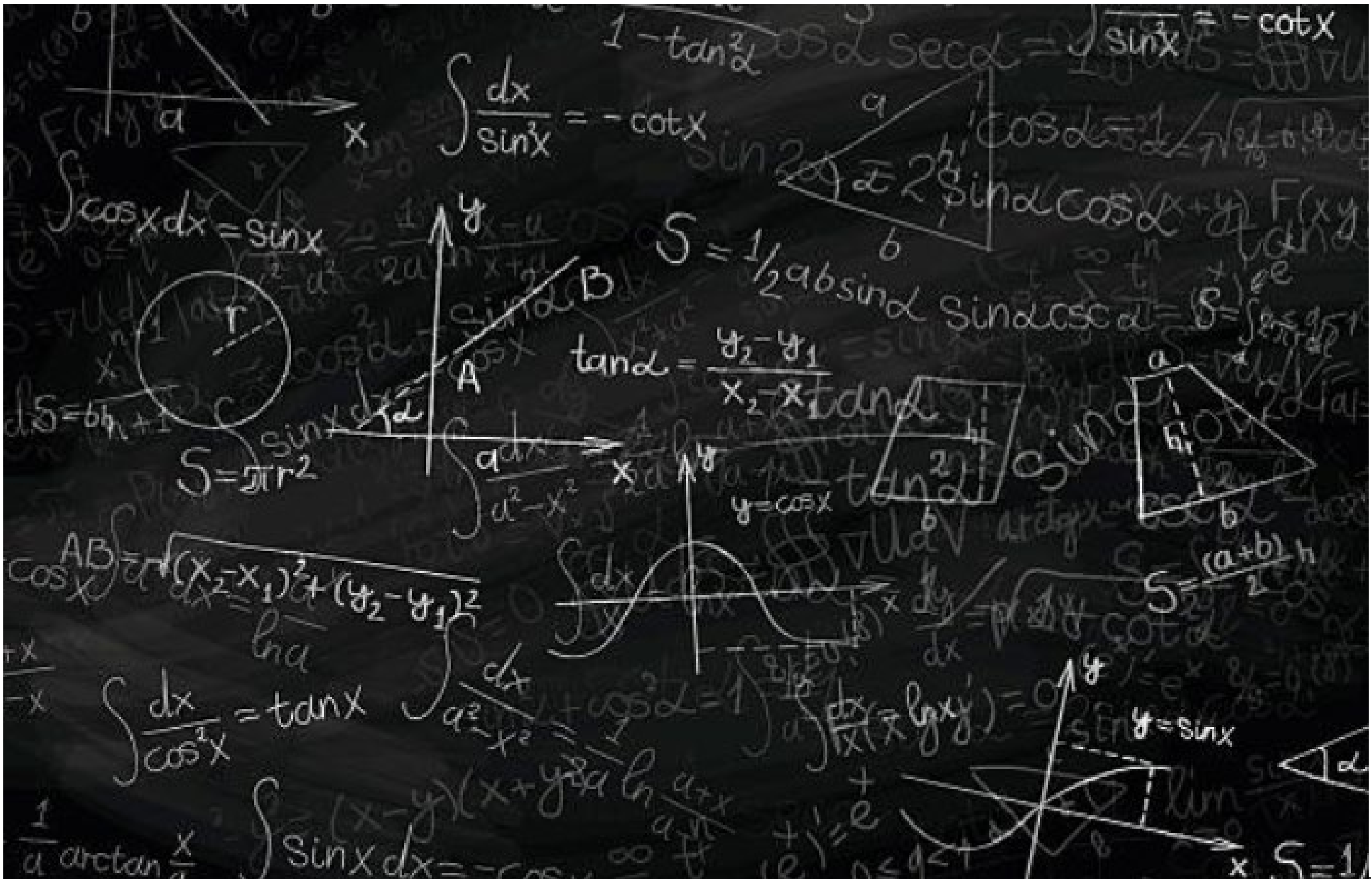
MARE Mathematics

Innovative Approach in Mathematical Education for Maritime Students



Teacher's Manual

Complex Numbers

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MareMathics

Innovative Approach in Mathematical Education for Maritime Students

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https://marematics.pfst.hr/

Manual for teachers

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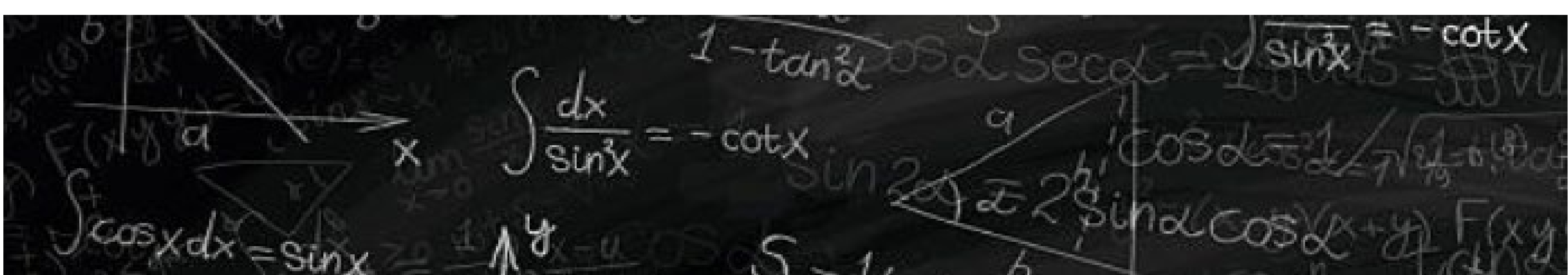
Reviewed by Toni Milun, Croatia



MARE Mathics

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The Manual is the outcome of the collaborative work of all the Partners for the development of the MareMathics Project.

Partners in the project:

University of Split Faculty of Maritime Studies
CROATIA

TTÜ ESTONIAN MARITIME ACADEMY
Tallinn University of Technology, Estonian Maritime Academy
ESTONIA

LJA
Latvian Maritime Academy
LATVIA

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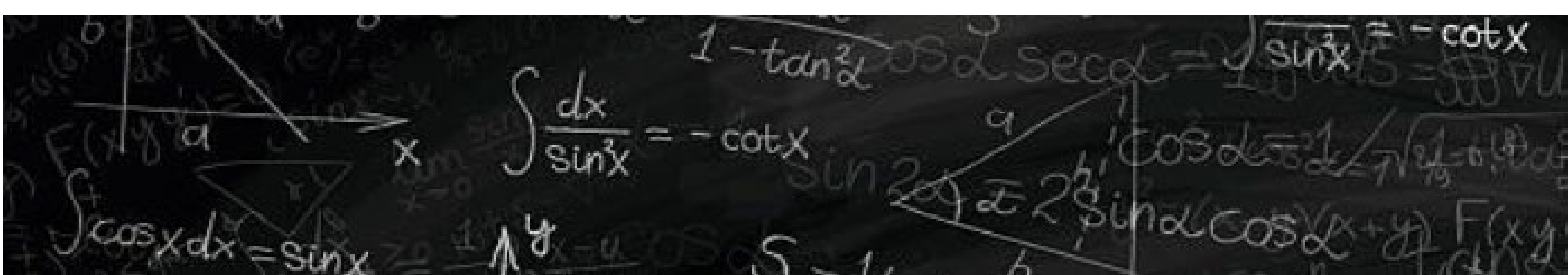

CONTENTS

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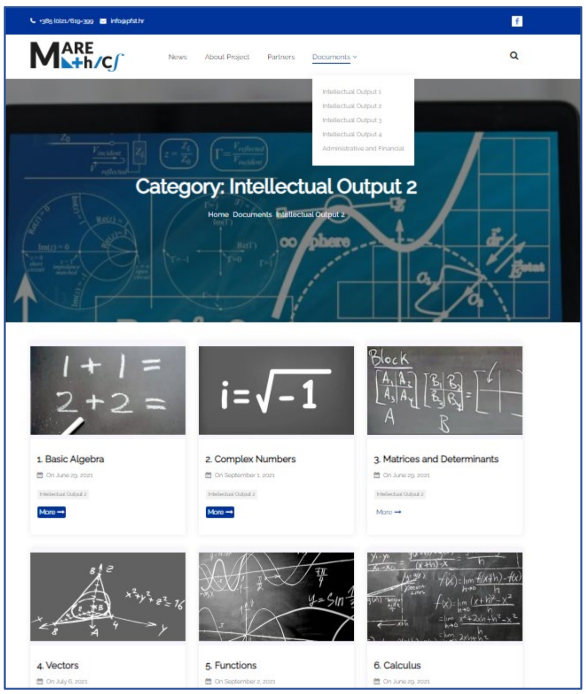
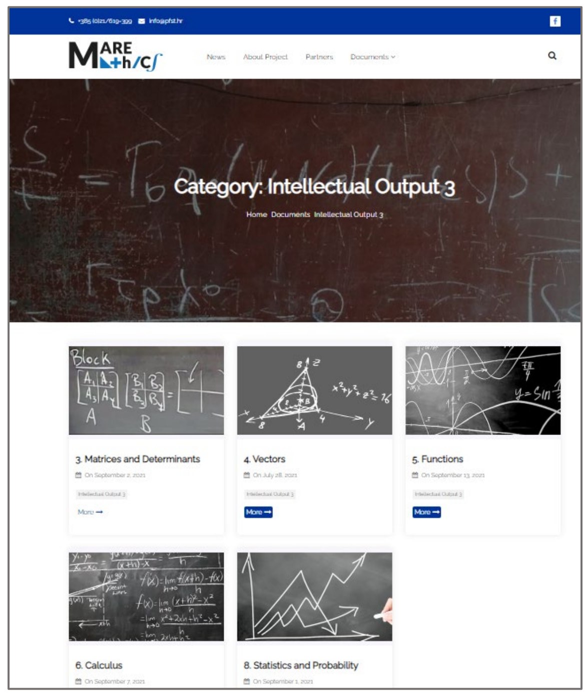
COMPLEX NUMBERS: Teaching and learning plan.

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The resources are picked from project ***MareMathics*** and available on the https://marematics.pfst.hr/.





Lesson. Complex numbers

Name of Unit	Workload	Handbook
Complex numbers	**Lecture + exercise: 90 min**	**Unit 1 Complex numbers**

DETAILED DESCRIPTION

In this lesson we are learning about complex numbers. Complex numbers are needed to solve equations like $x^2 = -4$.

Learning Outcomes:

Introducing concept of complex numbers

1) Showing students operations with complex numbers
2) Introducing algebraic and polar forms
3) Students know how to do operations with complex numbers

Key words of this Unit:

Imaginary unit, the complex plane, addition and subtraction, absolute value, multiplication, angles and polar coordinates, conjugation and division.

Previous knowledge of mathematics:

Polar coordinates, basic trigonometric formulas and angles

Relatedness with solving problems in the maritime field

Concept of phasors and alternating current, components of AC circuit

MareMathics Teacher Toolkit and Digital Resources:

- Video
- Presentation
- Quiz
- Worksheet
- GeoGebra Intellectual Output 4 – MareMathics (pfst.hr)

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LESSON FLOW

	Time	Sequence	Content	Teacher activities	Student activities	Points of discussion
I.	5 min	Introduction	Pre teaching	Motivation	Active listening and contributing to questions	Type of numbers previously known
II.	5 min	Presentation	Video	Frontal	Active listening and contributing to questions	
III.	30 min	Teaching	presentarion	Frontal . Moderator		
IV.	35 min	Worksheet	Solving exercises	Explain task and supports	Complete the worksheet	
V.	10 min	Quiz	Doing a quiz	Moderator		
VI.	5 min	Summary	Post teaching Giving homework.	Frontal	Active listening	Read material about use of complex numbers.

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Resources	• **Whiteboard** • Lesson (unit 1 and unit 2) https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-2-Complex-Numbers-1.pdf • Video https://marematics.pfst.hr/?p=3762 • https://livettu-my.sharepoint.com/:w:/g/personal/julia_tammela_ttu_ee/ETaLVVY2F_dBpw-mIe4zMFMBQKar5hilZoo4BJhpznsHJg?e=BRGXNq **Authors: MareMathics** • https://quizizz.com/admin/quiz/621b858d35ecf8001db2e91a?source=quiz_page • Geogebra https://www.geogebra.org/m/zjm4w3zn, **Authors: MareMathics**

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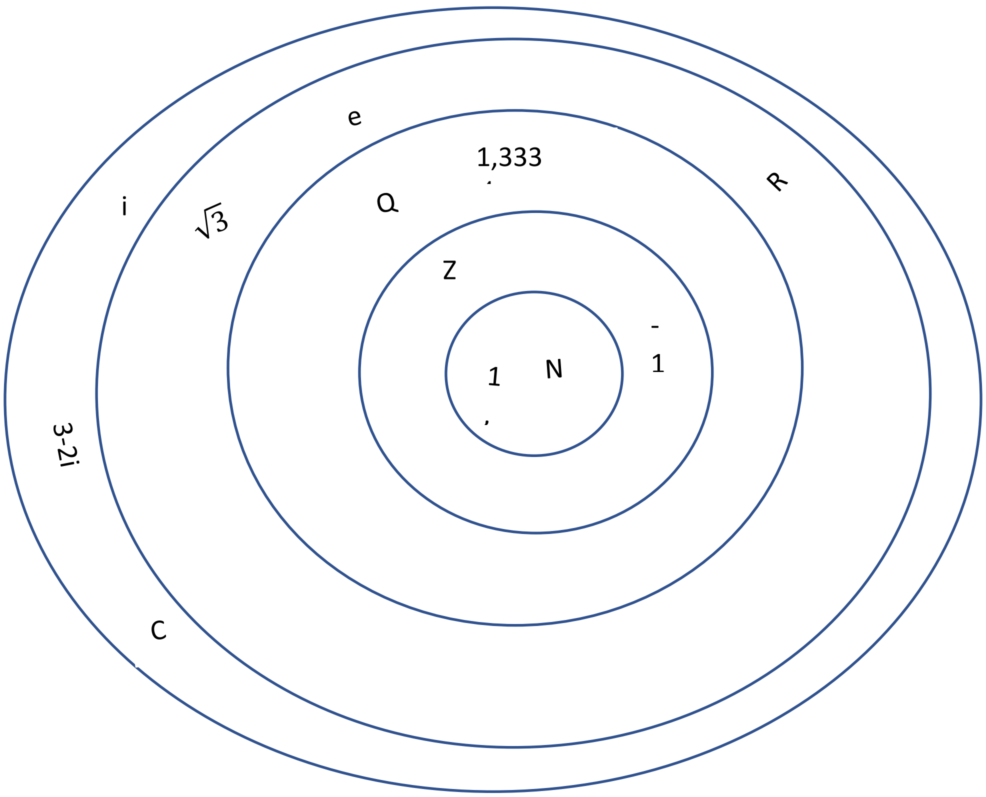
I. Introduction

Teacher appoints topic of the lesson. Initiates discussion what type of numbers students previously knows.

Students help teacher in creating a scheme while remembering all the things they have previously learned.



II. Presentation video

Teacher shows students video introducing complex numbers and answers questions if needed.

Students watch video and ask questions. How can we find $\sqrt{-9}$?

https://maremathics.pfst.hr/?p=3762#complex_numbers

III. Teaching

Teacher provides students new material and examples. https://maremathics.pfst.hr/?p=106

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Students listen to teacher explanation and write down new material. Ask additional questions if needed.

1) The imaginary unit *i*

The imaginary unit is defined as $i = \sqrt{-1}$, where $i^2 = -1$.

2) Complex Numbers and Imaginary Numbers

The set of all numbers in the form *a* + *bi* with real numbers *a* and *b* and *i* the imaginary unit, is called the set of complex numbers. The real number *a* is called the real part and the real number *b* is called the imaginary part of the complex number *a* + *bi*. If $b \neq 0$ then the complex number is called an imaginary number. An imaginary number in the form *bi* is called a pure imaginary number.

3) Operations with Complex Numbers

1) $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2) $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3) $(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$

EXAMPLE 1:

1) $(2 + 5i) + (3 - 9i) = 2 + 5i + 3 - 9i = (2 + 3i) + (5i - 9i) = 5 - 4i$
2) $(1 + 3i) - (7 - 2i) = 1 + 3i - 7 + 2i = (1 - 7) + (3i + 2i) = -6 + 5i$
3) $(1 - 2i)(3 + 4i) = 3 + 4i - 6i - 8i^2 = 3 + 4i - 6i + 8 = 11 - 2i$

4) Complex Conjugates and Division

The complex conjugate of the number *a* + *bi* is *a* - *bi* and the complex conjugate of *a* – *bi* is *a* + *bi*. The multiplication of complex conjugates gives a real number.

$(a + bi)(a - bi) = a^2 + b^2$

$(a - bi)(a + bi) = a^2 + b^2$

Complex conjugates are used to divide complex numbers. The goal of the division procedure is to obtain a real number in the denominator. This real number becomes the denominator of *a* and *b* in the quotient *a* + *bi*. By multiplying the numerator and the denominator of the division by the complex conjugate of the denominator, you will obtain this real number in the denominator.

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$
$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$$

$c^2+d^2\neq 0$

EXAMPLE 2:

1) $\frac{2-5i}{3+i}=\frac{(2-5i)(3-i)}{(3+i)(3-i)}=\frac{6-2i-15i+5i^2}{9-i^2}=\frac{6-2i-15i-5}{9+1}=\frac{1-17i}{10}$

2) $\frac{1+4i}{5-2i}=\frac{(1+4i)(5+2i)}{(5-2i)(5+2i)}=\frac{5+2i+20i+8i^2}{25-4i^2}=\frac{5+2i+20i-8}{25+4}=\frac{-3+22i}{29}$

5) Polar Form of a Complex Number

The complex number *z* = *a* + *bi* is written in polar form as $z=r(\cos\varphi+i\sin\varphi)$

where $a=r\cos\varphi$, $b=r\sin\varphi, r=\sqrt{a^2+b^2}$ and

$$\varphi=\begin{cases}\boldsymbol{2\pi-arctan\left|\frac{b}{a}\right|,\ if\ a>0,\ b<0}\\ \boldsymbol{arctan\left|\frac{b}{a}\right|,\ if\ a>0, b\geq 0}\\ \boldsymbol{\pi-arctan\frac{b}{a},\ if\ a<0,\ b>0}\\ \boldsymbol{\pi+arctan\frac{b}{a},\ if\ a<0, b<0}\\ \boldsymbol{\frac{\pi}{2}, if\ b>0,\ a=0}\\ \boldsymbol{\frac{3\pi}{2}, if\ b<0,\ a=0}\end{cases}$$

The value of *r* is called the modulus of the complex number and the angle ϕ is called the argument of the complex number z with $0\leq\varphi<2\pi$.

EXAMPLE 3: Write z =1 + $\sqrt{3}i$ in polar form. a =1, b = $\sqrt{3}$

1) $r=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$

2) a > 0, b > 0, $\varphi=arctan\left|\frac{b}{a}\right|=arctan\left|\frac{\sqrt{3}}{1}\right|=\frac{\pi}{3}$

3) z = 2(cos $\frac{\pi}{3}$ + *i*sin$\frac{\pi}{3}$)

6) Product of Two Complex Numbers in Polar Form

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Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ be two complex numbers in polar form. Their product, $z_1 z_2$ is $z_1 z_2 = r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$

To multiply two complex numbers, multiply moduli and add arguments.

EXAMPLE 4: Find $z_1 z_2$, if $z_1 = 4(\cos 30° + i\sin 30°)$ and $z_2 = 2(\cos 60° + i\sin 60°)$.

$$z_1 z_2 = 4(\cos 30° + i\sin 30°) \cdot 2(\cos 60° + i\sin 60°) = 4 \cdot 2[\cos(30° + 60°) + i\sin(30° + 60°)] = 8(\cos 90° + i\sin 90°)$$

7) Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ be two complex numbers in polar form. Their quotient, $\frac{z_1}{z_2}$ is $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)]$

To divide two complex numbers, divide moduli and subtract arguments.

EXAMPLE 5: Find $\frac{z_1}{z_2}$, if $z_1 = 10(\cos 58° + i\sin 58°)$ and $z_2 = 2(\cos 30° + i\sin 30°)$.

$$\frac{z_1}{z_2} = \frac{10(\cos 58° + i\sin 58°)}{2(\cos 30° + i\sin 30°)} = 5[\cos(58° - 30°) + i\sin(58° - 30°)] = 5(\cos 28° + i\sin 28°)$$

IV. GeoGebra

Teacher shows students applet introducing complex numbers and answers questions if needed.

V. Worksheet

Teacher gives students worksheet with exercises and explains the assignment. Asks some students to go solving exercises on a whiteboard. It helps to get some feedback if the new material was understood.

Students listens when teacher explains the assignment and start working on a worksheet. Some students solve exercises on the whiteboard while others work in their notebooks.

VI. Quiz

Teacher gives students instructions where to find a quiz and helps if someone have problems opening the link.

Students follow instructions and work on the quiz.

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VII. Summary

Teacher gives students some feedback about the quiz were overall results good or not. Concludes what new were learned today and gives homework. Asks students to get acquainted with the material about use of complex numbers in maritime studies.

https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-2-Complex-Numbers-2.pdf H

Students listen to the teacher and give an opinion whether this topic was hard or not. Write down homework.

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Appendix worksheet

Exercise 1

Add and write the result in standard form.

$a)$ $(3+5i)+(4+6i)$ $b)$ $(-4+6i)-(-7+5i)$

$c)$ $(-0{,}2-1{,}1i)+(-0{,}8-1{,}9i)$ $d)(134-2{,}5i)-(13-0{,}5i)$

Exercise 2

Add or subtract as indicated and write the result in standard form.

$a)$ $(1+i)+(2-3i)-(3+4i)$ $b)$ $(0{,}4-4{,}2i)-(1{,}5+0{,}6i)+3{,}3i$

$c)$ $(12-23\ i)+(23-34\ i)-(34\ 56\ i)$ $d)$ $[0{,}(3)+1{,}1(6)i]-[0{,}1(3)-0{,}(2)i]$

Exercise 3

Find each product and write the result in standard form.

$a)$ $(3+2i)(4-5i)$ $b)$ $(5-6i)(1-3i)$ $c)$ $(1-i)(1+i)$

$d)$ $(1-i)(3+4i)$ $e)$ $(-5i-4)(3-i)$ $f)$ $(2-2i)(4i+5)$

Exercise 4

Find each product and write the result in standard form.

$a)$ $(1+2\sqrt{3}\ i)(2-3\sqrt{3}\ i)$ $b)$ $2i(1-\sqrt{3}\ i)(1+\sqrt{3}\ i)$

$c)$ $(6-7i)(5+5)(3-5i)$ $d)$ $2i(7+10i)(2-4i)$

$e)$ $(2-3i)(-1-i)(3+4i)$ $f)$ $(5+4i)(-2-i)(5-4i)(-2+1)$

Exercise 5

Divide and express the result in standard form.

$a)11 + i$ $b)$ $3 + i3 - i$ $c)$ $2i - 31 - 3i$ $d)$ $3 - 5i2 + 3i$

$e)\ 1+\sqrt{3}\ i1-\sqrt{3}\ i\ f)1+\sqrt{15}\ i1-\sqrt{3}\ i\ g)\ \sqrt{6}-i\sqrt{6}\ -2i\ h)\ 1+2i1+\sqrt{2}i$

Exercise 6

Write the complex number in polar form.You may express the argument in

degrees or radians.

a) 1 b) 3i c) -2i d) -i

e) 6i f) -2 g) i h) -5i

Exercise 7

Write the complex number in polar form.You may express the argument in

degrees or radians.

a) 3 + i b) -3 - i c) 6 + 6i d) 6 - 6i

e) -6 + 8i f) 2,7 - 3,2i g) 1,8 + 0,52i

h) 2,7 -1,32i

Exercise 8

Find the product and quotient of the complex numbers. Leave answers in polar form.

a) z = 4(cos 70° + isin 70°) w = 2(cos 40° + isin 40°)

b) z = 8(cos 80° + isin 20°) w = 4(cos 80° + isin 20°)

c) z = 14(cos 3π/2 + isin3π/2) w = 7(cos 5π4 + isin 5π/4)

d) z = 15(cos 4π/3 + isin4π/3) w = 5[(cos