3

The region, revolving about x-axis, is bounded on the top by the curve $y = 3\sin x$ and on the bottom by the curve $y = \sin x$ between $x = 1$ and $x = \pi$. Therefore, to calculate the volume of the solid of revolution (see figure 6.4) we use:

$$\boldsymbol{V = \pi \int_a^b [(f_2(x))^2 - (f_1(x))^2]dx}$$

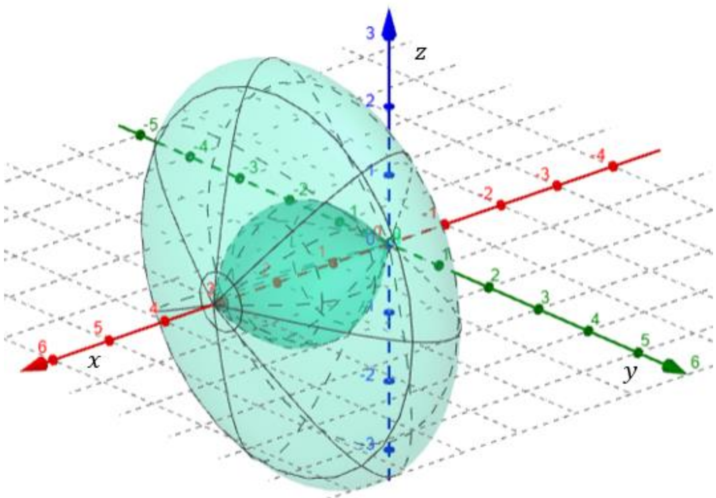

Figure 6.4

$$V = \pi \int_0^\pi [(3\sin x)^2 - (\sin x)^2]dx = \pi \int_0^\pi [9\sin^2 x - \sin^2 x]dx =$$

$$= \pi \int_0^\pi 8\sin^2 x dx = 8\pi \int_0^\pi \frac{1}{2}(1 - \cos 2x)dx =$$

$$= 4\pi \int_0^{\pi} (1 - \cos 2x)dx = 4\pi \left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi} =$$

$$= 4\pi \left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] = 4\pi^2 \approx 39.48 \text{ cub. units}$$

3. Calculate the volume of the solid obtained by revolving about the y-axis the area bounded by the hyperbola $xy = 4$ and the lines $y = 1$, $y = 4$, and $x = 0$.

Solution

The region is shown in figure 6.5

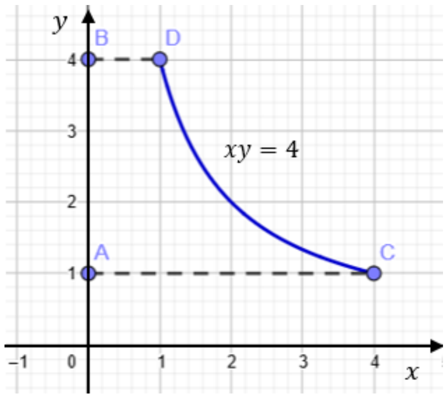

Figure 6.5

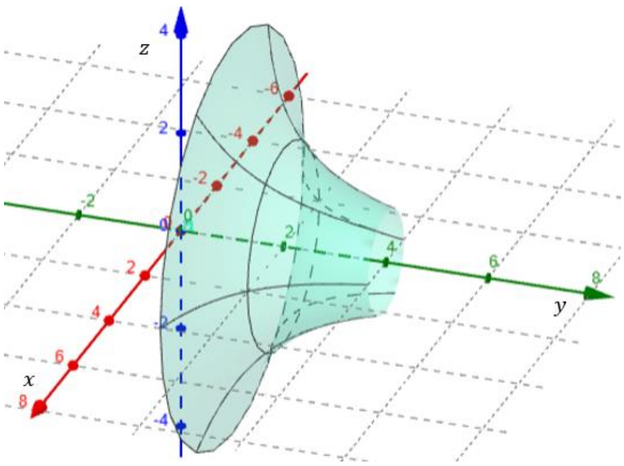

Figure 6.6

The region revolving about the y-axis is bounded on the right by the curve $y = 4/x$ and on the left by the y-axis between $y = 1$ and $y = 4$. Therefore, to calculate the volume of the solid of the revolution (see figure 6.6) we use:

$$\boldsymbol{V = \pi \int_c^d (x(y))^2 dy}$$

The solution is

$$V = \pi \int_1^4 \left(\frac{4}{y}\right)^2 dx = \pi \int_1^4 \frac{16}{y^2} dx =$$

$$= 16\pi \int_1^4 y^{-2} dx = 16\pi \frac{y^{-1}}{-1}\Bigg|_1^4 =$$

$$= -16\pi \frac{1}{y}\Bigg|_1^4 = -16\pi \left(\frac{1}{4} - 1\right) = 12\pi \ \text{cub. units}$$

4. Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = \ln x$ and the lines $y = 0$ and $x = e$ about the y-axis.

Solution

We construct the region:

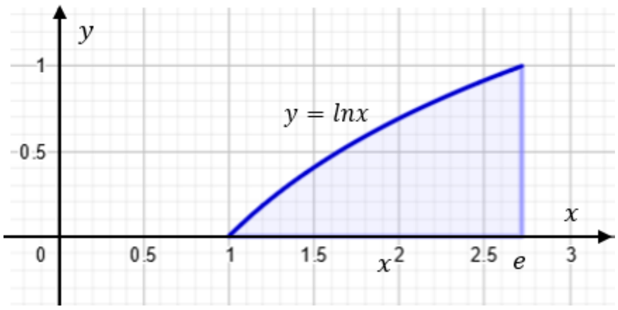

Figure 6.7

The region is bounded by the line $x = \mathrm{e}$ on the right and by $\mathrm{y} = \ln x$ on the top of the area. In this case, to calculate the volume of the solid obtained by rotating the region about the y-axis (see figure 6.8), we use the following formula:

$$\boldsymbol{V = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2\right] dy}$$

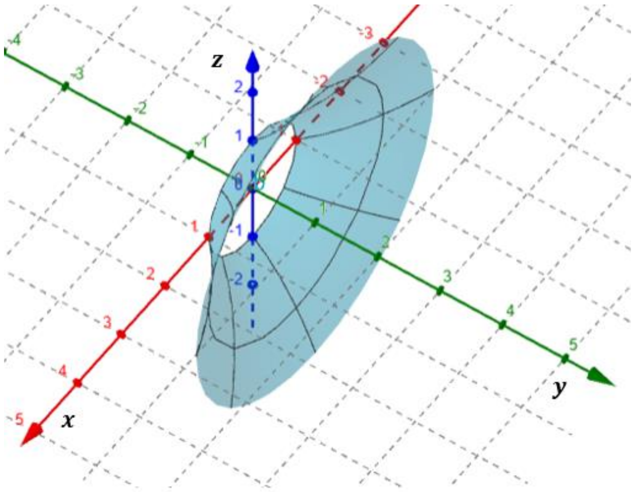

Figure 6.8

To compose an integral, we express the function x with respect to the argument y, that is, $x = e^y$

$$V = \pi \int_0^1 [\mathrm{e}^2 - (e^y)^2]dy = \pi \int_0^1 [\mathrm{e}^2 - e^{2y}]dy =$$

$$= \pi \left(\mathrm{e}^2 \mathrm{y} - \frac{1}{2} e^{2y} \right) \Big|_0^1 =$$

$$= \pi \left(\mathrm{e}^2 - \frac{1}{2} e^2 - \left(0 - \frac{1}{2} \right) \right) =$$

$$= \frac{\pi}{2} (e^2 + 1) \approx 13.18 \quad \text{cub. units}$$

5. Calculate the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x + y = 2$, and $y = 0$ a) about the x-axis; b) about the y-axis.

Solution

The region is shown in figure 6.9.

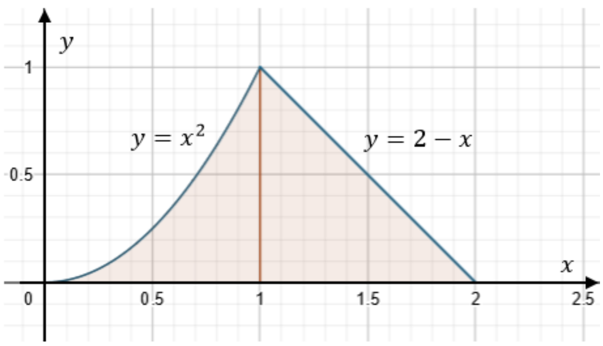

Figure 6.9

Case a)

The region revolves about the x-axis (see figure 6.10).

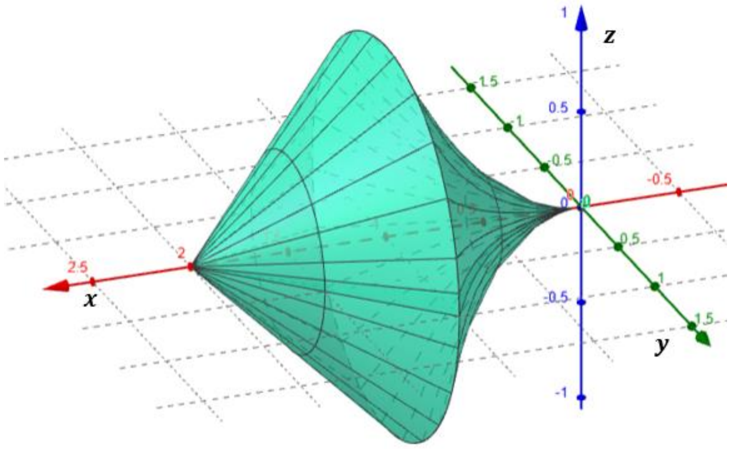

Figure 6.10

In this case, the volume of the obtained solid of revolution is equal to the sum of volumes of two solids V_1 and V_2

$$V = V_1 + V_2$$

where V_1 is the volume of the solid generated by the parabola $y = x^2$ rotated around the x-axis on the interval $0 \leq x \leq 1$ and V_2 is the volume of the solid generated by the line $x + y = 2$ rotated around the x-axis on the interval $1 \leq x \leq 2$.

Therefore,

$$V = V_1 + V_2 = \pi \int_0^1 (x^2)^2 \, dx + \pi \int_1^2 (2-x)^2 \, dx =$$

$$= \pi \int_0^1 x^4 \, dx + \pi \int_1^2 (4 - 4x + x^2) \, dx =$$

$$= \pi \frac{x^5}{5}\bigg|_0^1 + \pi \left(4x - 2x^2 + \frac{x^3}{3}\right)\bigg|_1^2 =$$

$$= \frac{\pi}{5} + \pi\left(8 - 8 + \frac{8}{3}\right) - \pi\left(4 - 2 + \frac{1}{3}\right) =$$

$$= \frac{8\pi}{15} \approx 1.68 \text{ cub. units}$$

Case b)

The region revolves about the y-axis (see figure 6.11).

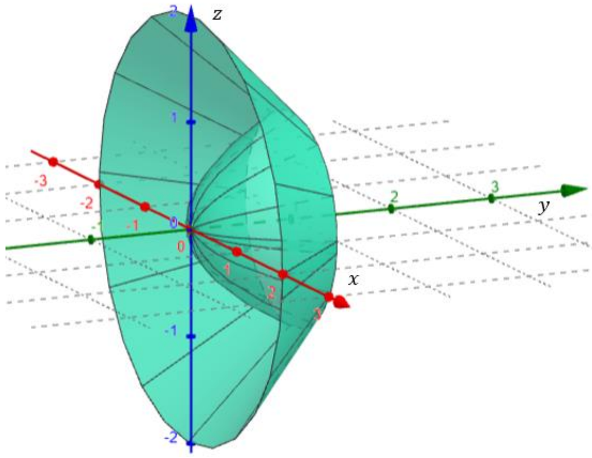

Figure 6.11

The region is located between $y = 0$ and $y = 1$ and is bounded by the line $x + y = 2$ on the right. On the left the parabola $y = x^2$ bounds the region. We express the functions in terms of the argument y:

$$\text{from } x + y = 2 \quad \text{follows} \quad x = 2 - y,$$

$$\text{from } y = x^2 \quad \text{follows} \quad x = \sqrt{y}.$$

Then

$$V = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2 \right] dy =$$

$$= \pi \int_0^1 \left[(2 - y)^2 - \left(\sqrt{y}\right)^2 \right] dy = \pi \int_0^1 [4 - 4y + y^2 - y] dy =$$

$$= \pi \int_0^1 [4 - 5y + y^2] dy = \pi \left(4y - \frac{5y^2}{2} + \frac{y^3}{3} \right) \Bigg|_0^1 =$$

$$= \pi \left(4 - \frac{5}{2} + \frac{1}{3} \right) = \frac{11\pi}{6} \approx 5.76 \text{ cub. units}$$

6. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the part of asteroid given in parametric form $x = \cos^3 t$ and $y = 2\sin^3 t$, $t \in [-\frac{\pi}{2}; \frac{\pi}{2}]$.

Solution

The region is shown in figure 6.12. We apply the following formula to calculate the volume of the solid (see figure 6.13)

$$\boldsymbol{V = \pi \int_{t_1}^{t_2} \big(x(t)\big)^2 y'(t)dt}$$

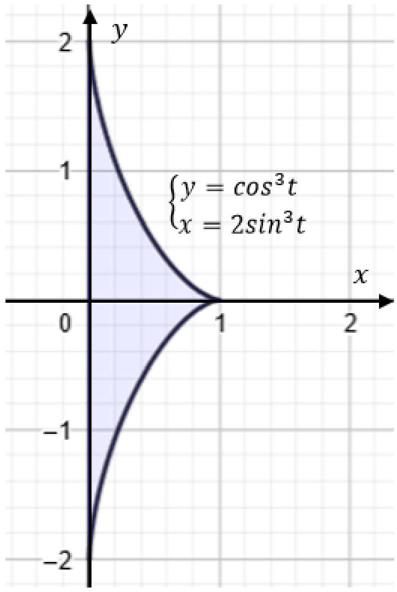

Figure 6.12

To create an integral for calculation of the volume of the solid (see figure 6.13) we need to find the derivative of the function y with respect to the argument t

$$y' = (2\sin^3 t)' = 6\sin^2 t\cos t$$

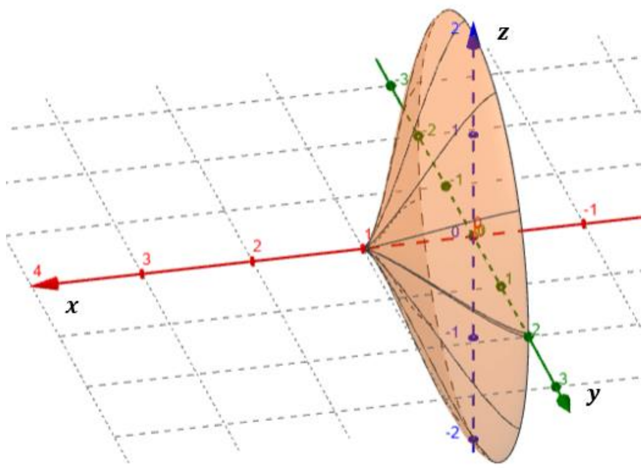

Figure 6.13

$$V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^3 t)^2 (6\sin^2 t\cos t)dt =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 t \sin^2 t \cdot \cos t dt =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 t)^3 \sin^2 t \cdot d(\sin t) =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - 3\sin^2 t + 3\sin^4 t - \sin^6 t) \cdot \sin^2 t d(\sin t) =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t - 3\sin^4 t + 3\sin^6 t - \sin^8 t) d(\sin t) =$$

$$= 6\pi \left(\frac{\sin^3 t}{3} - \frac{3\sin^5 t}{5} + \frac{3\sin^7 t}{7} - \frac{\sin^9 t}{9} \right) \Bigg|_0^{\frac{\pi}{2}} =$$

$$= 6\pi \left(\frac{1}{3} - \frac{3}{5} + \frac{3}{7} - \frac{1}{9} \right) - 0 = \frac{32\pi}{105} \approx 0.96 \text{ cub. units}$$

7. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the part of line given in parametric form

$$\begin{cases} x = 2\tan t \\ y = 2\cos^2 t \end{cases}; \; x = -2, x = 2, y = 0.$$

Solution

The area is shown in figure 6.14. The solid of revolution is presented in figure 6.15.

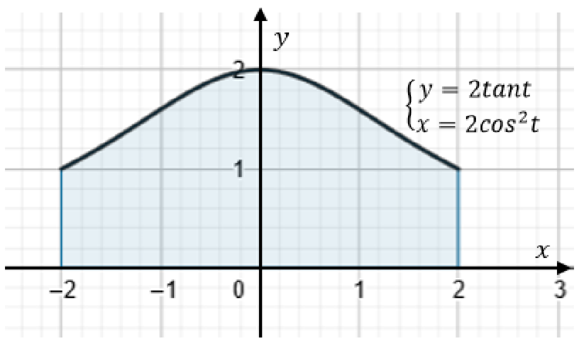

Figure 6.14

The given area is symmetric according to the y-axis, therefore we will find the volume for the interval $0 \leq x \leq 2$ and multiply the result by 2:

$$V = 2 \cdot V_1$$

where V_1 is volume for $0 \leq x \leq 2$. We use the formula

$$\boldsymbol{V=\pi\int_{t_1}^{t_2}\left(y(t)\right)^2x'(t)dt}$$

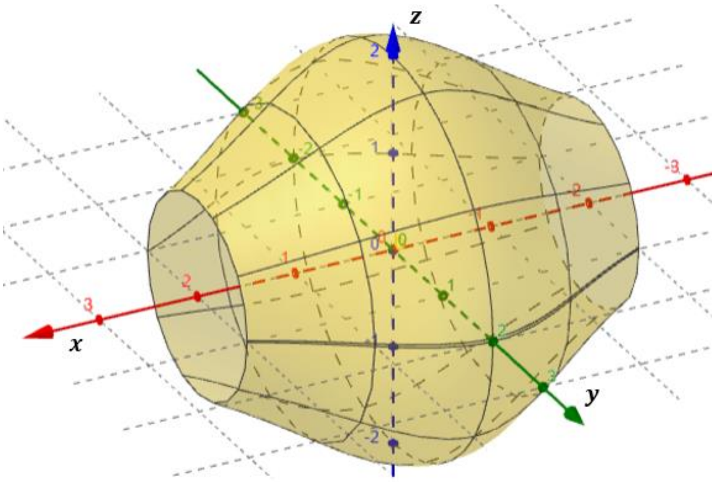

Figure 6.15

The values of a parameter t corresponding to $x=0$ and $x=2$ are the following:

If $x=0$ then $2\tan t=0$ and $t=0$.

If $x=2$ then $\tan t=1$ and $t=\pi/4$.

We differentiate

$$x'(t)=(2\tan t)'=\frac{2}{\cos^2 t}$$

Then

$$V_1=\pi\int_0^{\frac{\pi}{4}}(2\cos^2 t)^2\cdot\frac{2}{\cos^2 t}dt=8\pi\int_0^{\frac{\pi}{4}}\cos^4 t\cdot\frac{1}{\cos^2 t}dt=$$

$$=8\pi\int_0^{\frac{\pi}{4}}\cos^2 t\,dt=8\pi\int_0^{\frac{\pi}{4}}\frac{1+\cos 2t}{2}\,dt=$$

$$=4\pi\int_0^{\frac{\pi}{4}}(1+\cos 2t)dt=4\pi\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{4}}=$$

$$=4\pi\left(\frac{\pi}{4}+\frac{1}{2}\right)=\pi^2+2\pi\approx 16.15\ \text{cub. units}$$

The volume of the whole solid is

$V=2V_1\approx 32.3$ cub. units

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7.10 Application of Definite Integral: Area of Surface of Revolution

DETAILED DESCRIPTION:

Definite integrals can be applied to calculate the area of surface of revolution. The chapter demonstrate the application of a special formula of calculation of this area for cases if an arc revolves about the x-axis or about the y-axes. The formula can be transformed for curves that are given by the parametric equations. The content is supplied by the examples with a graphs constructed with GeoGebra applet. The exercises that relay to the topic are attached at the end of the lesson.

AIM: **to demonstrate the calculation of the area of surface of revolution in Cartesian coordinate system.**

Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can apply computer aids to construct geometric shapes and surfaces.
3. Students can calculate the area of surface of revolution.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of a functions; the construction of the graphs of a functions; the three dimensional construction of surfaces; algebra and trigonometry formulas.

Relationship to real maritime problems: Calculation of surface of revolution is an important part at the design of different parts of mechanical equipment. For instance, to increase the operational efficiency of centrifugal pump it is useful the calculation of surfaces of revolution for blade construction as an integral part of pump. The satellite dish has the shape of a solid of revolution. The calculation of its surface is necessary to detect the amount of paint required to cover the surface.

Content

1. Formula for calculation of a surface of revolution
2. Revolution about the y-axis
3. The surface area of a solid of revolution for parametrically given curve
4. Exercises
5. Solutions

Appendix: calculation of integral

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7.10.1 Formula for calculation of a surface of revolution

The function $y = f(x)$ is represent on the Cartesian coordinate plane (see figure 1.1). An arc of the function over the interval $[a, b]$ revolves about the x-axis and it forms the surface (see figure 1.2). The area of a surface of revolution we can calculate by the formula

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

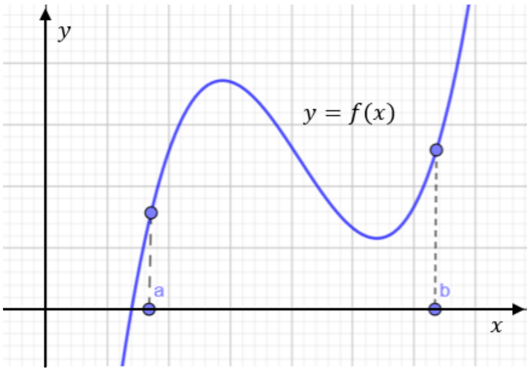

Figure 1.1

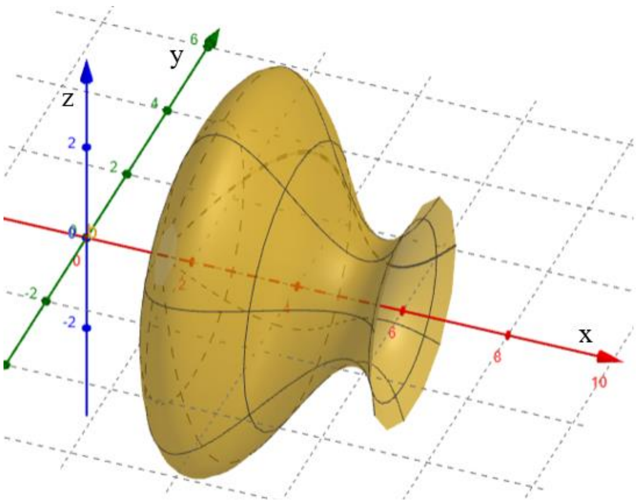

Figure 1.2

Example 1.1 A truncated cone is formed by the straight line $3x + 4y = 0$ revolving about the x-axis over the interval $[4,8]$. Calculate the lateral surface area of the cone!

Solution

We construct the graph of a function (see figure 1.3) and the cone (see figure 1.4).

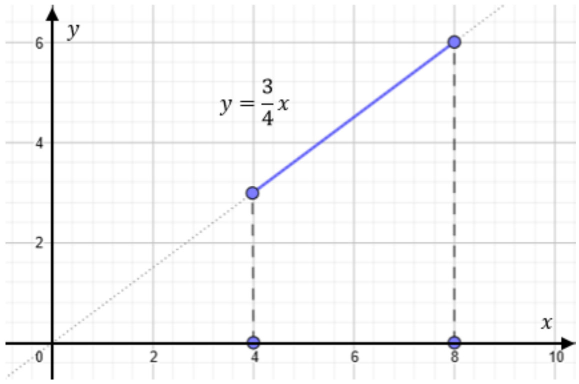

Figure 1.3

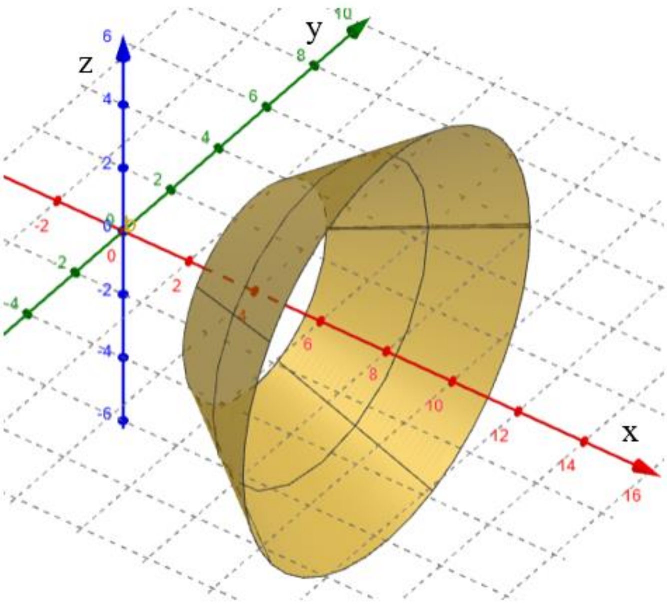

Figure 1.4

It is necessary to differentiate given function to apply the formula

$$y' = (\frac{3}{4}x)' = \frac{3}{4}$$

The lateral surface area of a cone is

$$S = 2\pi \int_4^8 \frac{3}{4}x\sqrt{1+\left(\frac{3}{4}\right)^2}\,dx = \frac{3}{2}\pi\int_4^8 x\sqrt{\frac{25}{16}}\,dx =$$

$$= \frac{15}{8}\pi\int_4^8 x\,dx = \frac{15}{8}\pi\left.\frac{x^2}{2}\right|_4^8 = \frac{15}{16}\pi(64-16) = 5\pi \text{ square units}$$

7.10.2 Revolution about the y-axis

If an arc of a function $y = f(x)$ (see figure 1.1) revolves about the y-axis, we have to express the inverse function with respect of the argument y, that is, $x = g(y)$. We detect the appropriate projection interval $[c, d]$ of the arc on the y-axis (see figure 2.1). Then the formula of the area of a surface of revolution (see figure 2.2) is

$$\boldsymbol{S = 2\pi \int_{c}^{d} g(y)\sqrt{1 + (g'(y))^2}\, dy}$$

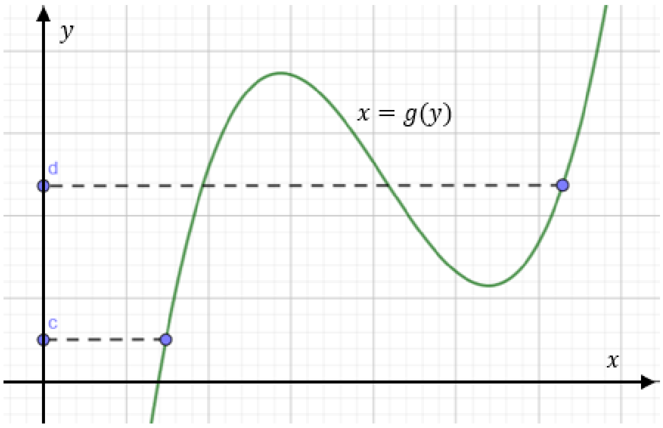

Figure 2.1

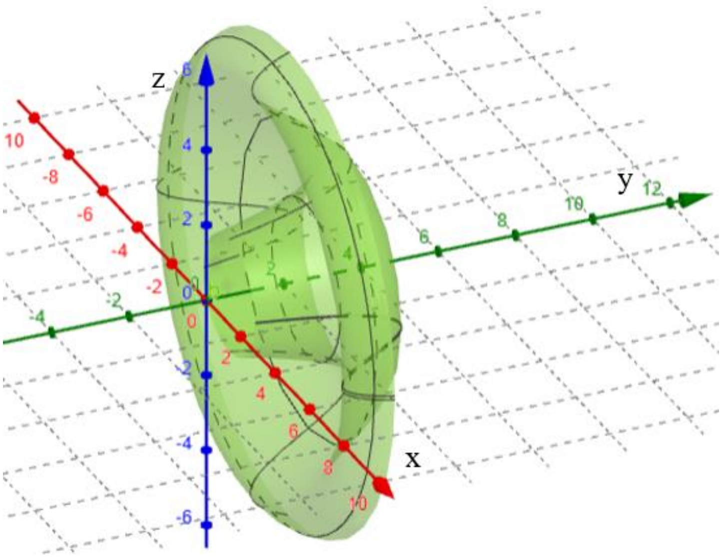

Figure 2.2

Example 2.1

Find the area of the surface obtained by rotating the curve $y = x^2$ on the interval $[0, 2]$ around the y–axis.

Solution

The arc and the surface are presented on figures 2.3 and 2.4. We rewrite the equation of the curve as a function with respect to the argument y.

$$x = g(y) = \sqrt{y}$$

The derivative of a function with respect to the argument y is

$$x' = \frac{1}{2\sqrt{y}}$$

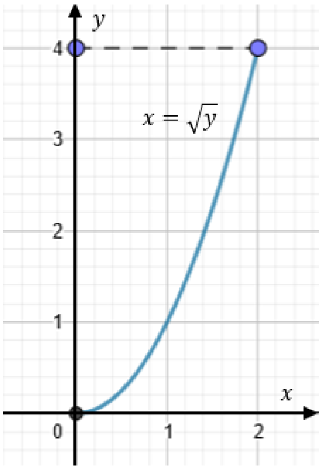

Figure 2.3

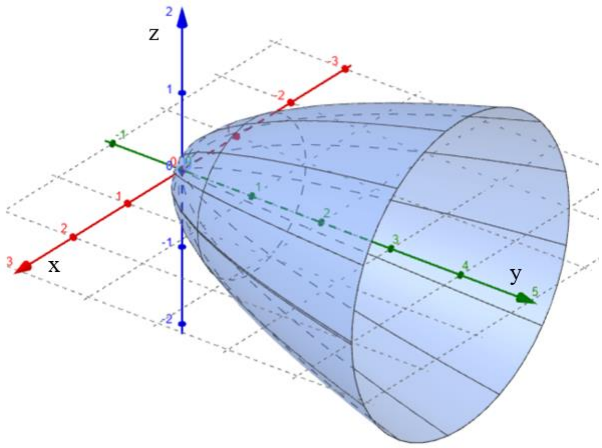

Figure 2.4

We detect the projection of the arc to the y-axis as interval [0, 4]. Then the integral for calculation of a surface of revolution is

$$S = 2\pi \int_0^4 \sqrt{y} \cdot \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy = 2\pi \int_0^4 \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}} dy =$$

$$= 2\pi \int_0^4 \sqrt{y} \cdot \frac{\sqrt{4y+1}}{2\sqrt{y}} dy = \pi \int_0^4 \sqrt{4y+1}\, dy =$$

$$= \frac{\pi}{4} \int_0^4 \sqrt{4y+1}\, d(4y+1) =$$

$$= \frac{\pi}{4} \frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^4 = \frac{\pi}{6}\left(\sqrt{17}^3 - 1\right) \approx 36.18 \text{ sq. units}$$

7.10.3 The surface area of a solid of revolution for parametrically given curve

The formula for calculating the surface area in the Cartesian coordinates are

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

Let us recall the formula for calculation of the length of an arc that is defined by the function $y = f(x)$ above the interval $[a, b]$

$$\boldsymbol{L = \int_a^b ds}$$

By expressing the differential of the arc ds we have

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

So the formula for calculation of a surface of revolution we can rewrite in the following way

$$\boldsymbol{S = 2\pi \int_a^b f(x) ds}$$

Let the curve is defined by the parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Let the parameter t belongs to the interval $t \in [\alpha, \beta]$. In this case the differential of the arc we can calculate in the following way

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

The formula for calculation the surface area when the curve is revolving around the x-axis is

$$\boldsymbol{S = 2\pi \int_\alpha^\beta y(t)\, ds = 2\pi \int_\alpha^\beta y(t)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt}$$

Let us solve similar problem as formulated in the example 1.1. Here we will express the line segment in parametric form.

Example 3.1

The segment of a straight line $y = \frac{x}{2}$ above the interval $[2, 6]$ (see figure 3.1) is rotating around the x-axis. It forms the lateral surface of a truncated cone. Calculate the total surface of the cone!

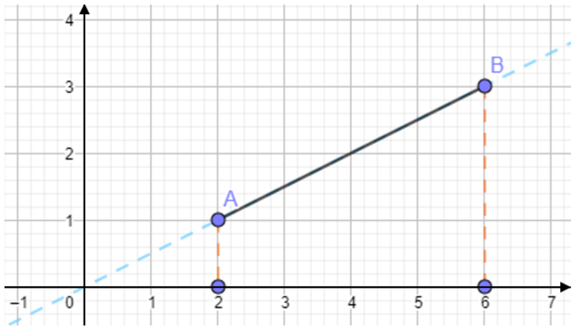

Figure 3.1

Solution

To calculate the total area of a surface of truncated cone (see figure 3.2), it is necessary to calculate its lateral surface area and the area of the upper and lower bases. The upper and lower bases are the circles with the radius $r = 1$ and $R = 3$. The base area will be calculated using the circle area formula. The lateral surface area will be calculated by an integral.

Let us transform the expression of the function $y = \frac{x}{2}$ into a parametric form

$$\begin{cases} x = t \\ y = t/2 \end{cases}; \quad 2 \leq t \leq 6$$

We now differentiate both functions and compose the formula for calculation of the lateral area of the surface formed by segment rotating around the x −axis (see figure 3.2)

$$\dot{x} = 1; \quad \dot{y} = \frac{1}{2}$$

$$S = 2\pi \int_{2}^{6} \frac{t}{2} \sqrt{1 + \frac{1}{4}} dt$$

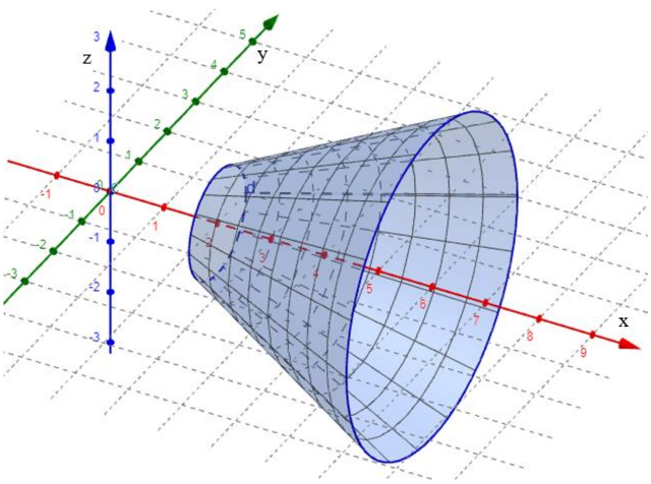

Figure 3.2

The area of a surface of revolution is

$$S = 2\pi \int_{2}^{6} \frac{t}{2}\sqrt{\frac{5}{4}}dt = \frac{\sqrt{5}\pi}{2}\int_{2}^{6} tdt \ =$$

$$= \frac{\sqrt{5}\pi}{2} \cdot \frac{t^2}{2}\bigg|_2^6 = \frac{\sqrt{5}\pi}{4}(36-4) = 8\sqrt{5}\pi \ \text{ sq. units}$$

The total surface area of a given cone is

$$S_T = S + \pi r^2 + \pi R^2 = 8\sqrt{5}\pi + 10\pi \approx 87.61 \ \text{ sq. units}$$

Example 3.2

Parametrically given curve revolves around the x-axis and form the surface. Calculate the area of this surface if the parameter t changes in the interval $[0, 1]$. Analytic expression of the curve is

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases}$$

Solution

Let us differentiate the functions and compose the integral

$$\dot{x} = 3t^2; \ \ \dot{y} = 2t$$

$$S = 2\pi \int_{0}^{1} t^2\sqrt{9t^4 + 4t^2}\, dt$$

We will simplify the expression under the square root and use substitution

$$S = 2\pi \int_{0}^{1} t^2 \cdot t \cdot \sqrt{9t^2 + 4}\, dt = \begin{vmatrix} u^2 = 9t^2 + 4 & udu = 9tdt \\ u_1 = 2 & u_2 = \sqrt{13} \end{vmatrix} =$$

$$= \frac{2\pi}{81}\int_{2}^{\sqrt{13}} (u^2 - 4) \cdot u \cdot u\, du = \frac{2\pi}{81}\int_{2}^{\sqrt{13}} (u^4 - 4u^2)\, du =$$

$$= \frac{2\pi}{81}\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\bigg|_2^{\sqrt{13}} = \frac{2\pi}{81}\left(\frac{\sqrt{13}^5}{5} - \frac{4\sqrt{13}^3}{3} - \frac{2^5}{5} + \frac{4 \cdot 2^3}{3}\right) \approx 4.19 \text{ sq. units}$$

The surface we can see in the figure 3.3.

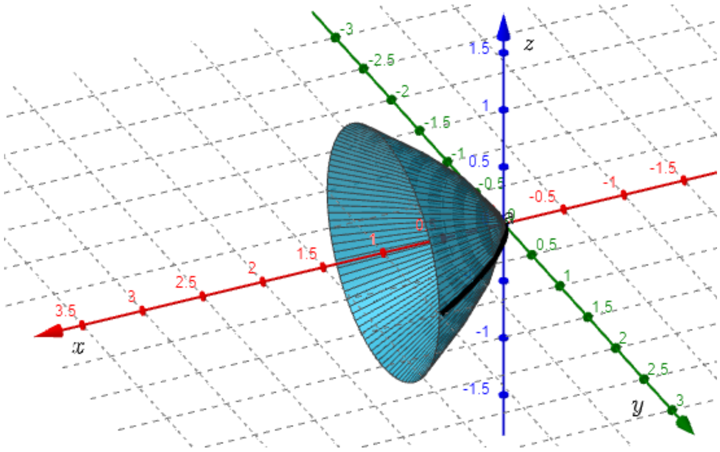

Figure 3.3

7.10.4 Exercises

1. Calculate the area of the surface obtained by the circle $x^2 + y^2 = 4$ rotating around the x-axis.
2. Calculate the area of the surface obtained by the arc of a function $y = e^{-x}$ about the interval $[0, 3]$ rotating around x-axis.
3. Find the area of the surface obtained by rotating the curve $y = \arccos x$ on the interval $[-1, 1]$ around the y–axis.
4. The arc of the cycloid rotates around the x-axis. Find the area of this surface if the parametric equations of the cycloid are

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

5. Find the area of the arc of astroid revolving around the y-axis

$$\begin{cases} x = 3\cos^3 t \\ y = 3\sin^3 t \end{cases}; \quad 0 \leq t \leq \frac{\pi}{4}$$

7.10.5 Solutions

1. Calculate the area of the surface obtained by the circle $x^2 + y^2 = 4$ rotating around the x-axis.

Solution

We construct the circle.

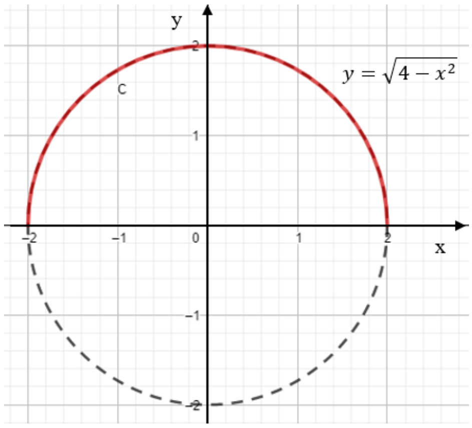

Figure 5.1

Let us choose the upper part of the circle line on the interval where $-2 \leq x \leq 2$ (see figure 5.1). The equation of this curve is $y = \sqrt{4 - x^2}$

This curve forms the sphere while the arc is rotating around the x-axis (see figure 5.2).

We differentiate the function and simplify the expression $1 + y'^2$

$$y' = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

$$1 + y'^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

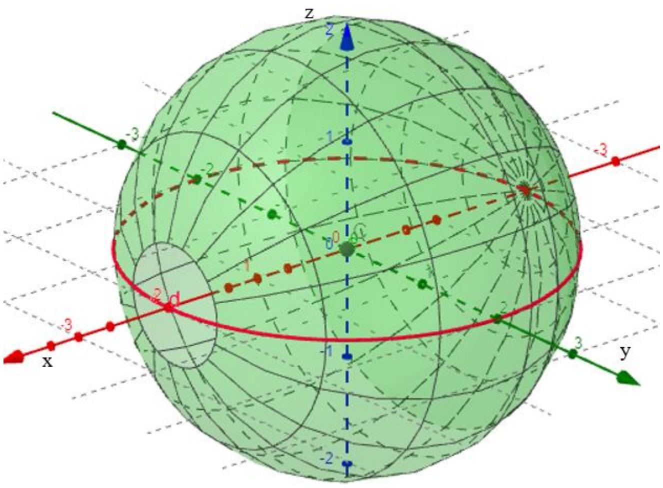

Figure 5.2

For calculation of a surface area of the sphere we apply the formula

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx}$$

The integral for a given function is

$$S = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx =$$

$$= 4\pi \int_{-2}^{2} dx = 4\pi x\Big|_{-2}^{2} = 16\pi \ \text{sq. units}$$

2. Calculate the area of the surface obtained by the arc of a function $y = e^{-x}$ about the interval $[0, 3]$ rotating around x-axis.

Solution

We construct the graph of a given function (see figure 5.3).

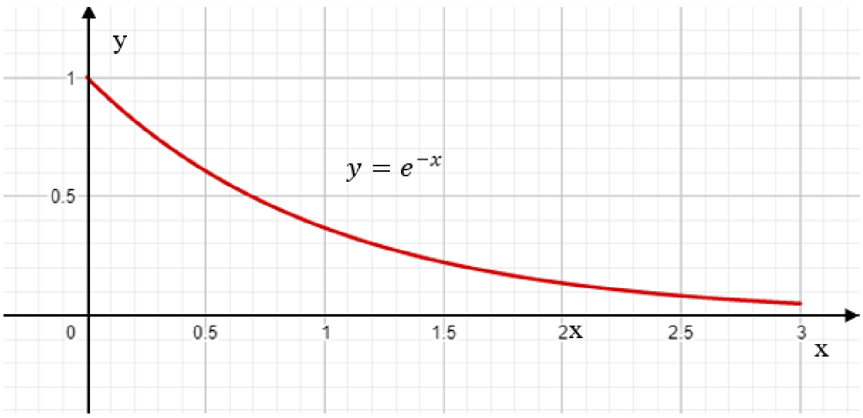

Figure 5.3

The derivative of the function $y = e^{-x}$ is

$$y' = (e^{-x})' = -e^{-x}$$

We compose the integral for calculation of the area of a revolution surface (see figure 5.4)

$$S = 2\pi \int_0^3 e^{-x}\sqrt{1+e^{-2x}}\,dx = \left| \begin{matrix} t = e^{-x} & dt = -e^{-x}dx \\ t_1 = e^0 = 1 & t_2 = e^{-3} \end{matrix} \right| =$$

$$= -2\pi \int_{1}^{e^{-3}} \sqrt{1+t^2}\, dt =$$

$$= -2\pi \cdot \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t + \sqrt{1+t^2}\right|\right)\Bigg|_{1}^{e^{-3}} =$$

$$= -\pi \left(e^{-3}\sqrt{1+e^{-6}} + ln\left|e^{-3} + \sqrt{1+e^{-6}}\right| - \sqrt{2} - ln\left|1+\sqrt{2}\right|\right)$$

$$\approx 6.9 \ \text{sq. units}$$

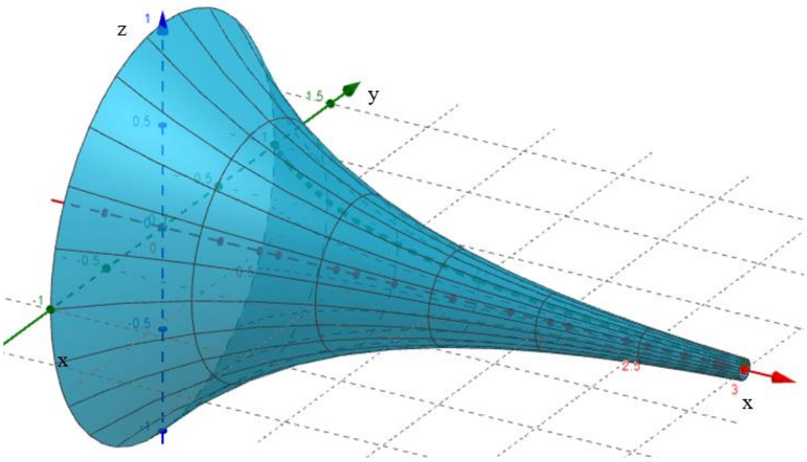

Figure 5.4

Comment. The integral

$$\int \sqrt{1+t^2}\, dt = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t + \sqrt{1+t^2}\right|\right) + C$$

can be solved using integration by parts (see Appendix at the end of this chapter).

3. Find the area of the surface obtained by rotating the curve $y = \arccos x$ on the interval$[-1, 1]$ around the y–axis.

Solution

The curve is given on the segment $[-1, 1]$ on x-axis (see figure 5.5). Let us find the projection of the curve onto the y-axis – it is the segment $AC = [0, \pi]$. We express the inverse function x with respect to the argument y and find the derivative

$$x = x(y) = \cos y$$

$$x' = -\sin y$$

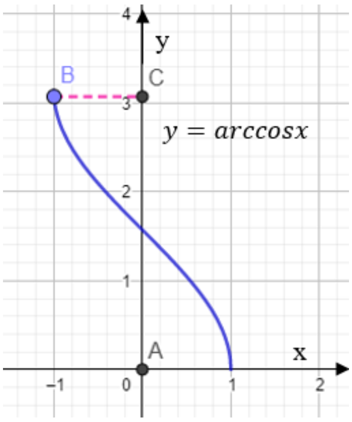

Figure 5.5

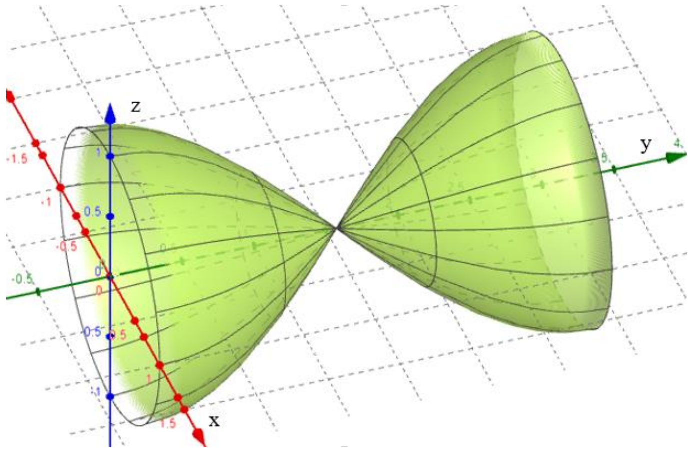

Figure 5.6

The curve rotating about the y-axis create symmetrical two-part surface (see figure 5.6). Therefore, we split the surface into two equal parts and calculate the surface area of one part, where the argument y changes from 0 to $\frac{\pi}{2}$. For calculation the surface area we use the formula

$$\boldsymbol{S = 2\pi \int_c^d g(y)\sqrt{1 + (g'(y))^2}\, dy}$$

The integral for calculation of the surface area of one part is

$$S_1 = 2\pi \int_0^{\frac{\pi}{2}} \cos y \sqrt{1 + \sin^2 y}\, dy =$$

$$= \left| \begin{matrix} t = \sin y; & dt = \cos y dy \\ t_1 = \sin 0 = 0; & t_2 = \sin \frac{\pi}{2} = 1 \end{matrix} \right| = 2\pi \int_0^1 \sqrt{1 + t^2} dt =$$

$$= \pi \left(t\sqrt{1 + t^2} + ln\left| t + \sqrt{1 + t^2} \right| \right) \Big|_0^1 =$$

$$= \pi\left(\sqrt{2} + ln\left|1 + \sqrt{2}\right|\right) \approx 10.34 \ \text{sq. units}$$

The total area of a given surface of revolution is

$$S = 2\pi\left(\sqrt{2} + ln\left|1 + \sqrt{2}\right|\right) \approx 20.68 \ \text{sq. units}$$

4. The arc of the cycloid rotates around the x-axis. Find the area of this surface if the parametric equations of the cycloid are

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

Solution

An arc of the cycloid is given if the range of a parameter $t \in [0, 2\pi]$ (see the arc AB in the figure 5.7).

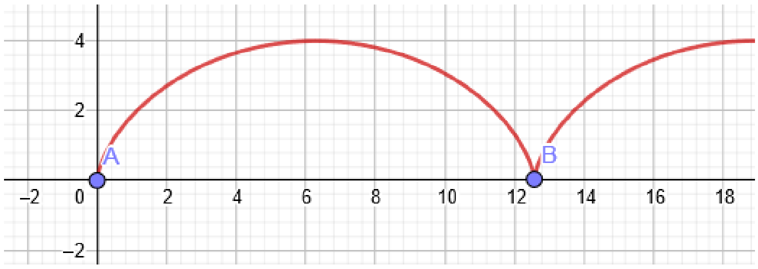

Figure 5.7

To calculate the surface of revolution (see figure 5.8) we apply the formula

$$\boldsymbol{S = 2\pi \int_{\alpha}^{\beta} y(t)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt}$$

At first we will calculate the derivatives

$$\dot{x} = 2(1 - \cos t)$$
$$\dot{y} = 2\sin t$$

Now we will simplify the expression applying algebra and trigonometry formulas

$$\dot{x}^2 + \dot{y}^2 = 4(1 - \cos t)^2 + 4\sin^2 t =$$

$$= 4(1 - 2\cos t + \cos^2 t + \sin^2 t) =$$

$$= 4(2 - 2\cos t) = 16\sin^2\frac{t}{2}$$

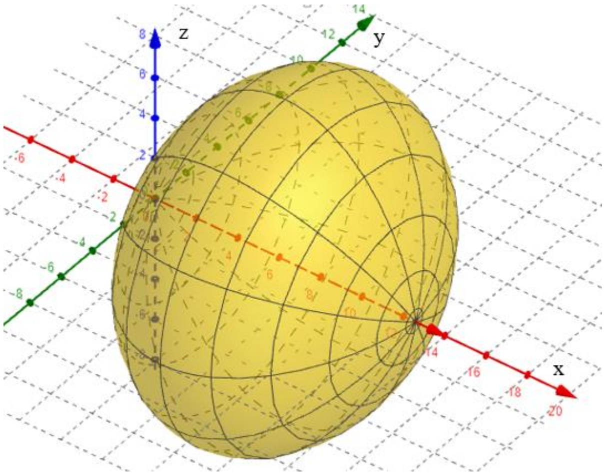

Figure 5.8

The surface area formed by the first arc of cycloid revolving around the x-axis is

$$S = 2\pi \int_0^{2\pi} 2(1 - \cos t)\sqrt{16\sin^2\frac{t}{2}}\,dt =$$

$$= 4\pi \int_0^{2\pi} 2\sin^2\frac{t}{2} \cdot 4\sin\frac{t}{2}\,dt =$$

$$= \left| \begin{matrix} u = \cos\frac{t}{2}; & du = -\frac{1}{2}\sin\frac{t}{2}\,dt \\ u_1 = \cos 0 = 1 & u_2 = \cos\pi = -1 \end{matrix} \right| =$$

$$= -64\pi \int_1^{-1} (1 - u^2)\,du = 64\pi\left(\frac{u^3}{3} - u\right)\Bigg|_1^{-1} =$$

$$= 64\pi \cdot \frac{4}{3} \approx 268.1 \ \text{sq. units}$$

5. Find the area of the arc of astroid revolving around the y-axis

$$\begin{cases} x = 3\cos^3 t \\ y = 3\sin^3 t \end{cases}; \quad 0 \le t \le \frac{\pi}{4}$$

Solution

Let us construct the astroid and show the given arc (see figure 5.9). We calculate the derivatives and simplify the expression to be written under the square root

$$\dot{x} = -9\cos^2 t \cdot \sin t$$
$$\dot{y} = 9\sin^2 t \cdot \cos t$$

$$\dot{x}^2 + \dot{y}^2 = 81\cos^4 t\sin^2 t + 81\sin^4 t\cos^2 t =$$

$$= 81\cos^2 t\sin^2 t(\sin^2 t + \cos^2 t) =$$

$$= 81\sin^2 t\cos^2 t$$

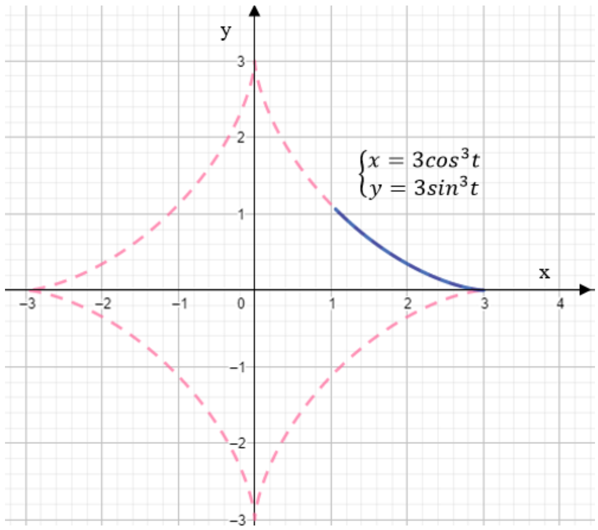

Figure 5.9

We create an integral to calculate the area of a surface (see figure 5.10)

$$S = 2\pi \int_0^{\frac{\pi}{4}} 3\cos^3 t\sqrt{81\sin^2 t\cos^2 t}\, dt =$$

$$= 2\pi \cdot 27 \int_0^{\frac{\pi}{4}} \cos^3 t \sin t \cos t\, dt =$$

$$= \left| \begin{matrix} u = \cos t & du = -\sin t dt \\ u_1 = \cos 0 = 1 & u_2 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{matrix} \right| =$$

$$= -54\pi \int_1^{\frac{\sqrt{2}}{2}} u^4\, du = -54\pi \frac{u^5}{5}\Bigg|_1^{\frac{\sqrt{2}}{2}} =$$

$$= -\frac{54\pi}{5}\left(\frac{\sqrt{2}}{8} - 1\right) \approx 14.05 \text{ sq. units}$$

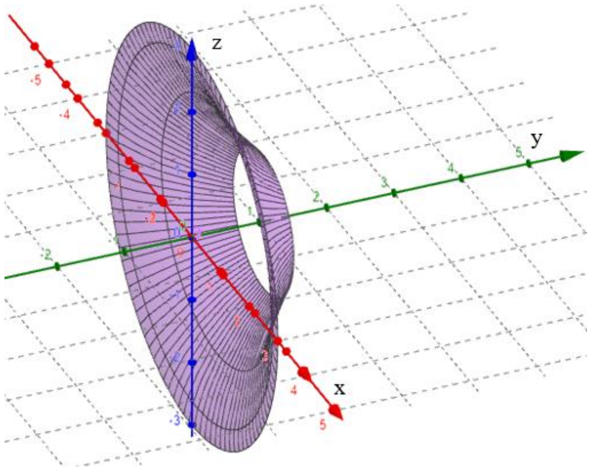

Figure 5.10

Appendix: calculation of integral

In the solution of exercises 2 and 3 we used a special formula

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t+\sqrt{1+t^2}\right|\right) + C$$

The integral can be evaluated by the method of integration by parts. Let us denote the given integral by Int and apply the method

$$Int = \int \sqrt{1+t^2}\,dt = \begin{vmatrix} u = \sqrt{1+t^2} & du = \dfrac{t}{\sqrt{1+t^2}}dt \\ dv = dt & v = t \end{vmatrix} =$$

$$= t\sqrt{1+t^2} - \int \frac{t\cdot t}{\sqrt{1+t^2}}dt = t\sqrt{1+t^2} - \int \frac{t^2+1-1}{\sqrt{1+t^2}}dt =$$

$$= t\sqrt{1+t^2} - \int \frac{t^2+1}{\sqrt{1+t^2}}dt + \int \frac{dt}{\sqrt{1+t^2}}dt =$$

$$= t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + ln\left|t+\sqrt{1+t^2}\right|$$

Looking at the given expression, its beginning and end, we have obtained the equation

$$Int = t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + ln\left|t+\sqrt{1+t^2}\right|$$

or

$$Int = t\sqrt{1+t^2} - Int + ln\left|t + \sqrt{1+t^2}\right|$$

We can express unknown Int from the equation

$$2Int = t\sqrt{1+t^2} + ln\left|t + \sqrt{1+t^2}\right|$$

$$Int = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t + \sqrt{1+t^2}\right|\right)$$

or

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + ln\left|t + \sqrt{1+t^2}\right|\right) + C$$

7.11 Application of definite integrals

Example 1. In a real situation, the movement of the ship at any time moment is affected by the energy of the water waves (heave, pitch and roll motions), the wind, as well as the speed of the ship itself. The resulting motion of a ship is sinusoidal. Suppose that the speed of a ship is given by the function $v(t) = \rho\cos\left(\frac{\pi}{2}t\right)$ knots per hour, where ρ is some specific constant. Compute the average speed of the ship between 13 hours.

Solution. The formula for calculation of the average value of a function $f(x)$ on the interval $[a,b]$ is

$$\boldsymbol{AVR = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

The given time interval is $t \in [0,13]$. We apply the formula to calculate the average speed of a ship

$$v_{avr} = \frac{1}{13}\int_0^{13} \rho\cos\left(\frac{\pi}{2}t\right)\,dt =$$

$$= \frac{\rho}{13}\cdot\frac{1}{\pi}\sin\left(\frac{\pi}{2}t\right)\Big|_0^{13} = \frac{\rho}{13\pi}\ \text{(kph)}$$

Answer. The average speed of a ship is $\frac{\rho}{13\pi}$ knots per hour.

Example 2. Root mean square voltage

Either DC or AC is used to operate different electric equipment. To determine the AC equivalent of DC in a circuit, the root mean square (RMS) voltage is calculated:

$$\boldsymbol{V_{RMS} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2\,dt}}$$

where $v(t)$ is a periodic function representing the character of the voltage over the period T. The RMS value of the AC voltage can be measured by the voltmeter.

Task. The voltage is defined as a saw-tooth function over the interval $[0,2\pi]$ with the peak voltage value $V_p = 170\,V$:

$$v(t) = \begin{cases} \frac{V_p}{\pi}t, & t \in (0,\pi) \\ -V_p + \frac{V_p}{\pi}(t-\pi), & t \in (\pi,2\pi) \end{cases}$$

Calculate the RMS value of the given function.

Solution

The graph of the function is presented in the figure 1.

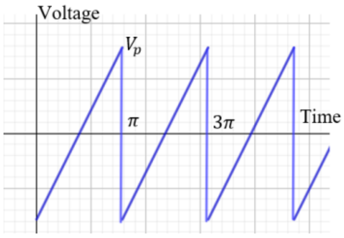

Figure 1

It can be assumed that function is symmetric according the origin of the coordinate system. It will be enough to calculate the voltage RMS in interval $[0, \pi]$

$$V_{RMS} = \sqrt{\frac{1}{\pi}\int_0^T \left(\frac{170}{\pi}t\right)^2 dt} =$$

$$= \sqrt{\frac{170^2}{\pi^3}\int_0^\pi t^2\, dt} =$$

$$= \sqrt{\frac{170^2}{\pi^3}\cdot\frac{t^3}{3}\bigg|_0^\pi} = 170\sqrt{\frac{\pi^3}{\pi^3\cdot 3}} =$$

$$= \frac{170}{\sqrt{3}} \approx 98.15\ V$$

Answer. The RMS voltage in the circuit is 98.15 volts.

Example 3. The cylindrical container with the lengths L must be covered by a metallic sheet of width L. How long must be the sheet if the cross section of the container can be described in polar coordinates as $r = 2(1 + cos\varphi)$?

Solution. To calculate the length of the metallic sheet it is necessary to find the length of the polar curve (see figure 2).

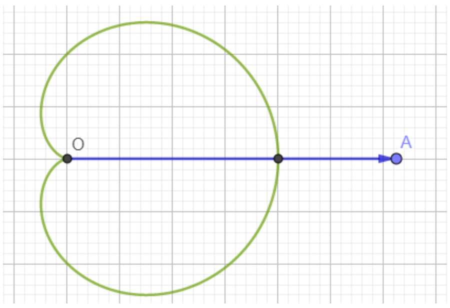

Figure 2

Let us apply the formula for calculation of the length of a curve

$$\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\varphi}$$

The half of the given curve is in the limits

$$0 \le \varphi \le \pi$$

The length of the whole curve

$$L = 2\int_0^{\pi} \sqrt{(2 + 2\cos\varphi)^2 + (2\sin\varphi)^2}\, d\varphi =$$

$$= 4\int_0^{\pi} \sqrt{1 + 2\cos\varphi + \cos^2\varphi + \sin^2\varphi}\, d\varphi =$$

$$= 4\int_0^{\pi} \sqrt{2 + 2\cos\varphi}\, d\varphi = 4\int_0^{\pi} \sqrt{4\cos^2\frac{\varphi}{2}}\, d\varphi =$$

$$= 8\int_0^{\pi} \cos\frac{\varphi}{2}\, d\varphi = 32\sin\frac{\varphi}{2}\Big|_0^{\pi} = 32$$

Answer. From the metallic sheet must be cut the 32 units long piece.

Example 4. The chain is hanging between two points in a distance 50 m. Such chain has a shape called the catenary. The catenary curve is described by the hyperbolic cosine function

$$f(x) = a\cosh\frac{x}{a}$$

The parameter a depends on the gravity, density of a chain, cross section of a thread, and tension forces. Calculate the length of given chain!

Solution. We can apply the formula of arc length

$$\boldsymbol{l = \int_a^b \sqrt{1 + y'^2} dx}$$

The derivative of hyperbolic cosine is

$$f'(x) = a\sinh\frac{x}{a}\cdot\frac{1}{a} = \sinh\frac{x}{a}$$

Let us construct a corresponding curve in the Cartesian coordinate system (see figure 3). The curve is symmetric with respect to the y-axis.

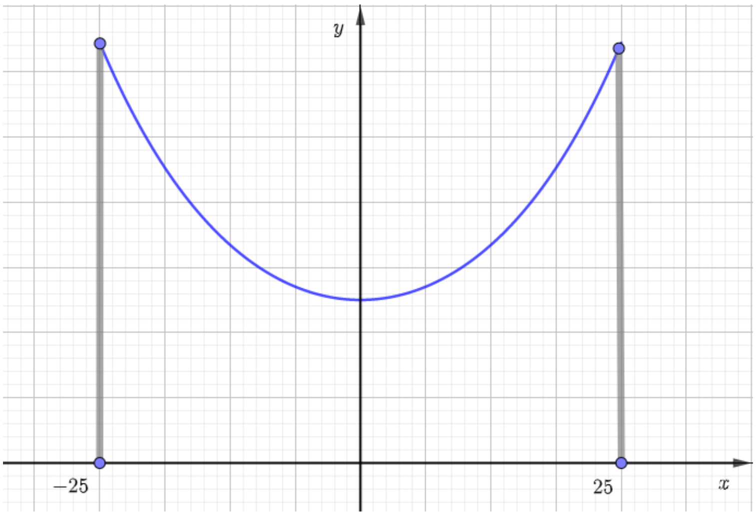

Figure 3

Hence the length of the chain is

$$l = 2\int_0^{25} \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx$$

$$= 2\int_0^{25} \sqrt{\cosh^2 \frac{x}{a}}\, dx =$$

$$= 2\int_0^{25} \cosh \frac{x}{a}\, dx =$$

$$= 2a \sinh \frac{x}{a} \Big|_0^{25} =$$

$$= 2a \sinh \frac{25}{a}$$

Answer. The chain is $2a \sinh \frac{25}{a}$ units long.

Example 5. The portholes are designed in such a way that when a ship cruises into the middle of the seas, one can get the best possible view from the ship's porthole. For this, the height of the portholes is strategically decided.

Task. A circular porthole on the vertical side of a ship has a radius 1 feet. If the centre of the porthole is 5 feet below the surface of the water, what is the fluid force on the window?

Solution. Cartesian coordinate system can be used for description of given situation. The centre of the porthole is 5 feet below the surface of the water (see figure 4).

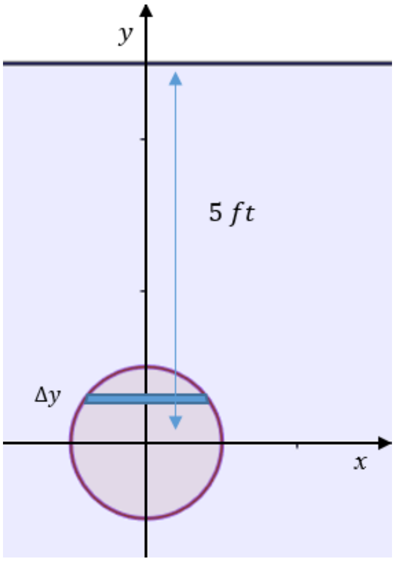

Figure 4

We describe the porthole by the equation where $R = 1\ ft$

$$x^2 + y^2 = R^2$$

We will construct horizontal slices, because the pressure is varying from the bottom of the porthole to the top. The height of one such slice is Δy (see figure 4). The length of the slice is expressed using the function $L(y)$ from the equation of a circle

$$L(y) = \sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right) = 2\sqrt{1-y^2}$$

The slice has the distance y from the x-axis and distance $h = 5 - y$ from the level of water. Density of sea water is $w = 63.93\ lbs/ft^3$. We apply the integral formula for fluid force on the porthole that is submerged vertically

$$\boldsymbol{F(y) = pressure \cdot area = (w \cdot depth) \cdot area = w \int_a^b hL(y)dy}$$

In the given case the integral is

$$F(y) = 63.93 \int_{-1}^{1} (5-y)2\sqrt{1-y^2}dy$$

We separate the integral in two parts

$$F(y) = 127.86 \int_{-1}^{1} 5 \cdot \sqrt{1-y^2}dy - 127.86 \int_{-1}^{1} y\sqrt{1-y^2}dy$$

The integrand of second integral is an odd function. Integral has symmetric limits thus the value of it is 0. We use trigonometric substitution in the first integral

$$\text{let } y = \sin u \text{ then } dy = \cos u\ du$$

New limits of integration are

$$y_1 = -1 \text{ then } \quad -1 = \text{sin}u; \quad u_1 = -\frac{\pi}{2}$$

Similarly

$$u_2 = \frac{\pi}{2}$$

We substitute

$$127.86 \int_{-1}^{1} 5 \cdot \sqrt{1-y^2}dy = 639.3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\, \text{cos}u\, du =$$

$$= 639.3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\, du = \frac{639.3}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2u) du =$$

$$= \frac{639.3}{2}\left(u - \frac{1}{2}\sin 2u\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 319.65\,\pi \quad lbs$$

Answer. The fluid force on the porthole is 319.65 π pound units of mass.

Example 6. Definite integrals are useful to describe a real situation. Many real applications begin with data that are not represented by a function but that are stored in a table. In this case, the definite integral used in the calculation formula is approximated by the Trapezoidal Rule:

$$\int_a^b \boldsymbol{f(x)\,dx \approx \frac{b-a}{2n}(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b))}$$

Given interval $[a, b]$ is divided in n equal parts to split the region under the graph of a function. Every part is replaced by a trapezoid (see figure 5).

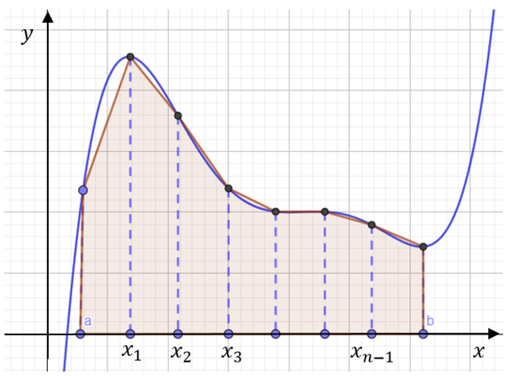

Figure 5

Task. A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown in a table. How many gallons were pumped during that hour?

Pumping rates

time	0	5	10	15	20	25	30	35	40	45	50	55	60
gallons	0	40	45	52	44	46	55	52	50	48	46	50	52

Solution. Let $R(t),\ \ 0 \le t \le 60$ be the pumping rate as a continuous function of time for the hour. We can partition the hour into short subintervals of length $\Delta t = 5$ on which the rate is nearly constant and form the sum

$$\sum_{i=0}^{12} R(t_i)\Delta t$$

as an approximation to the amount pumped during the hour. This reveals the integral formula for the number of gallons pumped to be

$$gallons = \int_0^{60} R(t)\,dt$$

We have no formula for $R(t)$ in this instance, but the 13 equally spaced values in table enable us to estimate the integral with the Trapezoidal Rule:

$$\int_0^{60} R(t)\,dt \approx \frac{5}{2}(0 + 2\cdot 40 + 2\cdot 45 + \cdots + 2\cdot 50 + 52) =$$

$$= \frac{5}{2}\cdot 2(40 + 45 + 52 + 44 + 46 + 55 + 52 + 50 + 48 + 46 + 50 + 26) =$$

$$= \ 5\cdot 554 = 2770$$

Answer. The total amount pumped during the hour is about 2770 gallons.

Example 7. According to the requirements of MARPOL on every vessel has been installed sewage treatment unit. Air compressors blow the air through the sewage continuously. Bio-active substances must be supplied in the unit periodically. The time period of supplement can be computed according to the mean decay time of sewage

$$\bar{t} = \int_0^{\infty} t\cdot ke^{-kt}dt,$$

where *k* is the constant that characterises the velocity of the decay. In the formula is used an improper integral (see the topic: Improper integrals).

Solution

$$\bar{t} = \int_0^{\infty} t\cdot ke^{-kt}dt = \lim_{T\to\infty}\int_0^{T} t\cdot ke^{-kt}dt =$$

$$= \left| \begin{array}{l} \text{let} \quad u = t; \qquad dv = ke^{-kt}dt \\ \text{then } du = dt; \;\; v = k\int\limits_0^T e^{-kt}dt = -\dfrac{ke^{-kt}}{k} \end{array} \right| =$$

$$= \lim_{T\to\infty}\left(-te^{-kt}\Big|_0^T + \int\limits_0^T e^{-kt}dt\right) =$$

$$= \lim_{T\to\infty}\left(Te^{-kT} - \frac{e^{-kt}}{k}\Bigg|_0^T\right) =$$

$$= \lim_{T\to\infty}\left(Te^{-kT} - \frac{e^{-kT}}{k} + \frac{1}{k}\right) = \frac{1}{k}$$

Answer. Expected decay time of sewage per time unit is $\bar{t} = \frac{1}{k}$.

Example 8. The blades of a bow thruster has a shape of polar rose with 4 petals. It is necessary to cover the blades with antifouling paint. For this purpose it is necessary to calculate the surface area of the bow thruster blades if the lengths of one blade is 75 cm.

Solution. The shape of the blades is given by the formula $r = 0.75\cos 2\varphi$ (see figure 6)

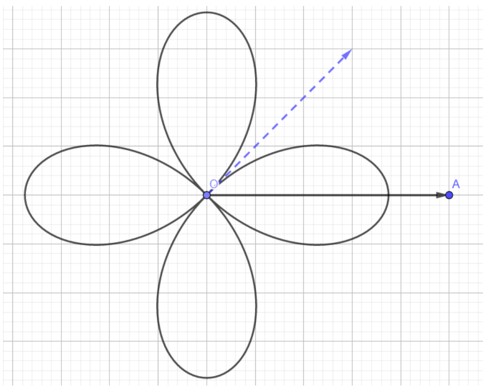

Figure 6

The polar rose is symmetric with respect to the polar axis OA and all petals are equal. Therefore it is enough to calculate a half of the area of one petal. We will calculate the angle between the polar axis and that ray where the value of the given function is zero:

$$0.75\cos 2\varphi = 0$$

$$\cos 2\varphi = 0;\; 2\varphi = \frac{\pi}{2};\;\; \varphi = \frac{\pi}{4}$$

The surface area of polar rose can be calculated in the following way

$$S = 4 \cdot 2 \cdot \frac{1}{2} \int_{0}^{\frac{\pi}{4}} (0.75\cos 2\varphi)^2 \, d\varphi =$$

$$= 4 \cdot \frac{9}{16} \int_{0}^{\frac{\pi}{4}} \cos^2 2\varphi \, d\varphi =$$

$$= \frac{9}{4} \int_{0}^{\frac{\pi}{4}} \frac{1 + \cos 4\varphi}{2} d\varphi =$$

$$= \frac{9}{8}\left(\varphi + \frac{1}{4}\sin 4\varphi\right)\Big|_{0}^{\frac{\pi}{4}} =$$

$$= \frac{9}{8}\left(\frac{\pi}{4} + 0\right) =$$

$$= \frac{9\pi}{32} \, (m^2) \approx 0.88 \, (m^2)$$

Answer. Surface area of one side of the blades of bow thruster is approximately 0.88 m^2.

Example 9. Determine the centre of mass of a wire of length L lying on the interval $[0, L]$ if the line density at point x is $\rho(x) = x$.

Solution. The centre of mass $\bar{x}$ can be calculated by a formula

$$\boldsymbol{\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_0^L x \cdot \rho(x) \, dx}{\int_0^L \rho(x) \, dx}}$$

where m is the mass of the wire and $M_{x=0}$ is the moment of the one-dimensional object around zero.

Given example presents the wire for that the mass is not distributed symmetrically. The density increases moving to the right along the wire.

The mass of wire is

$$m = \int_{0}^{L} x \, dx = \frac{x^2}{2}\Big|_{0}^{L} = \frac{L^2}{2}$$

The moment about the origin is

$$M_{x=0} = \int_{0}^{L} x^2 \, dx = \frac{x^3}{3}\Big|_{0}^{L} = \frac{L^3}{3}$$

We calculate the centre of mass

$$\bar{x} = \frac{L^3}{3} : \frac{L^2}{2} = \frac{2}{3} L$$

Answer. The centre of mass is located two-thirds of the way along the wire from the left end.

Example 10. Calculation of the centre of mass or centroid is one of the most important issues in the stability theory of ships. For instance, the centroid of the operating water plane is the point about which the ship will list and trim. This point is called the centre of flotation and it acts as a fulcrum or pivot point for a floating ship.

Task. Find the centroid of the half disc of a radius R.

Solution. Let us place the given half disc in the centre of coordinate system (see figure 7).

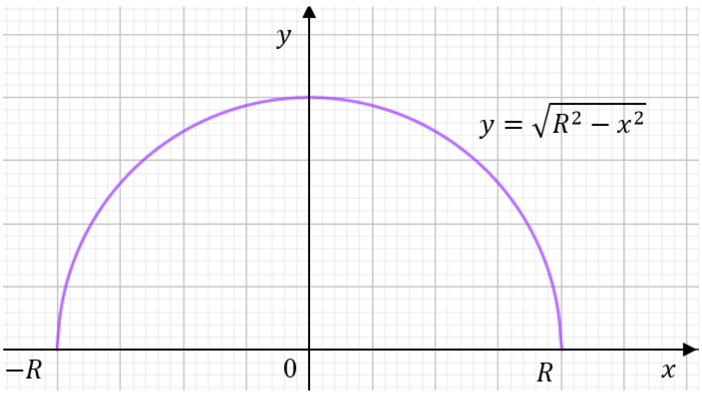

Figure 7

The coordinates of the centroid of the figure $0 \le y \le f(x)$ given over the interval $a \le x \le b$ can be calculated by the following formulas

$$\boldsymbol{\bar{x} = \frac{M_{x=0}}{S} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}}$$

$$\boldsymbol{\bar{y} = \frac{M_{y=0}}{S} = \frac{\frac{1}{2}\int_a^b (f(x))^2\, dx}{\int_a^b f(x)\, dx}}$$

By symmetry the x-coordinate of the centroid is $\bar{x} = 0$. The area of the half disc is $S = \frac{\pi R^2}{2}$. We calculate the y-coordinate

$$\bar{y} = \frac{M_{y=0}}{S} = \frac{\frac{1}{2}\int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx}{\frac{\pi R^2}{2}} =$$

$$= \frac{1}{\pi R^2} \int_{-R}^{R} (R^2 - x^2)\, dx =$$

$$= \frac{1}{\pi R^2} \left(R^2 x - \frac{x^3}{3} \right) \Bigg|_{-R}^{R} =$$

$$= \frac{1}{\pi R^2}\left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right) =$$

$$= \frac{4R^3}{3\pi R^2} = \frac{4R}{3\pi}$$

Answer. The centroid of a half disc is $\left(0,\ \frac{4R}{3\pi}\right)$.

Example 11. What is the surface area of an uncovered 10 meters long tank in the form of right cylinder and with the flat basis of radius 2 meters? How many kilograms of paint would be needed to paint the outside of the tank? One kilogram of paint will cover 10 square meters.

Solution. Let us put the tank in the coordinate system. Let the axis of symmetry of the tank coincides with x-axis (see figure 8). The surface of revolution is created by the straight line $y = 2$ rotating around the x-axis about the interval $[0, 10]$.

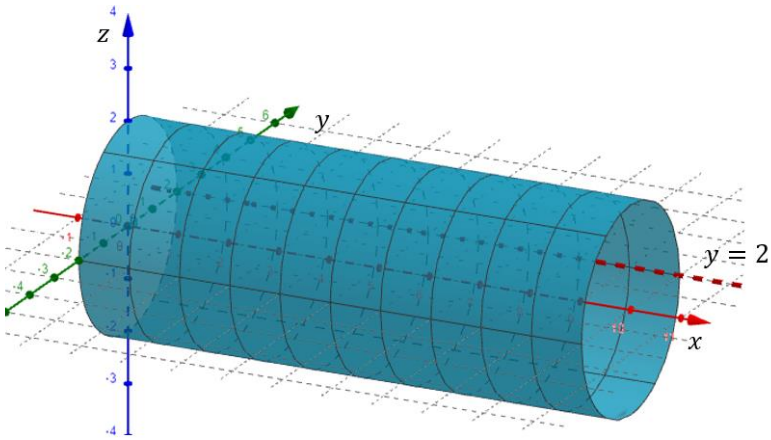

Figure 8

The formula for calculation of the surface area is

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

According to this formula the surface area of the cylinder is

$$S = 2\pi \int_0^{10} 2\sqrt{1 + (2')^2} dx = 4\pi \int_0^{10} dx = 40\pi\ (m^2)$$

The area of both basis we will calculate as the area of a circle

$$S_2 = 2 \cdot \pi R^2 = 8\pi\ (m^2)$$

The amount A of the paint to cover whole tank is

$$A = (40\pi + 8\pi):10 = 4.8\pi \approx 15.07\ (kg)$$

Answer. To paint the tank it is necessary to use 15. 07 kg of paint.

Example 12. It is necessary to detect the weight of a homogeneous iron part that is turned in the shape of ellipsoid. The axes of the part are 16, 6, and 6 centimetres long.

Solution. According to the fact that two axes of the given ellipsoid are identical, a given part can be considered as a 3-dimensional shape formed by an ellipse rotating around its longest axis. Let us apply the canonical equation of the ellipse (see figure 9)

$$\frac{x^2}{64} + \frac{y^2}{9} = 1$$

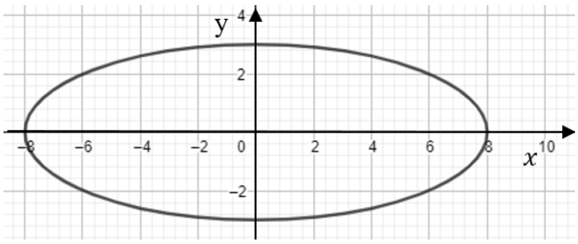

Figure 9

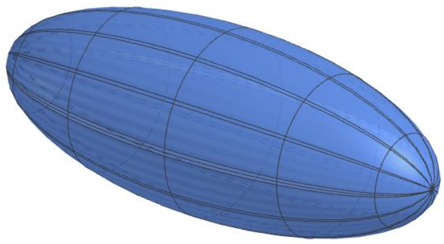

Figure 10

The mass of the solid of revolution given on the interval $[a, b]$ with the density $\rho(x)$ (see figure 10) we can calculate by the formula

$$\boldsymbol{m = \pi \int_a^b \rho(x)\, y^2 dx}$$

The homogeneous iron part has a constant density $\rho = 7.2\ g/cm^3$. Hence, the mas of the given part is

$$m = \pi \int_{-8}^{8} 7.2 \cdot 9 \left(1 - \frac{x^2}{64}\right) dx =$$

$$= 2\pi \cdot 64.8 \int_0^8 \left(1 - \frac{x^2}{64}\right) dx =$$

$$= 129.6\pi \left(x - \frac{x^3}{192}\right)\Bigg|_0^8 =$$

$$= 129.6\pi \left(8 - \frac{8}{3}\right) =$$

$$= 691.2\pi \approx 2171.47\ (g)$$

Answer. The part weighs 2171.47 grams.

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Example 13. A tanker is a specialized vessel for the transportation of liquid cargo and liquefied gases at very low temperatures. Tankers include gas carriers - for the carriage of liquefied gases and chemical carriers - for the carriage of liquid chemical cargo. By design, these are single-deck vessels, in which the cargo tanks are tanks or barrels. Some barrels have the form of sphere (see figures 11, 12).

Figure 11

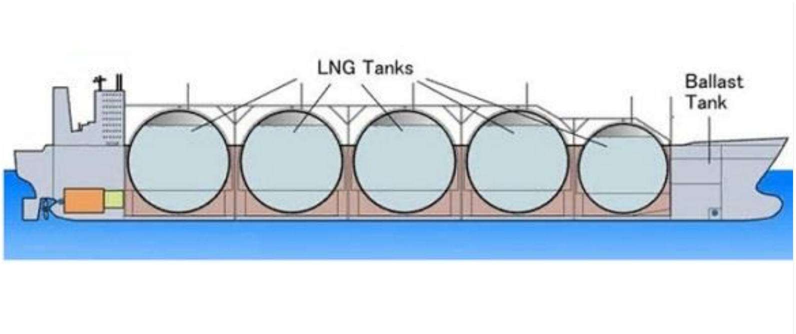

Figure 12

Task. Let the radius of a spherical tank is R and the part of the tank not filled with liquefied gas is a spherical segment with a height h. Find the volume of the liquid gas in the tank.

Solution. The volume of the liquefied gas in the tanks can be calculated as a volume of the solid of revolution.

We should find the volume of the solid formed by rotating curve $x^2 + y^2 = R^2 \quad (x \geq 0)$ between $y = -R$ and $y = R - h$ about the y-axis (see figure 13).

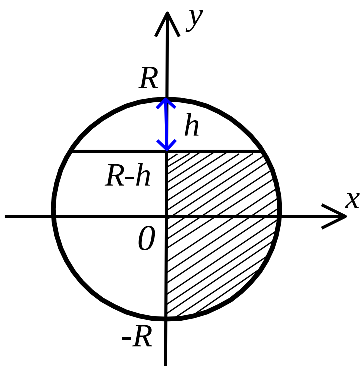

Figure 13

We use the formula

$$\boldsymbol{V_y = \pi \int_c^d \left(x(y)\right)^2 dy}$$

From the equation $x^2 + y^2 = R^2$ we express $x^2 = R^2 - y^2$.

Then

$$V_y = \pi \int_{-R}^{R-h} (R^2 - y^2)dy =$$

$$= \pi \left(R^2 y - \frac{y^3}{3} \right)\Bigg|_{-R}^{R-h} =$$

$$= \pi \left(R^2(R-h) - \frac{(R-h)^3}{3} \right) - \pi \left(-R^3 + \frac{R^3}{3} \right) =$$

$$= \pi \left(R^3 - R^2 h - \frac{R^3 - 3R^2 h + 3Rh^2 - h^3}{3} + \frac{2R^3}{3} \right) =$$

$$= \pi \left(\frac{4R^3}{3} - Rh^2 + \frac{h^3}{3} \right)$$

Or

$$V_{gas} = \pi \frac{4R^3}{3} - \pi h^2 \left(R - \frac{h}{3} \right)$$

Answer. The Volume of a gas in the spherical tank is $\pi \frac{4R^3}{3} - \pi h^2 \left(R - \frac{h}{3} \right)$ units of volume.

Example 14. The volume of tanks in which crude oil and oil products are transported to the port by rail (see figure 14) can also be calculated as the volume of a body of revolution.

Figure 14

Tanks with many different head types are manufactured for specific application. The heads can be flat, hemispherical, semi-ellipsoidal, thorispherical, and of other types. The tank head is often convex on one end and concave on the other end.

Task. Let the tank can be described as a cylinder of length $L+2a$ and radius R, and the ends of cylinder there are ellipsoidal (see figure 15). Calculate the volume of the tank!

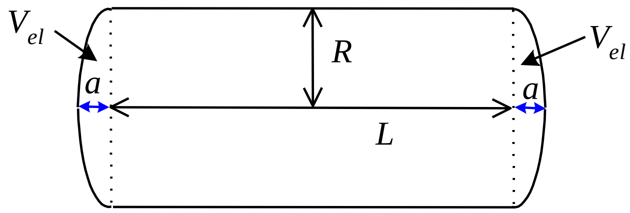

Figure 15

Solution. The volume of tank is equal

$$\mathrm{V} = \mathrm{V_{cylinder}} + 2V_{el}$$

$$\mathrm{V_{cylinder}} = \pi R^2 L$$

The volume of the head of a tank V_{el} can be find as the volume of the solid formed by rotating the area bounded by quarter of ellipse $\frac{x^2}{a^2}+\frac{y^2}{R^2}=1$ and $x=0$ between $x=0$ and $x=a$ about the x-axis (see figure 16).

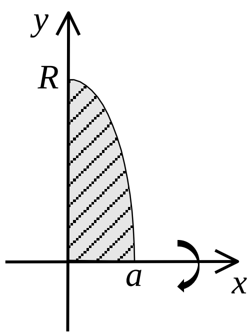

Figure 16

We can use the formula

$$\boldsymbol{V_x = \pi \int_a^b \left(y(x)\right)^2 dx}$$

Let us express the term y^2 from the equation $\frac{x^2}{a^2}+\frac{y^2}{R^2}=1$

$$y^2 = \mathrm{R}^2\left(1-\frac{x^2}{a^2}\right)$$

Then

$$\mathrm{V_{el}} = \pi \int_0^a \mathrm{R}^2\left(1-\frac{x^2}{a^2}\right) dx =$$

$$= \pi R^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx =$$

$$= \pi R^2 \left(\mathrm{x} - \frac{x^3}{3a^2}\right)\Bigg|_0^a =$$

$$= \pi R^2 \left(\mathrm{a} - \frac{a^3}{3a^2}\right) =$$

$$= \pi R^2 \left(\mathrm{a} - \frac{\mathrm{a}}{3}\right) =$$

$$= \frac{2\pi R^2 a}{3}$$

As a result, the volume of tank is equal

$$\mathrm{V} = \pi R^2 L + \frac{4\pi R^2 a}{3} = \pi R^2 \left(L + \frac{4a}{3}\right)$$

Answer. The volume of the tank is $\pi R^2 \left(L + \frac{4a}{3}\right)$ units of volume.

Example 15. Suppose that a water tank is shaped like a right circular truncated cone with the smaller base at the bottom, and has height 8 meters and radius 3 meters at the top, and radius 1 meter at the bottom. If the tank is full, how much work is required to pump all the water out over the top?

Solution. The work can be calculated by the formula:

$$\boldsymbol{W = \int_a^b F(x)\, dx}$$

where the continuous function $F(x)$ represents the force moving an object along x-axis about the interval $[a, b]$.

Let us construct the cross-section of the tank (see figure 17)

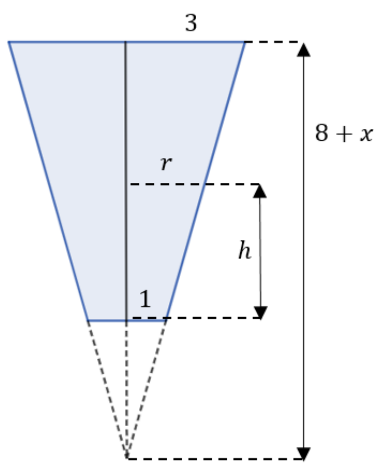

Figure 17

A thin disc of water at the height h above the bottom of the tank has the radius r. We will express radius in the terms of h by similar triangles. Let the height of whole cone is $8+x$ meters long (see figure 18). Then

$$\frac{x}{1}=\frac{x+h}{r}=\frac{8+x}{3}$$

We get $x=4$ and $r=\frac{4+h}{4}$

The volume of the disc is

$$dV=\pi r^2 dh=\pi\frac{(4+h)^2}{16}dh$$

The weight or the force of gravity on the mass of water in the disc is

$$dF=\rho g\,\pi\frac{(4+h)^2}{16}dh$$

Here ρ is the density of a water in kilograms per cubic meter (approximately $\rho\approx 1000$) and g is the constant of gravity. The water in disc must be raised a distance $8-h$ meters. The work required to do this

$$dW=\rho g\,\pi\frac{(4+h)^2}{16}(8-h)dh$$

The total work we calculate in the following way

$$W=\int_0^8 \rho g\,\pi\frac{(4+h)^2}{16}(8-h)dh=$$

$$=\frac{\rho g\,\pi}{16}\int_0^8 (4+h)^2(8-h)\,dh=$$

$$=\frac{\rho g\,\pi}{16}\int_0^8 (128+48h-h^3)\,dh=$$

$$=\frac{\rho g\,\pi}{16}\left(128h+24h^2-\frac{h^4}{4}\right)\Bigg|_0^8=$$

$$=\frac{\rho g\,\pi}{16}\cdot 1536\approx$$

$$\approx\frac{1000\cdot 9.8\cdot 3.14\cdot 1536}{16}\approx$$

$$\approx 19.7\cdot 10^5\ \ (\mathrm{N}\cdot\mathrm{m})$$

Answer. The work required to pump out the water from the tank is $19{,}7\cdot 10^5$ Newton-meters.

7.12 Worksheet of self-control

Application of basic formulas of indefinite integrals

Identify the appropriate answer/answers for every integral and fill in the formula applied

Nr	Exercise	Identificator of correct answer	Formula
1	$\int 4x^3\,dx$		
2	$2\int \left(\sqrt{x}+\sin x\right)dx$		
3	$\int \frac{5}{9+x^2}\,dx$		
4	$\int \left(6^x+\frac{2}{x}\right)dx$		
5	$\int x^4\cdot\sqrt[3]{x^4}\,dx$		
6	$\frac{1}{\sqrt{3}}\int \frac{3dx}{\sin^2 x}$		
7	$\int \sqrt{\frac{8}{x^2-8}}\,dx$		
8	$\int 7\cosh x\,dx$		
9	$\int \frac{3^x}{3}\,dx$		
10	$\int \frac{6dx}{11\sqrt{36-x^2}}$		

Possible answers

A. $\frac{1}{3}\cdot 3^x\cdot\frac{1}{\ln 3}+C$; B. $-\sqrt{3}\cot x+C$; C. $5\arctan x+C$; D. x^4+C; E. $7\sinh x+C$;

F. $\frac{1}{\sqrt{x}}-2\cos x+C$; G. $2\sqrt{2}\ln\left|x+\sqrt{x^2-8}\right|+C$; H. $1\frac{2}{3}\arctan\frac{x}{3}+C$;I. $\frac{6}{11}\arcsin\frac{x}{6}+C$;

J. $\frac{4}{3}x^{3/2}+2(-\cos x)+C$; K. $\frac{x^5}{5}\cdot\frac{\left(\sqrt[3]{x}\right)^5}{5}+C$; L. $\frac{6^x}{\ln 6}+2\ln x+C$;M. $2\left(\frac{2\sqrt{x^3}}{3}-\cos x\right)+C$;

N. $\frac{3^x}{\ln 27}+C$; O. $12x^2+C$; P. $\frac{3}{19}\cdot x^{\frac{19}{3}}+C$; Q. $\frac{3}{\sqrt{3}}(-\cot x)+C$; R. $3x^6\cdot\frac{\sqrt[3]{x}}{19}+C$

Answers

No	Exercise	Identificator of correct answer	Formula
1	$\int 4x^3\,dx$	D	$\int x^n dx = \frac{x^n}{n+1} + C$
2	$2\int (\sqrt{x} + \sin x)\,dx$	J, M	$\int x^n dx = \frac{x^n}{n+1} + C$; $\int \sin x dx = -\cos x + C$
3	$\int \frac{5}{9+x^2}\,dx$	H	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$
4	$\int \left(6^x + \frac{2}{x}\right) dx$	L	$\int a^x dx = \frac{a^x}{lna} + C$; $\int \frac{dx}{x} = \ln x + C$
5	$\int x^4 \cdot \sqrt[3]{x^4} dx$	P, R	$\int x^n dx = \frac{x^n}{n+1} + C$
6	$\frac{1}{\sqrt{3}} \int \frac{3dx}{\sin^2 x}$	B, Q	$\int \frac{dx}{\sin^2 x} = -\cot x + C$
7	$\int \sqrt{\frac{8}{x^2-8}} dx$	G	$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$
8	$\int 7\cosh x\,dx$	E	$\int \cosh x\,dx = \sinh x + C$
9	$\int \frac{3^x}{3} dx$	A, N	$\int a^x dx = \frac{a^x}{\ln a} + C$
10	$\int \frac{6dx}{11\sqrt{36-x^2}}$	I	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

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7.13 Worksheet of self-control

Test about the differentials

Part I Calculate the given differentials and mark the correct answer!

1.	$d(\sin x)$	A	$(\cos x)'dx$	B	$\cos x$	C	$\cos x\,dx$	D	$\sin 2x\,dx$
2.	$d(4x^3-8x)$	A	$3\cdot 4(x^2-8)\,dx$	B	$(12x^2-8)\,dx$	C	$(x^4-8)\,dx$	D	$4(3x^2-2)dx$
3.	$d(5^x)$	A	$5^x \ln 5\,dx$	B	$x5^{x-1}\,dx$	C	$\frac{5^x}{5}\,dx$	D	$5\,dx$
4.	$d(\cos 3x)$	A	$3\cos x\,dx$	B	$-3\sin x\,dx$	C	$-\sin 3x\,dx$	D	$-3\sin 3x\,dx$
5.	$d(\mathrm{artctan}x)$	A	$\frac{dx}{1+x^2}$	B	$\frac{1}{cos^2x}dx$	C	$\frac{1}{1+x}dx$	D	$\frac{dx}{1+\cos x^2}$

Part II Convert the given expression into the differential of a function. Mark the correct answer!

1.	$\frac{12}{\sqrt{1-x^2}}dx$	A	$d\left(\sqrt{1-x^2}\right)$	B	$d\left(24\sqrt{1+x^2}\right)$	C	$12d(\arcsin x)$	D	$d\left(\frac{\arcsin x}{12}\right)$
2.	$16x^3dx$	A	$d(4x^4)$	B	$16d(x^3)$	C	$d(48x^2)$	D	$d(8x^4)$
3.	$\frac{21\,dx}{\cos^2 7x}$	A	$21d(\tan x)$	B	$d(3\tan 7x)$	C	$3d(\tan^2 7x)$	D	$d\left(\frac{7}{\cos^3 x}\right)$
4.	$\frac{dx}{\sqrt{x}}$	A	$d(2\sqrt{x})$	B	$d(x^{1/2})$	C	$\frac{3}{2}d\left(\frac{1}{\sqrt{x^3}}\right)$	D	$\frac{1}{2}d(x)$
5.	$\frac{\ln^4 x}{x}dx$	A	$d\left(\frac{\ln^5 x}{x}\right)$	B	$d(4\ln x)$	C	$d(5\ln^5 x)$	D	$\frac{1}{5}d(\ln^5 x)$

Answers

Part I

1 – C; 2 – B; 3 – A; 4 – D; 5 – A

Part II

1 – C; 2 – A; 3 – B; 4 – A; 5 – D

Solution

Part I

Apply the formula $\boldsymbol{dy = y'\,dx}$ $if\ \ y = y(x)$

Example 1

$$d(\sin x) = (\sin x)'dx = \cos x\ dx$$

Example 2

$$d(4x^3 - 8x) = (4x^3 - 8x)'dx = (12x^2 - 8)\ dx$$

Example 3

$$d(5^x) = (5^x)'dx = 5^x \ln 5 dx$$

Example 4

$$d(\cos 3x) = (\cos 3x)'dx = -\sin 3x \cdot 3\ dx = -3\sin 3x\ dx$$

Example 5

$$d(\arctan x) = (\arctan x)'dx = \frac{1}{1+x^2}dx$$

Part II

Apply the formula $\boldsymbol{y'\,dx = dy}$ $if\ \ y = y(x)$

Example 1

$$\frac{12}{\sqrt{1-x^2}}dx = 12(\arcsin x)'dx = 12\ d(\arcsin x)$$

Example 2

$$16x^3dx = 4(4x^3)dx = 4(x^4)'\ dx = d(4x^4)$$

Example 3

$$\frac{21\ dx}{\cos^2 7x} = 3(\tan 7x)'dx = d(3\tan 7x)$$

Example 4

$$\frac{dx}{\sqrt{x}} = \frac{2dx}{2\sqrt{x}} = 2\left(\sqrt{x}\right)'\ dx = d\left(2\sqrt{x}\right)$$

Example 5

$$\frac{\ln^4 x}{x}dx = \frac{1}{5}\cdot\frac{5\ln^4 x}{x}dx = \frac{1}{5}(\ln^5 x)'dx = \frac{1}{5}d(\ln^5 x)$$

7.14 Worksheet of self-control

Integration by parts

$$\int u\,dv = u \cdot v - \int v\,du$$

1) For given integrals **choose** the functions for $\boldsymbol{u}$-substitution:

x; $8x$; $2x+1$; x^2; $\sin x$; $\cos x$; $\cos^2 6x$; e^x; $\ln x$; $\lg x$; $\arcsin x$; $\arctan x$

2) **Solve** following integrals **1; 3; 4; 6**

Nr	Integral	$\boldsymbol{u}$-function
1	$\int x^2 \cos x\, dx$	$u = x^2$
2	$\int 8x \sin x\, dx$	
3	$\int x \arctan x\, dx$	
4	$\int (2x+1)e^x dx$	
5	$\int 8x \ln x\, dx$	
6	$\int \frac{\lg x}{x^2}\, dx$	
7	$\int e^x \sin x\, dx$	
8	$\int \frac{6x}{\cos^2 6x} dx$	
9	$\int (2x+1)\ln x\, dx$	
10	$\int \arcsin x\, dx$	
11	$\int x^2 \ln x\, dx$	

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Answers

1)

Nr	Integral	$\boldsymbol{u}$-function
1	$\int x^2\cos x\ dx$	$u = x^2$
2	$\int 8x\ \sin x\ dx$	$u = 8x$
3	$\int x\ \arctan x\ dx$	$u = \arctan x$
4	$\int (2x+1)e^x dx$	$u = 2x + 1$
5	$\int 8x\ \ln x\ dx$	$u = \ln x$
6	$\int \frac{\lg x}{x^2}\ dx$	$u = \lg x$
7	$\int e^x\sin x\ dx$	$u = \sin x\ or\ u = e^x$
8	$\int \frac{6x}{\cos^2 6x}dx$	$u = x$
9	$\int (2x+1)\ln x\ dx$	$u = \ln x$
10	$\int \arcsin x\ dx$	$u = \arcsin x$
11	$\int x^2\ln x\ dx$	$u = \lg x$

2) Solutions

$$\mathbf{1.}\int x^2\cos x\ dx = \left|\begin{matrix} u = x^2 & du = 2xdx \\ dv = \cos x dx & v = \int \cos x\ dx = \sin x\end{matrix}\right| =$$

$$= x^2\sin x - \int 2x\sin x\ dx = \left|\begin{matrix} u = 2x & du = 2dx \\ dv = \sin x dx & v = \int \sin x dx = -\cos x\end{matrix}\right| =$$

$$= x^2\sin x - \left(-2x\cos x + 2\int \cos x dx\right) =$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + C$$

3. $$\int x \arctan x \, dx = \left| \begin{array}{ll} u = \arctan x & du = \dfrac{dx}{1+x^2} \\ dv = x dx & v = \displaystyle\int x dx = \dfrac{x^2}{2} \end{array} \right| =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{1+x^2} dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2} =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C$$

4. $$\int (2x+1)e^x dx = \left| \begin{array}{ll} u = 2x+1 & du = 2dx \\ dv = e^x dx & v = \displaystyle\int e^x dx = e^x \end{array} \right| =$$

$$= (2x+1)e^x - 2 \int e^x dx =$$

$$= (2x+1)e^x - 2e^x + C$$

6. $$\int \frac{\lg x}{x^2} dx = \left| \begin{array}{ll} u = \lg x & du = \dfrac{1}{x \ln 10} dx \\ dv = \dfrac{dx}{x^2} & v = \displaystyle\int \dfrac{dx}{x^2} = \dfrac{-1}{x} \end{array} \right| =$$

$$= -\frac{1}{x} \lg x + \frac{1}{\ln 10} \int \frac{1}{x} \cdot \frac{1}{x} dx =$$

$$= -\frac{1}{x} \lg x - \frac{1}{x \ln 10} + C$$

7.15 Worksheet of self-control

Test about the integrals of composite functions

The argument of the composite functions is a linear function

$$\int \boldsymbol{f(ax+b)dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\int \boldsymbol{f(ax+b)d(ax+b)} = \frac{\mathbf{1}}{\boldsymbol{a}}\boldsymbol{F(ax+b)+C}$$

Mark the correct answer

		A	B	C	D
1	$\int e^{3x}\,dx$	$\frac{1}{3}e^{3x}+C$	$3e^{3x}+C$	$e^{3x}+C$	$3e^{x}+C$
2	$\int sin\frac{x}{4}dx$	$4sin\frac{x}{4}+C$	4cosx	$-\frac{1}{4}sin\frac{x}{4}+C$	$-4cos\frac{x}{4}+C$
3	$\int (x+10)^5\,dx$	$5(x+10)^4+C$	$\frac{(x+10)^6}{6}+C$	$(x+10)^6+C$	$\frac{1}{5}(x+10)^5+C$
4	$\int \frac{2dx}{x-7}$	$(x-7)^2+C$	$\frac{-(x-7)^2}{2}+C$	$2ln\lvert x-7\rvert+C$	$2(x-7)+C$
5	$\int \frac{4dx}{cos^2(8x+\pi)}$	$4sin^2(8x+\pi)+C$	$\frac{32}{sin^2(8x+\pi)}+C$	$\frac{4}{8}tan(8x+\pi)+C$	$\frac{1}{2}tan^2(8x+\pi)+C$
6	$\int \frac{dx}{7x+1}$	$7ln\lvert x+1\rvert+C$	$\frac{1}{7}ln\lvert 7x+1\rvert+C$	$\frac{(7x+1)^{-2}}{-2}+C$	$ln\lvert 7x+1\rvert+C$
7	$\int \frac{dx}{16x^2+1}$	$\frac{1}{4}arctan4x+C$	$\frac{1}{8}ln\left\lvert\frac{4x+1}{4x-1}\right\rvert+C$	$arctan\frac{4x}{16}+C$	$\frac{1}{16}ln\left\lvert\frac{x+0.25}{x-0.251}\right\rvert+C$

8 $\int 5cos6x\,dx$

A $30cosx + C$

B $\frac{5}{6}cos6x + C$

C $5sin6x + C$

D $-5cos6x + C$

9 $\int 2^{2-4x}\,dx$

A $-4 \cdot 2^{2-4x} + C$

B $-4 \cdot 2^{2-4x} ln2$

C $(2-4x)2^{1-4x} + C$

D $-\frac{2^{2-4x}}{4ln2} + C$

10 $\int \frac{dx}{(12-x)^{10}}$

A $\frac{-11}{(12-x)^{11}} + C$

B $\frac{-1}{9(12-x)^9} + C$

C $\frac{(12-x)^{-10}}{-10} + C$

D $10(12-x)^9 + C$

11 $\int \frac{11dx}{\sqrt{2-9x}}$

A $\frac{22\sqrt{2-9x}}{-9} + C$

B $\frac{-99}{2(2-9x)^2} + C$

C $11\sqrt{2-9x} + C$

D $\frac{-11}{9}(2-9x)^{1/2} + C$

12 $\int \frac{dx}{1-x}$

A $ln|1-x| + C$

B $ln|x-1| + C$

C $-ln|1-x| + C$

D $-ln|x+1| + C$

Answers

1 – A; 2 – D; 3 – B; 4 – C; 5 – C; 6 – B; 7 – A; 8 – B; 9 – D; 10 – B; 11 – A; 12 – C

7.16 Worksheet of self-control

Puzzle – Quote by mathematician

Solve the following tasks. The sequence of the answers reveals a famous quote by Leopold Kronecker.

1 the integers	2 the work	3 made	4 Pythagoras
5 the rhombus	6 of	7 else	8 zero
9 human	10 triangle	11 God	12 reflects
13 infinity	14 all	15 logic	16 man
17 is true	18 the numbers	19 belief	20 the purpose
21 his	22 random	23 constructions	24 is

1. Calculate $$99\int_{-2}^{-1}(3+2x)^8dx$$	2. Detect the exponent n of the expression $$\frac{x^4\cdot\sqrt{x^3}}{x^{0.5}\cdot x^2}=x^n$$
3. Calculate the lower limit a of the integral $$\int_a^{e^3}\frac{12\,dx}{x}=36$$	4. Calculate $$\int_0^{\pi/6}28\,cosx\,dx$$
5. Calculate the value of the differential if $x=5$ and $\Delta x=0.1$ $$d(x^3-5x+1)$$	6. Find the value of a multiplier C $$C\int_0^{\pi/4}\frac{dx}{2cos^2x}=12$$
7. What is the constant A for the trigonometric identity? $$sin^2x=\frac{1-cos2x}{A}$$	8. Calculate $$42\int_{\pi/4}^{\pi/2}\frac{cos^6x}{sin^8x}dx$$
9. Calculate $$\int_1^3\frac{3x^3-x^2-x+3}{x+1}dx$$	

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