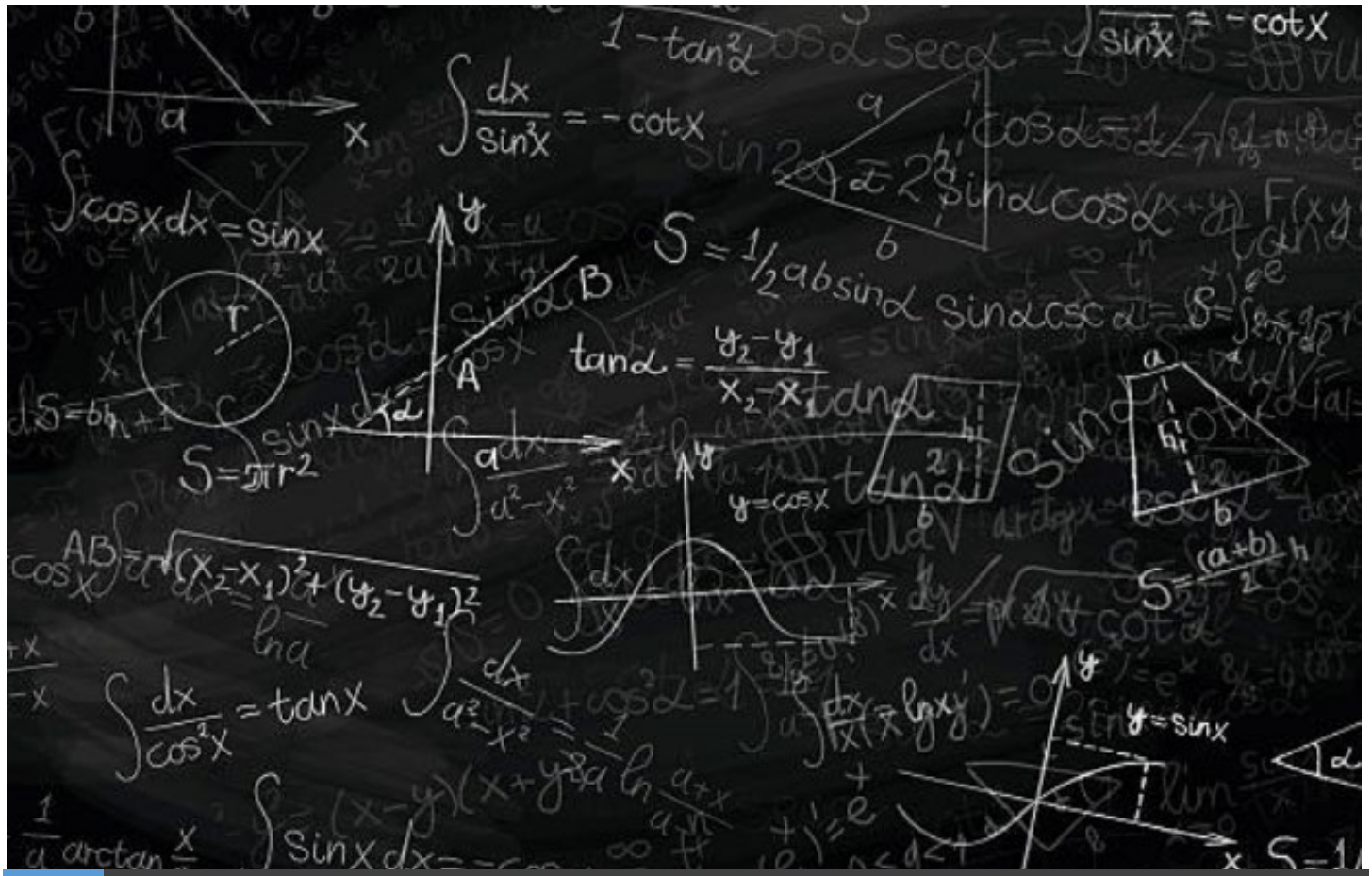
MARE Math/CS

Innovative Approach in Mathematical Education for Maritime Students



Teacher's Manual

Differential equations

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MareMathics

Innovative Approach in Mathematical Education for Maritime Students

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https://maremathics.pfst.hr/

Manual for teachers

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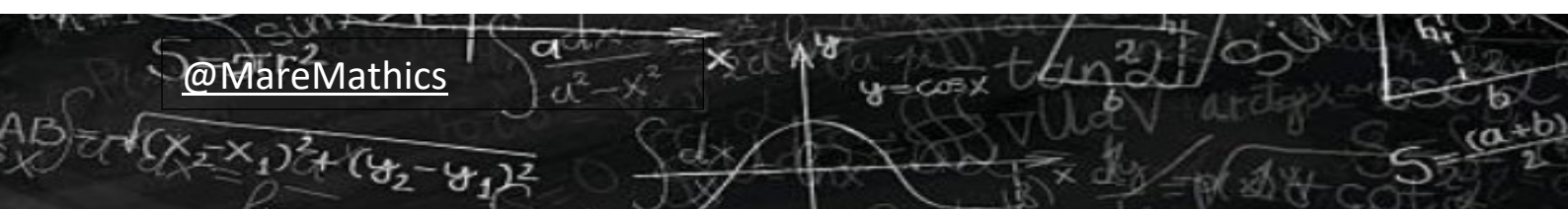
Reviewed by partners from Tallinn University of Technology, Estonian Maritime Academy



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The Manual is the outcome of the collaborative work of all the Partners for the development of the MareMathics Project.

Partners in the project:

University of Split Faculty of Maritime Studies **CROATIA**

Tallinn University of Technology, Estonian Maritime Academy **ESTONIA**

Latvian Maritime Academy **LATVIA**

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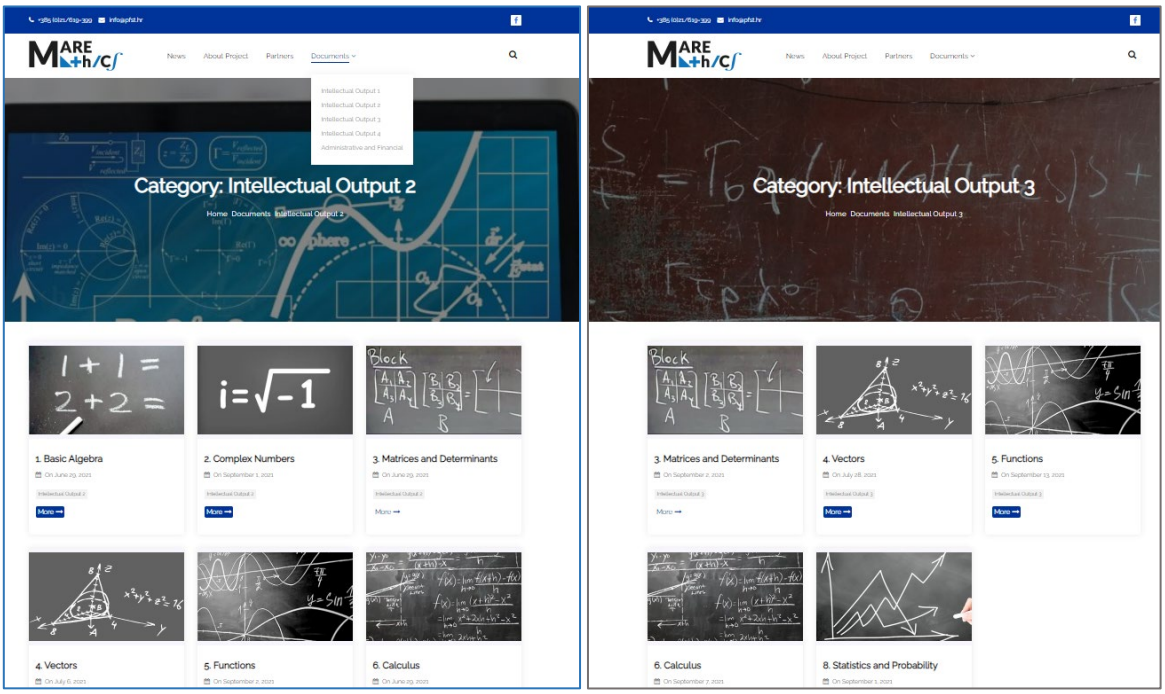
CONTENTS

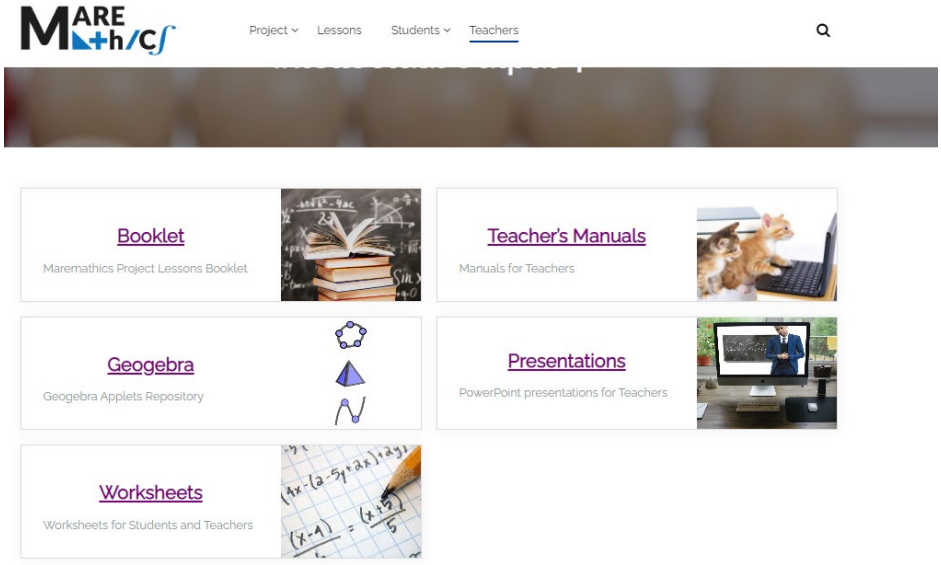
Differential equations: Teaching and Learning Plan

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The topic Differential equations consist of several lessons. The sets of exercises at the end of lessons are recommended for classes to acquire the methods of solution of differential equations and its applications.

The resources are picked from project ***MareMathics*** and available on the https://maremathics.pfst.hr/.





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Name of Unit	Workload	Handbook
Differential equations	**Lectures: 540 min**	**Unit 8. Differential equations**

INTRODUCTION

This unit is devoted to ordinary differential equations of first and second order. Major types of equations such as first order separable equations, linear equations and second order linear equations will be considered. Examples of differential equations with detailed solving are presented through the topics to provide guidance on how to approach and solve various problems. The Laplace transform will be considered in applications for solving differential. Some applications of differential equations are considered in the relation to maritime problems.

AIM: To gain understanding what differential equations are and their importance for applications; to acquire skills in solving basic types of first and second order ordinary differential equations and understanding the concepts behind these solutions.

Learning outcomes

At the end of these lectures, students should be able to

1. understand the main concepts of ordinary differential equations
2. determine the type of a differential equation
3. solve first-order separable differential equations and linear differential equations, using an appropriate solution method
4. solve initial value problem (Cauchy problem) for first-order differential equations
5. solve second-order linear differential equations with constant coefficients, using an appropriate solution method
6. solve initial value problem and boundary value problem for second-order differential equations
7. solve linear differential equations, using Laplace transform.

Students should know also some applications of differential equations

Key words of this Unit:

differential equations, ordinary differential equations, initial value problem, boundary value problem, Laplace transform.

Previous knowledge of mathematics: algebra, logarithmic functions and their properties, complex numbers, differentiation, methods of integration, improper integrals, ability to solve systems of linear equations, the expansion of rational functions into elementary fractions.

Relationship to real maritime problems: Differential equations are a model of the World. They are widely used for mathematical modelling of processes and phenomena in a variety of fields of science and technology: in mechanics, physics, chemistry, biology, economics and also in modeling many maritime problems. For example, a vessel's motion, rolling, pitching and heaving motions are described by differential equations. Differential equations are used to describe cooling processes, vibrations, deformation of beams and plates which are

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important in ship design and in load calculations for cargo transport. Processes in electrical circuits of electric systems of vessels and electro-automatics are also described by differential equations. There are also many marine ecological issues where differential equations are useful, for example, to describe many chemical reactions, to describe the propagation and extinction of various bacteria and microorganisms in the sea, to describe propagation and extinction of fish population that is important for fish catch control.

Assessment strategies:

Homework for students and Test at the end of unit

MareMathics Teacher Toolkit and Digital Resources:

- Powerpoint presentation to introduce and present theoretical materials
- MareMathics Booklet https://maremathics.pfst.hr/?3d-flip-book=booklet
- Lessons https://maremathics.pfst.hr/?p=123
- Videos
- Multi-choice questioner
- Quizzes /WorkSheets

Useful websites:

https://www.khanacademy.org/math/ap-calculus-ab/ab-differential-equations-new/ab-7-1/v/differential-equation-introduction

https://www.khanacademy.org/math/differential-equations/

https://math24.net/topics-differential-equations.html

Contents:

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Lesson 1: The concept of differential equations. Main concept of First-order differential equations.

Name of Unit	Workload	Handbook
The concept of differential equations. Main concept of First-order differential equations. First-order separable differential equations	Lecture: 90 min	Unit 8. Differential equations 8.1. The concept of differential equations 8.2. First order differential equations 8.2.1. Main concept 8.2.2. Separable variable equations

DESCRIPTION:

First Lesson contains introductory questions about the differential equations and first order differential equations. In this Lesson we consider the concept of differential equations and the main concept of first-order differential equations. The first-order separable variable equations and first-order linear differential equations and their solving methods will be considered in detail. Some applications of differential equations are considered in the relation to maritime problems.

AIM: To gain understanding what differential equation is and what means the solution of differential equation; to gain understanding the main concepts of first-order ordinary differential equations; to consider first-order separable differential equations and to acquire skills in solving this type differential equations; to show some applications of first-order separable differential equations.

Learning outcomes

At the end of this lecture, students should be able to

1. understand the main concepts of ordinary differential equations
2. understand the meaning of general solution, particular solution of a differential equation
3. to determine first-order separable differential equations and solve them

Key words of this Lesson:

Ordinary differential equations, order of differential equation, general solution, particular solution, initial value problem, separable variable differential equations

Previous knowledge of mathematics: algebra, logarithmic functions and their properties, differentiation, methods of integration.

Assessment strategies:

Assessing students' knowledge about the differentiation during the lesson, Quizz

Teacher Toolkit and Digital Resources:

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- Powerpoint presentation to introduce the concepts of differential equations and main concepts of first order ordinary differential equation
- Graphing calculator Geogebra Classic to demonstrate a set of solutions for a first order differential equation (Desmos graphing calculator or another tool)
- Videos https://maremathics.pfst.hr/?p=3764
- Quiz https://maremathics.pfst.hr/?p=123
- Lesson websites

  https://maremathics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-1.pdf
  https://maremathics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-2.pdf
- Useful websites

  https://www.khanacademy.org/math/ap-calculus-ab/ab-differential-equations-new/ab-7-1/v/differential-equation-introduction

  https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction

  https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exponential-models-diff-eq/v/newtons-law-of-cooling?modal=1

  https://www.youtube.com/watch?v=eJPpFRGqHQo

  https://math24.net/topics-differential-equations.html

Content of lesson:

1. Introduction
2. The concept of differential equations. Ordinary differential equations.
3. Some examples of applications of ordinary differential equations (without solving problems)
4. The main concept of first-order differential equations
5. First-order Separable equations
6. Application of First-order Separable equations (with solving some problem)
7. Quiz or multi-choice questioners

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LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Discussion	Introduction to the topic ; discussion of prior knowledge together with students	Frontal and questioning	Active listening and contributing to questions. Discussion.	What is the physical meaning of derivative?
10 min	Presentation Example 2 in Unit 8.1 Example 1 in Unit 8.1 Example 3 in Unit 8.1 Discussion	Main concept of differential equations: Partial differential equations and ordinary differential equations; Order of an ordinary differential equation. Solution of a differential equation. General and particular solutions.	Frontal and questioning	Active listening. Contributing to questions	• What is the order of a given differential equation? • How can we check if a given function is a solution of given differential equation? • Is a given function a particular or a general solution of a given differential equation?
5 min	Presentation Some Examples from Unit 8.5	Some examples of applications of ordinary differential equations to real life problem (without solving them);	Frontal and questioning Video	Active listening Contributing to questions	•
10 min	Presentation Example 4 in Unit 8.2.1 Example 5 in Unit 8.2.1 Example 6 in Unit 8.2.1	Main Concept of first order ordinary differential equations; The initial value problem; General and particular solutions for first-order differential equations	Frontal	Active listening Contributing to questions	• How can we solve the given first order differential equation? • How can we find only one solution that satisfies given additional conditions?

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25 min	Presentation Examples 7 in Unit 8.2.2 Examples 8 in Unit 8.2.2 Video https://marematics.pfst.hr/?p=3764#sep-diff	First-order Separable equations: definition of a Separable variable differential equation; explanation of solving method	Frontal. Explains and discusses with students the solution for two examples	Active listening	
15 min	Exercises	Solution of exercises.	Posing the problem; Correcting solving	Students work on tasks; Discussion; Contributing solving process	
15 min	Presentation Video Problem posing Example 3 in Unit 8.5 Example 4 in Unit 8.5 Examples 6 in Unit 8.5	Application of First-order Separable equations in the maritime field. Explanation of the solution for one of the problem (teacher can use PowerPoint presentation)	Problem posing Frontal	Active listening; Questioning; Answering questions; Discussing	
10 min	Quiz or multi-choice questioners	Quiz or multi-choice questioners	Posing the problem; Correcting solving;	Answering questions of questioners	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** **Lesson 8.1 (Unit 8) https://maremathics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-1.pdf** • **Video (flare):** • **Quiz**
Learning objectives	By the end of the lesson: • ***all*** students should understand the meaning of differential equation and ordinary of a differential equation • ***all*** students should understand the meaning of general solution and particular solutions of a differential equation • ***all*** students should understand the meaning of an initial value problem • ***all*** students should determine the first-order separable differential equations and solve them

1. The lesson starts with introducing of the topic and together with students discussing prior knowledge.

Lecturer can ask students about meaning of derivative of a function and its physical meaning and to discuss it with students.

It is important to revise the definition of one augment function derivative and its physical meaning, making the accent, that if the function $y(t)$ describes the time-dependent process, than $\frac{dy}{dt}$ describes the rate of the process (speed) and $\left.\frac{dy}{dt}\right|_{t=t_0}$ describes the rate of the process at the time moment $t = t_0$

Lecturer discusses some examples with students.

2. Main concept of differential equations:

A. Definition of differential equations. Partial differential equations and ordinary differential equations, examples.

Example 1 and Example 2 in Unit 8.1

Suggestion: As examples of partial differential equations, Lecturer can show following equations and show some video:

1) The equation of a string vibration (Example 2 in Unit 8.1) and video https://mathlets.org/mathlets/wave-equation/

2) The wave equation $\frac{\partial^2 U}{\partial t^2} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$, where unknown function is a function of 4 variables x,y,z,t. This equation describes properties of most wave

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phenomena such as *vibration of the rectangular membrane, water wave, air wave, earth wave*

3) The heat equation (known as the diffusion equation) $\frac{\partial U}{\partial t} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$

B. Order of an ordinary differential equation. Examples and discussion with students these examples. **Example 3 in Unit 8.1**

Posing the question *"what is the order of a given differential equation?"*. Discussion with students to get answers.

C. Definition of a solution of a differential equation. General and particular solutions of a differential equation. Example.

Solving and discussing with students the following example:

Example: *Given differential equation:* $y'' - 4y' = 0$

and set of functions : a) $y = e^{4x}$ b) $y = 3 + e^{4x}$

c) $y = x + e^{4x}$ d) $y = a + be^{4x}$

Select from the set of functions, the functions which are the solutions of the given equation and to define which of them represent a general solution and which of them are the particular solutions.

D. Some applications of ordinary differential equations to some maritime problems (without solving them).

Lecturer can select *any examples* from **Examples 1 ,2, 3,4,5,6 in Unit 8.5**

Suggestion: As examples of partial differential equations, Lecturer can show following equations and show following video

1) Equation of rolling of a vessel, occurring in calm water without resistance

Example 1 in Unit 8.5:

$$\theta'' + n_\theta{}^2\theta = 0$$

where $\theta = \theta(t)$ is the rolling amplitude at the time moment t,

n_θ is the circular frequency of free (natural) vibrations during the rolling without resistance.

Lecturer can demonstrate the following video of rolling of ship and discuss the maritime safety.

Video: https://www.youtube.com/watch?v=_uUa0xrdddY or https://www.youtube.com/watch?v=0xJsGgWUj0Y

2) The Euler–Bernoulli equation that is used in beam theory. This equation describes the

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relationship between the beam's deflection and the applied load.

Example 5 in Unit 8.5:

In the case of small deflections, the beam shape can be described by a fourth-order differential equation

$$EI\frac{d^4w}{dx^4} = q(x)$$

where *q(x)* is external load acting on the beam,

E is the modulus of elasticity of the beam,
I is the second moment of area of the beam's cross-section.

Beam theory which is an important tool in ship design.

Suggestion: Lecturer can find additional examples of application differential equations, for instance on the homepage:

https://math24.net/topics-differential-equations.html

3. Main concept of first –order ordinary differential equations:
   - **A.** Forms in which the first-order differential equation can be given. Examples.
   - **B.** General and particular solutions of first-order differential equations and it's geometric interpretation. Initial value problem.

     Solving and discussing with students the **Examples 5 and 6 in Unit 8.2.**:

     Posing the question *"how we can find unknown function y(x)?"*, discuss with students the way of solving the given equation.

     By using GeoGebra, construct the general solution of the differential equation for different constant C values to show that the general solution is a set of functions. Solve initial value problem for this equation, find particular solution and show that this solution is one function from the set).
4. First-order Separable equations. Definition and the method of solving. Explanation and discussion with students the solution for two examples. **Examples 7 and 8 in Unit 8.2.:**

Suggestion: As an example for explanation of solving method, teacher can also demonstrate the video records on https://maremathics.pfst.hr from Unit 8 instead of the explanation of example 7.

5. A teacher gives to students one or two differential equations. Students work on the task , compare, discuss.
6. Application of First-order Separable equations in the maritime field. Explanation and solving of one or two problems(teacher can use PowerPoint presentation). Posing the question *"how we can find unknown function?"*, discuss with students the solving of the given problem

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Suggestion: **A)** solve the Example 7 in Unit 8.2 with given values of R, r and H.

For example: There is a cylindrical container of radius R=1m and a small hole with a radius r=5 cm at the bottom of the container. The container if filled with liquid. At the time moment t=0, a circular hole in the bottom of the tank is opened and the liquid flows out of the tank.

- Find a function that describes the dependence of the liquid level in tank on the time. (Teacher can draw this function, using GeoGebra)
- During what time the liquid completely flows out of the container if at the initial time t=0 the liquid level was in the container H=2m, assuming that the flow rate is k=1. Find the liquid level in the vessel at time t=2 min.

B) Solve Example 8 from Unit 8.2 for the constant temperature of environment and without of source of heat .

For example, teacher can suggest to solve one of the following problem:

Problem 1. A cup with boiling water stands in a room where the temperature is 20◦C. If T(t) is the temperature of the water at time t in minutes, than the cooling of the water is described by differential equation $\frac{dT}{dt} = -k(T(t) - 20)$.

What is the sign of k? Solve this differential equation. If the coffee cools to 80◦C in 4 minutes, how does coffee takes to cool to 50◦C?

Problem 2. Write a differential equation whose solution is a function of time presented the temperature of a bottle of milk taken out of a 8 ◦C refrigerator and left in a 20◦C room. Solve the equation and graph the solution.

7. Quiz or multi-choice questioners to access students understanding and archiving goals of the lesson.

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Lesson 2: First-order Linear equations. Method of Variation of Constants. Bernoulli method

Name of Unit	Workload	Handbook
First-order Linear equations. Method of Variation of Constants. Bernoulli method	Lectures: 90 min	Unit 8. Differential equations 8.2. 3. First order linear differential equations

DESCRIPTION:

In this Lesson, the first-order linear differential equations and two their solving methods will be considered. Some applications of first-order linear differential equations are considered in the relation to maritime problems.

AIM: To consider first-order linear differential equations and to acquire skills in solving this type differential equations; to show some applications of first-order linear differential equations.

Learning outcomes

At the end of this lecture, students should be able to

1. determine first-order linear differential equations
2. solve the first-order linear differential equations, using method of Variation of Constants
3. solve the first-order linear differential equations, using Bernoulli method.
4. know some applications of first-order linear differential equations to maritime problems

Key words of this Lesson:

First-order linear differential equations, general solution, particular solution, initial value problem, method of Variation of a Constant, Bernoulli method.

Previous knowledge of mathematics: algebra, logarithmic functions and their properties, differentiation and ability to find derivatives, methods of integration, ability to solve Separable differential equations.

Assessment strategies:

Assessing students' knowledge during the lesson, Quizz

Teacher Toolkit and Digital Resources:

- PowerPoint presentation
- Geogebra Classic
- Videos
- Quizz
- MareMathics websites

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https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-9-Differential-Equations-1.pdf
https://marematics.pfst.hr/wp-content/uploads/2021/09/IO2-9-Differential-Equations-2.pdf

Content of lesson:

1. Introduction
2. Method of Variation of Constants
3. Bernoulli method
4. Application of First-order Separable equations (with solving some problem)
5. Quiz or multi-choice questioners

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LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Discussion	Introduction to linear differential equations	Frontal and questioning	Active listening and contributing to questions Discussion	
20 min	Presentation Example 8 in Unit 8.2 Discussion	Main concept of linear differential equations. Method of Variation of Constants.	Frontal and questioning	Active listening Contributing to questions	• How can we solve the corresponding Separable differential equation?
20min	Presentation Example 10 in Unit 8.2 Discussion	Bernoulli method	Frontal and questioning Video	Active listening Contributing to questions	• How can we solve the corresponding Separable differential equation?
10 min	Presentation Discussion	Some applications of First-order linear equations (with solving some problem)	Frontal	Active listening Contributing to questions	
20 min	Exercises	Solution of exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the tasks. Discussion. Contributing the solving process	
15 min	Quiz or multi-choice questioners	Quiz or multi-choice questioners	Posing the problem;	Answering questions of questioners	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** **Lesson 8.2 (Unit 8) https://marematics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-2.pdf** • **Video (https://marematics.pfst.hr/?p=3764#bernoulli):** • **Video (https://marematics.pfst.hr/?p=3764#var-constant):** • **Quiz**
Learning objectives	By the end of the lesson: • ***all*** students should determine the first-order linear differential equations • ***all*** students should be able to solve a first order linear equation • ***all*** students should know some applications of first-order linear differential equations to maritime problems

1. The lesson starts with introducing to the first order differential equations.
2. Main concept of linear differential equations; Method of Variation of Constants; Explanation and discussion with students the solution for an example. **Example 9 in Unit 8.2**
3. Bernoulli method; Explanation and discussion with students the solution to **Example 10 in Unit 8.2**

Suggestion: As examples for explanation of solving method, teacher can also demonstrate the video records on https://marematics.pfst.hr from Unit 8 instead of explanation of example 9 and 10.

4. Some application of First-order linear equations (with solving some problem)

Suggestion: For example teacher can consider the following problem (see . **Example 2 in Unit 8.5**):

Example: Consider an alternating-current electrical RC circuit with resistor of R=10 ohm and capacitor of C=30 μf and applied changing voltage $V(t) = 150\sin(\omega t)$. Find the total current in the circuit and find the current in the circuit at the time moment $t = 3$ sec provided that the circuit was switched on at the time moment $t = 0$ (It means that $i(0) = 0$)

The current in such circuits for some time after the circuit is switched on or off at time t_0 is described by the following differential equation:

$$R\frac{di}{dt} + \frac{1}{C}i = V(t)$$

where V(t) is the voltage source powering the circuit,

t is the time,

i(t) is the current admitted through the circuit,

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R is the effective resistance of the combined load, source, and components,

L is the inductance of the inductor component,

C is the capacitance of the capacitor component.

Before consider this example, it would be good to note that any modern vessel is not complete without electrical and electro-mechanical systems. An alternating-current electrical circuit is a component of such system.

5. A teacher gives two linear differential equations to students. Students work on the task. Discussion.
6. Quiz or multi-choice questioners to access students understanding and archiving goals of the lesson

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Lesson 3: The basic concept of Second-order linear differential equations. Second order linear homogeneous differential equations with constant coefficients

Name of Unit	Workload	Handbook
The basic concept of Second-order linear differential equations. Second order linear homogeneous differential equations with constant coefficients	Lectures: 90 min	Unit 8. Differential equations 8.3. Second order linear differential equations 8.3.1. Basic concepts for second-order differential equations. Second-order linear differential equations. 8.3.2. Second-order linear homogeneous differential equations with constant coefficients.

DESCRIPTION:

This Lesson is devoted to the Second-order linear homogeneous differential equations. At first, the main concept of Second-order differential equations will be considered in short. Second-order linear homogeneous differential equations with constant coefficients will be considered in detail.

AIM: To gain understanding of the main concepts of second -order differential equations and second -order linear differential equations with constant coefficient; to consider second-order linear homogeneous differential equations and to acquire skills in solving this type differential equations.

Learning outcomes

At the end of this lecture, students should be able to

1. understand the main concepts of second -order differential equations and second -order linear differential equations with constant coefficient
2. solve the second-order linear homogeneous differential equations
3. solve the initial value problems and boundary problems for second-order linear homogeneous differential equations
4. to know some applications of second-order linear differential homogeneous equations by to maritime problems

Key words of this Lesson:

second-order linear differential equations with constant coefficients, general solution, particular solution, initial value problem, boundary problem.

Assessment strategies:

Assessing students' knowledge during the lesson, Quiz

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Content of lesson:

1. Basic concepts of second-order differential equations. Second-order linear differential equations. Homogeneous and Nonhomogeneous linear differential equations.
2. Second-order Homogeneous linear differential equations with constant coefficients.
3. Some application of Second-order Homogeneous linear differential equations with constant coefficients (with solving of some problem)

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LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
15 min	Presentation Example 11 in Unit 8.3 Example 12 in Unit 8.3 Discussion	Introduction. Basic concepts of second-order differential equations. General solution and particular solution of second-order differential equations. Initial value conditions, Boundary conditions.	Frontal and questioning	Active listening and contributing to questions Discussion	What it does a second order differential equation means? How many constants does the general solution of a second order differential equation involve?
5 min	Presentation	Second order linear differential equations with constant coefficients. Homogeneous and Nonomogeneous second-order linear differential equations.	Frontal and questioning	Active listening Contributing to questions	
20min	Presentation Example 13 in Unit 8.3 Example 14 in Unit 8.3 Example 15 in Unit 8.3 Discussion	Second order Homogeneous linear differential equations and it's method of solving	Frontal and questioning	Active listening Contributing to questions	
20 min	Exercises	Solution of exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the task Discussion Contributing the solving process	
15 min	Presentation Example 1 in Unit 8.5 Example 2 in	Some application of second-order homogeneous linear equations with constant	Frontal	Active listening Contributing to questions	

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	Unit 8.5 Discussion	coefficients (with solving some problem)			
15 min	Quiz or multi-choice questioners	Quiz or multi-choice questioners	Posing the problem Revising topic/knowledge	Answering questions of questioners	

SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** **Lesson 8.3 (Unit 8) https://marematics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-3.pdf** • **Video (https://marematics.pfst.hr/?cat=6):** • **Quizz**
Learning objectives	By the end of the lesson: • ***all*** students should determine the second -order linear differential equations with constant coefficients and understand if the given equation is homogeneous differential equation or nonhomogenious differential equation • ***all*** students should be able to solve second -order linear homogeneous differential equations with constant coefficients • ***all*** students should be able to solve initial value problems and boundary for second -order linear homogeneous differential equations with constant coefficients • ***all*** students should know some applications of second-order linear homogeneous differential equations to maritime problems

1. The lesson starts from introducing the second order differential equations.
2. Main concept of second order differential equations; General and particular solution of second-order differential equations; Initial value problem and boundary problem; Particular solution of second order differential equation. Examples.

   Example 11 and Example 12 in Unit 8.3
3. Second-order linear differential equations. Homogeneous and Nonhomogeneous second-order linear differential equations. Examples.
4. Second order Homogeneous linear differential equations with constant coefficients. Definition, method of solving. Explanation and discussing with students the solution for three examples.

   Example 13, Example 14 and Example 15 in Unit 8.2

   Suggestion: As examples for explanation of solving method, teacher can also demonstrate the solution of higher order homogeneous differential equation, for example:

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1) $y''' + 4y'' = 0$
2) $y''' + 2y'' + y' = 0$
3) $y^{(5)} - 9y''' = 0$

5. A teacher gives to students three or five Second order Homogeneous linear differential equations. Students work on the tasks, compare solutions, discuss. A teacher checks the solutions.

6. Some application of second-order homogeneous linear equations with constant coefficients (with solving of problems)
Suggestion: For example, teacher can consider the following Examples: **Example 1 and Example 2 in Unit 8.5**

For the **Example 1** on a ship rolling, teacher can use the video and pictures form the 1st lesson.

7. Quiz or multi-choice questioners to access students understanding and archiving goals of the lesson

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Lesson 4 and Lesson 5 Second-order linear nonhomogeneous differential equations with constant coefficients. Method of variation of Constant.

Name of Unit	Workload	Handbook
Second-order linear nonhomogeneous differential equations with constant coefficients. Method of variation of Constant. Method of Undetermined Coefficients	Lectures: 180 min	Unit 8. Differential equations 8.3.3. Second-order linear homogeneous differential equations with constant coefficients. Method of Variation of Constant. Method of undetermined Coefficients

DESCRIPTION:

This Lesson is devoted to the Second-order Nonhomogeneous linear differential equations. The Method of Variation of Constant and the Method of Undetermined Coefficients will be considered in details.

AIM: To consider second-order linear **nonhomogeneous** differential equations and to acquire skills in solving this type differential equations by **method of Variation of Constants and method of Undetermined Coefficients**.

Learning outcomes

At the end of this lecture, students should be able to

1. solve the second-order nonhomogeneous linear differential equations by method of Variation of Constants
2. solve the second-order linear nonhomogeneous differential equations by method of Undetermined Coefficients
3. Choose the corresponding method of solving for given second-order nonhomogeneous linear differential equation

Key words of this Lesson:

second-order nonhomogeneous linear differential equations with constant coefficients, method of variation of constants, method of undetermined coefficients.

Assessment strategies:

Assessing students' knowledge during the lesson

Content of lessons:

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1. Basic concepts for second-order linear Nonhomogeneous differential equations with constant coefficients.
2. Method of Undetermined Constants for Second-order Homogeneous linear differential equations with constant coefficients
3. Method of Variation of Constants for Second-order Homogeneous linear differential equations with constant coefficients.

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LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
15 min	Presentation Discussion	Introduction. Basic concepts for second-order linear Nonhomogeneous differential equations with constant coefficients.	Frontal and questioning	Active listening and contributing to questions Discussion	
50 min	Presentation Example 17 in Unit 8.3 Example 18 in Unit 8.3 Example 19 in Unit 8.3 Example 20 in Unit 8.3 Discussion	The Method of Undetermined Coefficients .	Frontal and questioning	Active listening Contributing to questions	What kind functions can satisfy the differential equation?
50 min	Exercises	Solving exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the task Discussion Contributing the solving process	
20 min	Presentation Example 16 in Unit 8.3 Discussion	The Method of variation of constants	Frontal and questioning	Active listening Contributing to questions	
40 min	Exercises	Solving exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the task Discussion Contributing the solving process	
15 min	Quiz or multi-choice questioners	Quiz or multi-choice questioners	Posing the problem; Revising topic/knowledge	Answering questions of questioners	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** **Lessons (Unit 8)** **https://marematics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-3.pdf** **https://marematics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-4.pdf** https://marematics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-5.pdf **Quiz** https://docs.google.com/forms/d/e/1FAIpQLSefQ7lCmpel3drvY-U7JjuKMAyGtywD2JfpZMxhc95Qp7d0Mw/viewform?ts=6137068a
Learning objectives	By the end of the lesson: • ***all*** students should be able to solve second order nonhomogenious differential equations by using method of undetermined coefficients • ***all*** students should be able to solve nonhomogenious differential equations by using method of Variation of constants • ***all*** students should be able to select the proper method of solving for a given differential equation

1. The lesson starts from introducing the topic
2. Main concepts for second-order linear Nonhomogeneous differential equations with constant coefficients. The theorem about the general solution of a nonhomogeneous linear equations.
3. . The Method of variation of constants. Explanation and discussing with students the solution for examples.

   Example 18 in Unit 8.3
   Example 19 in Unit 8.3
   Example 20 in Unit 8.3

Suggestion: Before explaining the method of undefined coefficient, lecturer can discuss with students *what kind of functions can satisfy the following equations* (i.e try to predict the form of particular solution for given equations):

1) $y'' + 4y' + 4y = x^2 - 3x + 1$
2) $y'' + 2y' = 5e^{2x}$
3) $y'' + 9y''' = sinx$

4. A teacher gives some linear differential equations with special form function on the right- hand side of differential equation. Students work on the task, compare. Discussion.
5. The Method of variation of constants. Explanation and discussion with students solution of considered examples. **Example 21 in Unit 8.3**

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Suggestion: Before explaining the method of variation of constants, lecturer can discuss with students *if it is possible solve the following equation by using the method of undetermined constants*:

1) $y'' + 9y = \frac{1}{sin3x}$

2) $y'' - 2y' + y = \frac{e^x}{x}$

6. A teacher gives to students some linear nonhomogeneous differential equations for solving them by method of variation of constants. Students work on the task, compare. Discussion.

7. Quiz or multi-choice questioners to access students understanding and archive goals of the lesson

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Lesson 6: The basic concept of Laplace Transform. Application of Laplace transform for solving differential equations

Name of Unit	Workload	Handbook
The basic concept of Laplace Transform. Application of Laplace transform for solving differential equations	Lectures: 90 min	Unit 9. Differential equations Part 9.4: Application of the Laplace transform for solving differential equations solving

DESCRIPTION:

This Lesson is devoted to solving second-order linear nonhomogeneous differential equation by using the Laplace transform. Definition and properties of the Laplace transform are considered in brief.

AIM: To gain understanding about main concepts of Laplace transform; to gain understanding about solving second -order linear differential equations with constant coefficient by using the Laplace transform.

Learning outcomes

At the end of this lecture, students should be able to

1. understand the main concepts of the Laplace transform
2. solve the second-order linear nonhomogeneous differential equations with constant coefficients by using Laplace transform

Key words of this Lesson:

Laplace transform, second-order linear differential equations with constant coefficients

Previous knowledge of mathematics: algebra, differentiation and ability to find derivatives, the expansion of rational functions into elementary fractions.

Assessment strategies:

Assessing students' knowledge during the lesson

Content of lesson:

1. The definition of the Laplace transform. Main properties. Finding the transform for the given function.
2. The Inverse Laplace transform. Finding the original of the function.
   Second-order Homogeneous linear differential equations with constant coefficients.
3. Application of the Laplace transform for solving differential equation

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LESSON FLOW

Time	Sequence	Content	Teacher activities	Student activities	Points for discussion
5 min	Presentation Discussion	Introduction.	Frontal and questioning	Active listening and contributing to questions Discussion	
20 min	Presentation Example 21 in Unit 8.4 Example 22 in Unit 8.4	Introduction. Definition of the Laplace transform. Main properties. Finding of the transform of a given functions.	Frontal and questioning	Active listening Contributing to questions	
15 min	Exercises	Solution of exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the task Discussion Contributing the solving process	
10 min	Presentation Example 23 in Unit 8.4	The Inverse Laplace Transform. Finding the original of the given transform	Frontal and questioning	Active listening Contributing to questions	
20 min	Presentation Example 24 in Unit 8.4	Application of the Laplace transform for solving differential equations	Frontal and questioning	Active listening Contributing to questions	
20 min	Exercises	Solution of exercises.	Posing the problem; Asking questions; Correcting solving	Students work on the task Discussion. Contributing the solving process	

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SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

	• **Whiteboard** • **Lesson https://maremathics.pfst.hr/wp-content/uploads/2022/06/Differential-Equations-4.pdf f**
Learning objectives	By the end of the lesson: • ***all*** students Know the definition of the Laplace transform • ***all*** students should understand what transform of the function and original of the function means • ***all*** students should know properties of the Laplace transform which are necessary for solving differential equations and should be able to use them • ***all*** students should be able to solve first-order and second-order linear nonhomogenous differential equations with constant coefficients by using the Laplace transform

1. The lesson starts from introducing the Laplace transform and its applications for solving differential equations.
2. Main concept of the Laplace transform:
   A) Definition of the Laplace transform. Theorem of existence of Laplace integral. Heaviside function.
   B) Main properties of the Laplace transform which are necessary for solving differential equations (Linearity of the Laplace transform; theorem on a derivative of integral)
   C) Some examples of finding transform of given function by using definition of Laplace transform and by using properties of Laplace transform. **Example 21 and Example 22 in Unit 8.4**
3. A teacher gives to students two exercises. Students work on the task. Comparing, discussing.

Suggestion: Teacher can give, for example the following exercises:

1) Find the Laplace transform of the function $f(t) = e^t$ by using definition of Laplace transform

2) Find the Laplace transform of the function $\mathrm{f(t)} = \mathrm{e^{2t}cos3t} + 3\mathrm{e^{-5t}} + 4\mathrm{t^2} - 6$ by using table of Laplace transform and properties of Laplace transform

3. The Inverse Laplace Transform. Finding the original of the given transform. Examples. **Example 23 from Unit 8.4**
4. A teacher gives to students two functions and asks to find the Laplace transform for them. Students work on the task, comparing, discussing

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Suggestion: Teacher can give, for example the following exercises:

1) Find the original of the function $F(s) = \frac{3s+4}{s^2+2s-3}$
2) Find the original of the function $F(s) = \frac{5s-1}{s^2+6s+13}$
3) Find the original of the function $F(s) = \frac{2-s}{(s+2)(s-1)^2}$
4) Find the original of the function $F(s) = \frac{2s+3}{(s-1)(s^2+4s+13)^2}$

5. Application of the Laplace transform for solving differential equations. Explanation. Discussion. Examples.

1) *Example*: Solve the Cauchy problem $y'' + 6y' + 13y = e^{-x} \quad y(0) = 1, y'(0) = 0$ by using Laplace transform:
2) **Example 24 from Unit 8.4**

6. A teacher gives to students two functions and asks to find the Laplace transform for them. Students work on the task, comparing, discussing

Suggestion: Teacher can give, for instance, the following exercises:

Solve the Cauchy problem by using Laplace transform:

1) $y'' + 6y' + 13y = e^{-x} \quad y(0) = 1, y'(0) = 0$
2) $y'' - y' - 2y = sin2x \quad y(0) = 0, y'(0) = 0$
3) $y'' + 4y = 2 + e^{3x} \quad y(0) = 0, y'(0) = 1$

7. Teacher consider some application of the Laplace transform to the real-life problem, for example, a problem on LRC circuits (see Examples in Unit 8.5)

8. Quiz or multi-choice questioners to access students understanding and archiving goals of the lesson

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