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## MareMathics

Innovative Approach in Mathematical Education for Maritime  
Students

2019-1-HR01-KA203-061000

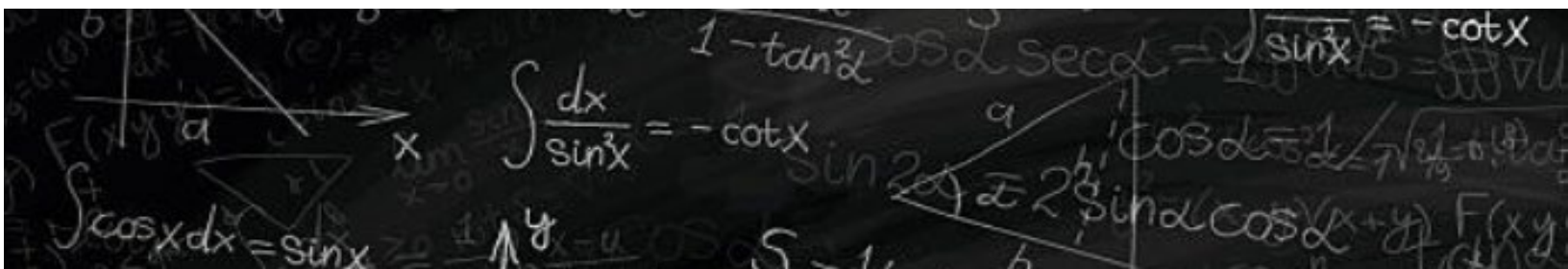
2020-2022

<https://maremathics.pfst.hr/>

# Manual for teachers

**Authors:** Goran Kovačević [gkovacev@pfst.hr](mailto:gkovacev@pfst.hr)

Reviewed by Agata Załęska-Fornal, Poland





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*The Manual is the outcome of the collaborative work of all the Partners  
for the development of the MareMathics Project.*

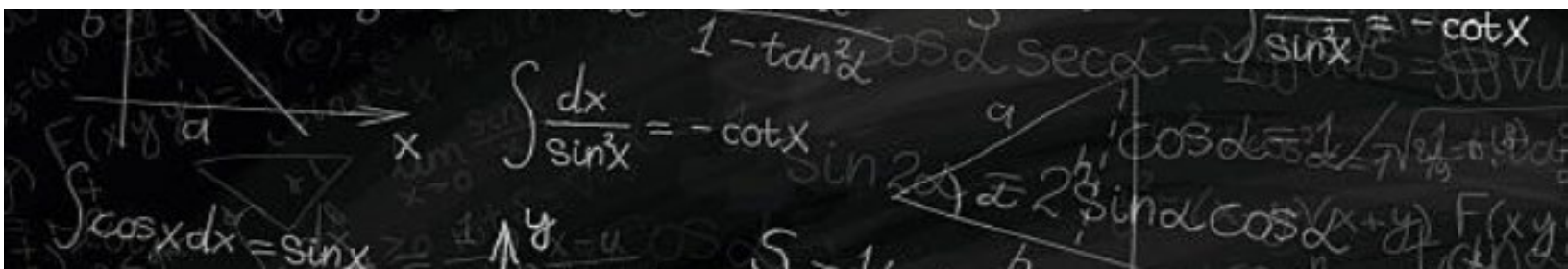
### **Partners in the project:**



Contact the coordinator:

Anita Gudelj at [agudelj@pfst.hr](mailto:agudelj@pfst.hr)

[maremathics@gmail.com](mailto:maremathics@gmail.com)



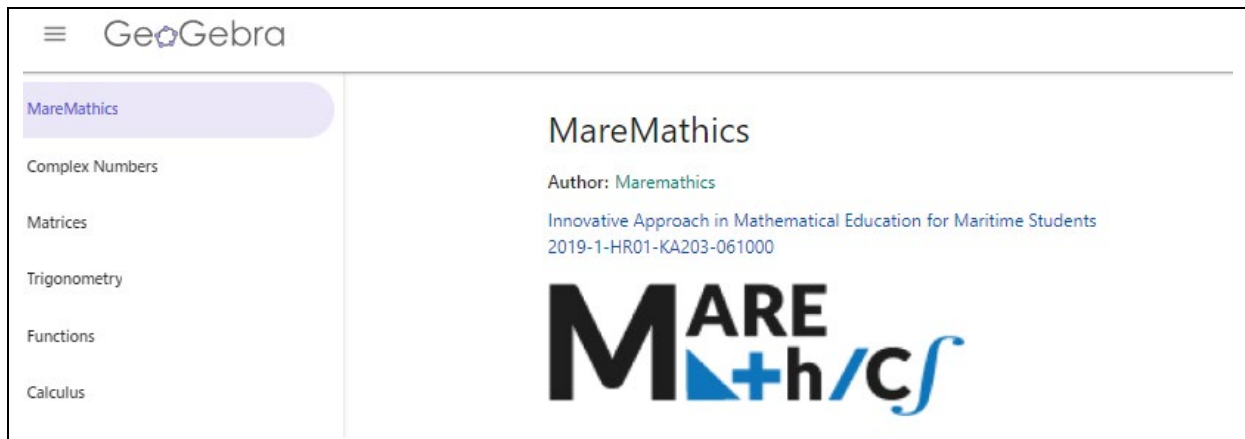
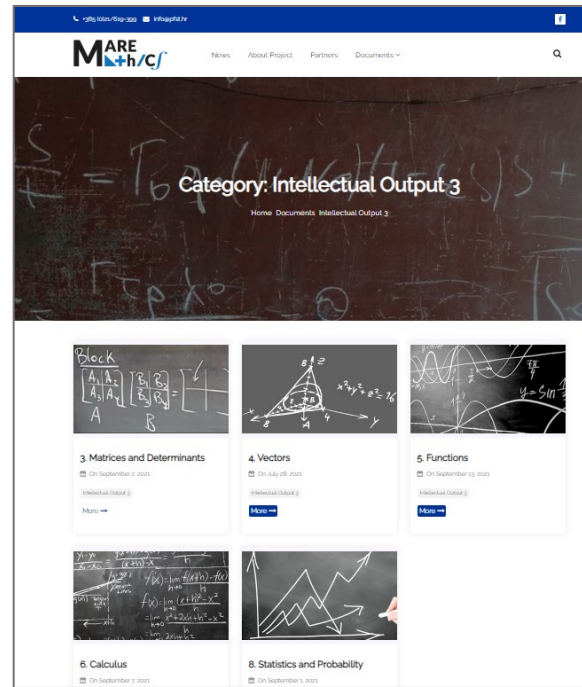
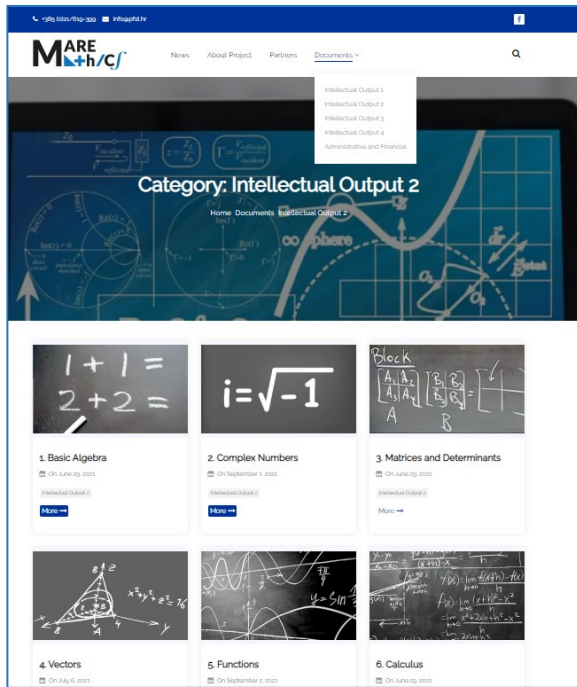
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## MATRICES AND DETERMINANTS: Teaching and Learning Plan

The goal of this material and related resources is to assist teachers in planning their lessons allowing achieving learning outcomes posted in the course's syllabus. It enables teachers to design student activities to encourage students to learn.

The resources are picked from project *MareMathics* and available on the <https://maremathics.pfst.hr/>.



## Lesson 1. The Matrix

| Name of Unit | Workload                             | Handbook                          |
|--------------|--------------------------------------|-----------------------------------|
| Matrices     | Lecture: 90 min<br>Exercises: 90 min | Unit 3. Matrices and determinants |

### DETAILED DESCRIPTION

This chapter introduces the concept of matrix. Firstly, some operations with matrices are defined, then the determinant of the square matrix and the inverse matrix of the regular matrix. After that, it is demonstrated how to solve matrix equations and how a system of linear equations can be solved using matrices. Each topic is covered in detail, and in addition to some solved examples, it also contains exercises. At the end of the chapter is a knowledge test. Additional applications in MATLAB, Excel and Geogebra can be used to check the obtained solutions or to make faster calculations (if the procedure for solving exercises is not required).

#### AIM:

Learn how to calculate the determinant, inverse matrix and how to solve a system of linear equations using matrices.

#### Learning Outcomes:

1. Know how and when arithmetic operations with matrices are defined.
2. Calculate the determinant using the basic properties of the determinant.
3. Determine the inverse matrix using the Gauss-Jordan method.
4. Solve a linear system using matrices.

:

#### Key words of this Unit:

Matrix, determinant, inverse matrix, matrix rank, linear algebraic equation

**Previous knowledge of mathematics:** The student should know the basic arithmetic operations (addition, subtraction, multiplication, and division) with real numbers.



**Application:**

- in geometry
- in processing of digital photography
- a model of consumer preference
- encryption and decoding of messages in cryptography
- analysis of an economic system
- regression line with the least square deviation from the given data set
- a problem of transport and distribution
- travelling salesman problem

**Contents:**

- 3.1 Matrix
- 3.2 Matrix arithmetic
- 3.3 Determinant of a square matrix
- 3.4 Inverse matrix
- 3.5 Matrix equations
- 3.6 Matrix rank
- 3.7 Systems of linear algebraic equations
- 3.8 Some examples of matrix applications

**Assessment strategies:**

???

**MarMathics Teacher Toolkit and Digital Resources:**

- Powerpoint presentation to introduce and work with matrices

[Geogebra Learning Tools](#)

- Videos
- Quizzes /WorkSheets
- These resources are for revision at the end of the lesson. They could also be used to introduce some of the concepts, but you may need to split the work into sections and provide extra examples and problems.



| LESSON FLOW      |  |   |   |  |   |
|------------------|--|---|---|--|---|
| Time             | Sequence   | Content   | Teacher activities  | Student activities   | Points for discussion                       |
| 15 min<br>10 min | Starter/Introduction<br>Presentation<br>3.1<br>Exercises 1-5 | What is matrix?<br>Some special types of matrices | Frontal then questioning<br>Group work  | Active listening and contributing to questions<br>Solving exercises      |   |
| 15 min<br>15 min | Presentation<br>3.2<br>Exercises 6-10                        | Matrix arithmetic                                 | Frontal then questioning<br>Group work  | Active listening and contributing to questions<br>Solving some exercises | Practical applications of matrix arithmetic |
| 30 min<br>30 min | Presentation<br>3.3<br>Exercises 11-12                       | Determinant of a square matrix                    | Frontal<br>Questioning<br>Explains tasks  | Active listening<br>Contributing<br>Solving some exercises               |   |
| 30 min<br>30 min | Presentation<br>3.4<br>Exercise 14                           | Inverse matrix                                    | Frontal<br>Discussion using solved examples   | Active listening and contributing to questions<br>Solving exercise       | Gauss-Jordan method                         |
| 20 min<br>30 min | Presentation<br>3.5<br>Exercises 15-20                       | Matrix equations                                  | Frontal<br>Discussion using solved examples<br>Group work   | Active listening<br>Discussion<br>Solving some exercises                 |   |
| 45 min<br>30 min | Presentation<br>3.6<br>Exercise 21                           | <b>Matrix rank</b>                                | Frontal<br>Discussion using solved examples   | Active listening<br>Discussion<br>Solving exercise                       | Gauss method                                |
| 60 min<br>60 min | Presentation<br>3.7<br>Exercises 22-27                       | Systems of linear algebraic equations             | Frontal<br>Questioning<br>Discussion using solved examples  | Active listening<br>Discussion<br>Solving some exercises                 | When the system has no solution?            |
| 30 min           | Presentation<br>3.8  | Some examples of matrix application               | Frontal<br>Discussion using solved examples   | Active listening<br>Discussion   | More examples?                              |
| 30 min           | Summary  | Post-teaching                                     | Guides students to conclude the lessons<br>Giving homeworks<br>Helping students to solve more difficult exercises |  |   |



## SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

|                            |  |
|----------------------------|--|
| <b>Resources</b>           | <ul style="list-style-type: none"> <li>• <b>Whiteboard</b></li> <li>• <b>Lesson 1</b> <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-1.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-1.pdf</a></li> <li>• <b>Exercises 1-5</b></li> </ul> |
| <b>Learning objectives</b> | <p>By the end of the lesson:</p> <ul style="list-style-type: none"> <li>• Students should be able to determine the dimensions of the given matrix and the required element and to solve all the exercises on their own.</li> </ul>   |

Students should have no problem learning this lesson. The teacher should possibly emphasize or repeat some details several times to make them easier for students to remember. This will make it easier for students to solve the exercises on their own.

## Lesson 2: Matrix arithmetic

|                  |   |
|------------------|---|
| <b>Resources</b> | <ul style="list-style-type: none"> <li>• <b>Whiteboard</b></li> <li>• <b>Lesson 2</b> <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-2.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-2.pdf</a></li> <li>• MareMathics_SummerSchool_Split_Matrices</li> <li>• Solving examples 4 and 7 using Geogebra</li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-intro">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-intro</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-row-col">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-row-col</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-col-row">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-col-row</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-ex3">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#matrix-multi-ex3</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi-part1">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi-part1</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi-part2">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appli-matrix-multi-part2</a></li> <li>• <b>Exercises 6-10</b></li> <li>• <b>Generating_a_matrix_polynomial</b> – Matlab tool primarily intended for teachers</li> </ul> |
|------------------|---|

**Learning objectives**

- To learn how to add, subtract, multiply and potentiate matrices

## SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

The teacher should possibly emphasize or repeat some details several times to make them easier for students to remember. This will make it easier for students to solve the exercises on their own. Videos can help with self-study because the teacher is unlikely to be able to show all of them to students in the classroom. The teacher can divide the students into groups so that each group solves one task from the exercises.

The Matlab tool can help teachers compose exam assignments.

### Lesson 3: Determinant of a square matrix

|                            |  |
|----------------------------|--|
| <b>Resources</b>           | <ul style="list-style-type: none"> <li>• Whiteboard</li> <li>• Lesson 3 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-3.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-3.pdf</a></li> <li>• Exercises 11-12</li> <li>• MareMathics Geogebra book (solution of the 11<sup>th</sup> exercise), <a href="https://www.geogebra.org/m/q3gg4jef">https://www.geogebra.org/m/q3gg4jef</a></li> <li>• Determinant, Generating the determinant of the desired value – Excel VBA tools for teachers and students</li> </ul> |
| <b>Learning objectives</b> | By the end of the lesson:<br>students should learn how to calculate the determinant of a square matrix of any order.   |

## SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

It is very important that students remember the Laplace expansion and then the most important properties of the determinant in order to understand the process of calculating the determinant. If students have learning difficulties, the teacher can solve some more examples.

It would be desirable for students to learn how to determine the determinant using GeoGebra.

Excel's VBA tools can be used to test solutions and compose new tasks.

## Lesson 4: Inverse matrix

|                            |   |
|----------------------------|---|
| <b>Resources</b>           | <ul style="list-style-type: none"> <li>• Whiteboard</li> <li>• Lesson 4 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-4.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-4.pdf</a></li> <li>• Exercises 13-14</li> <li>• Geogebra (solution of 13<sup>th</sup> exercise)</li> <li>• Inverse matrix (using Gauss-Jordan method) – Excel VBA tool for teachers and students</li> </ul> |
| <b>Learning objectives</b> | To learn: <ul style="list-style-type: none"> <li>• how the inverse matrix is determined by calculating determinants</li> <li>• Gauss-Jordan method procedure</li> </ul>   |

### SUGGESTED TEACHING STRATEGIES, INPUT AND RESOURCES

Every student must learn the Gaussian and Gauss-Jordan method. The videos in the Lesson 6 can help. If necessary, the teacher should solve additional examples. The solution can be tested with an Excel tool.

## Lesson 5: Matrix equations

|                            |  |
|----------------------------|--|
| <b>Resources</b>           | <ul style="list-style-type: none"> <li>• <b>Whiteboard</b></li> <li>• Lesson 5 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-5.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-5.pdf</a></li> <li>• Exercises 15-20</li> </ul> |
| <b>Learning objectives</b> | By the end of the lesson: <ul style="list-style-type: none"> <li>• Students should remember that matrix multiplication is associative but not commutative</li> <li>• <b>all</b> students should know how to solve simpler matrix equations.</li> </ul>   |

At the lecture, matrix arithmetic is first briefly repeated. After listening to the lecture, the students study the pdf file once again. The teacher then divides them into groups. Each group solves several tasks from the exercises. Finally, tasks are compared and controlled.

## Lesson 6: Matrix rank

|                            |  |
|----------------------------|--|
| <b>Resources</b>           | <ul style="list-style-type: none"> <li>• <b>Whiteboard</b></li> <li>• Lesson 6 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-6.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-6.pdf</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-1">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-1</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-2">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-2</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-3">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-intro-task-3</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex1">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex1</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex2">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex2</a></li> <li>• <a href="https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex3">https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#gauss-method-ex3</a></li> <li>• Exercise 21</li> <li>• <b>Generating_a_matrix_of_the_desired_rank</b> – Matlab tool primarily intended for teachers</li> </ul> |
| <b>Learning objectives</b> | <ul style="list-style-type: none"> <li>• To learn the procedure of the Gaussian method</li> <li>• To know how to determine the rank of any matrix using the Gaussian method</li> </ul>   |

The students should watch all videos the day before listening to Lesson 6. Then they listen to the lecture, read the pdf file and finally solve Exercise 21. Matlab's tool can be used in compiling tasks in which the rank of the matrix should be determined.

## Lesson 7: Systems of linear algebraic equations

|                  |  |
|------------------|--|
| <b>Resources</b> | <ul style="list-style-type: none"> <li>• <b>Whiteboard</b></li> <li>• <b>MareMathics_SummerSchool_Split_Matrices.pptx</b></li> <li>• Lesson 7 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-7.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-7.pdf</a></li> <li>• <b>Exercises 22-27</b></li> <li>• <b>Solving_a_linear_system</b> – Matlab tool for teachers and students</li> <li>• <b>Solving a system with a unique solution in which <math>m=n=r</math> (Gauss-Jordan)</b> - Excel VBA tool for teachers and students</li> </ul> |
|------------------|--|

|                            |  |
|----------------------------|--|
|                            | <ul style="list-style-type: none"> <li>• <b>Generating_a_system_with_a_unique_solution – Matlab tool primarily intended for teachers</b></li> <li>• <b>Generating_a_system_with_an_infinite_number_of_solutions – Matlab tool primarily intended for teachers</b></li> <li>• <b>Generating_an_impossible_system – Matlab tool primarily intended for teachers</b></li> </ul> |
| <b>Learning objectives</b> | <ul style="list-style-type: none"> <li>• To remember when a linear system has at least one solution and how to find them in that case</li> </ul>   |

Start with an example of applying a rank matrix on page 15 of a powerpoint presentation. Continue with the pdf file. Students read this file and then solve Exercises 22-27 on their own. Additional tools are used to solve assigned tasks faster or to assign and solve new tasks.



# Presentation Screens:

The screenshot displays a PowerPoint presentation with 20 slides. The slides cover the following topics:

- Slide 1:** MareMathics Summer School - Split, 20-24 September 2023.
- Slide 2:** Matrices: What are matrices and where do we use them? (Definition and applications in geometry, cryptography, etc.)
- Slide 3:** Example: A 2x3 real matrix with elements  $a_{11}=1, a_{12}=4, a_{13}=7, a_{21}=3, a_{22}=-1, a_{23}=-5$ .
- Slide 4:** Application: In geometry, processing of digital photography, encryption and decoding of messages in cryptography, analysis of an economic system, regression line with the least square deviation, traveling salesman problem.
- Slide 5:** Matrix multiplication and its application: Product  $F = A \cdot B$  if the number of rows of  $A$  is equal to the number of columns of  $B$ .
- Slide 6:** Matrix multiplication and its application: Problem 1. First, we need to know how to find matrix  $F = A \cdot B$  if  $A$  is a row matrix and  $B$  is a column matrix.
- Slide 7:** Matrix multiplication and its application: Problem 2. Find  $F = A \cdot B$  if  $A = \begin{bmatrix} 5 & 2 \\ 1 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 1 & 4 \\ -2 & -1 & 0 \end{bmatrix}$ .
- Slide 8:** Matrix multiplication and its application: Solution. Matrix  $F$  exists because number of rows of the second matrix ( $B$ ) is equal to the number of columns of the first matrix ( $A$ ).
- Slide 9:** Matrix multiplication and its application: How to find elements  $f_{ij}$  of the matrix  $F$ ? We use:  $f_{ij} = R_i \cdot C_j$ .
- Slide 10:** Matrix multiplication and its application: Then  $f_{11} = R_1 \cdot C_1 = [5 \ 2] \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 5 \cdot 3 + 2 \cdot (-2) = 11$ .
- Slide 11:** Matrix multiplication and its application: Remaining elements calculate by yourself. In the end it is obtained  $F = \begin{bmatrix} 14 & 11 & 13 & 20 \\ 12 & 22 & 3 & 28 \\ -8 & 19 & -19 & 4 \end{bmatrix}$ .
- Slide 12:** Matrix multiplication and its application: Solution. Matrix  $F$  is not defined (or, does not exist) because the number of rows of matrix  $B$  is not equal to the number of columns of matrix  $A$ .
- Slide 13:** Matrix multiplication and its application: Solution. Matrix  $F$  is not defined (or, does not exist) because the number of rows of matrix  $B$  is not equal to the number of columns of matrix  $A$ .
- Slide 14:** Matrix multiplication and its application: Application of matrix multiplication. Watch videos: 05e Application of matrix multiplication - part 1.mpg, 05b Application of matrix multiplication - part 2.mpg.
- Slide 15:** Matrix rank and its application: The figure below shows an electrical circuit, such as an electrical circuit on a ship.
- Slide 16:** Matrix rank and its application: We want to determine the current strength in all branches. Using the method of contour currents, this problem is reduced to the following problem of solving a system of three linear equations with 3 unknowns.
- Slide 17:** Matrix rank and its application: It's very important to find  $r(A)$  (that is rank of the matrix  $A$ ) and  $r(\tilde{A})$  (that is rank of the matrix  $\tilde{A}$ ). Why? The rank of each matrix is an integer  $\geq 0$  and it holds: Our system has at least one solution only if  $r(A) = r(\tilde{A})$ .
- Slide 18:** Matrix rank and its application: How to determine rank of some matrix  $A$ ? So, if we want to find out if our system has a solution, we must know how to determine matrix rank.
- Slide 19:** Matrix rank and its application: Elementary transformations (on rows or columns) of the real matrix  $A$ : a) replacing the position of any two rows (columns), b) multiplying any row (column) by a number other than zero, c) adding any row (column) to another row (column).
- Slide 20:** Matrix rank and its application: The rank of the real matrix  $A$  is determined using the Gaussian method. From the first row we get the equation  $10I_1 - 4I_2 + 0I_3 = -0.24$ . If we insert  $I_1 = 0.01$  and  $I_2 = 0.04$ ,  $I_3$  is obtained:  $I_3 = \frac{-0.024}{10} = -0.002$ .

## Lesson 8: Some examples of matrix applications

|                     |   |
|---------------------|---|
| Resources           | <ul style="list-style-type: none"><li>• Whiteboard</li><li>• Lesson 8 <a href="https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-8.pdf">https://maremathics.pfst.hr/wp-content/uploads/2021/09/IO2-3-Matrices-and-Determinants-8.pdf</a></li></ul> |
| Learning objectives | <ul style="list-style-type: none"><li>• To remember some more interesting applications of matrices</li><li>• To make at least one applet in Geogebra or one Matlab program with the application of matrices</li></ul>   |

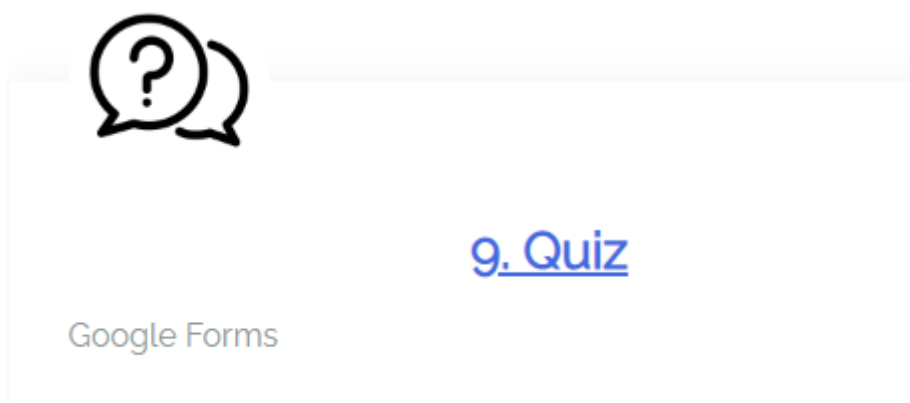
Students read a pdf file the day before the lecture. At the lecture, the teacher discusses with the students the applications of the matrices listed in the document. Eventually, some new applications are mentioned. For homework, students must, with the help of teacher, create several programs in Geogebra or Matlab with the application of matrices. Videos

<https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appi-matrix-multi-part1>

<https://maremathics.pfst.hr/index.php/2021/09/02/3-matrices-and-determinants-2/#appi-matrix-multi-part2>

could be helpful.

### QUIZ



If there is a problem with opening this quiz, an user can copy and run the following link in Crom or FireFox browser.

<https://forms.gle/Fk5e5PQVFLAXyjih7>

